Analytical solutions for diffraction problem of nonlinear acoustic wave beam in the stratified atmosphere

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The nonlinear wave equation and modified Khokhlov–Zabolotskaya type equation for high intensive acoustics wave beams propagating in stratified atmosphere with inhomogeneous of sound speed is set up. Some approaches to find analytical solutions of this equation are developed. The geometrical acoustics approximation and modified Raley integral for this problem is suggested. The asymptotical procedure is developed for describing of wave profile near the axis of wave beam. This method allows to take into account phase distortion due to diffraction and nonlinear effects and improve the nonlinear geometrical acoustics solution.

1 Introduction

The problem of intensive acoustic wave propagation in the stratified atmosphere is connected with many important applications. Among them the influence of seismic processes under Earth surface on the high layers of atmosphere and interaction and energy exchange between different geospheres can be mentioned. The main feature of this problem is that the presence of gravity leads to equilibrium density decreasing with increasing of height. As a result the amplitude of acoustic velocity increases exponentially with height, so the nonlinear effects become important even for small initial amplitudes.

2 The main equation for the intensive wave beam

The equation for intensive wave beam propagation in the stratified atmosphere can be derived from the hydrodynamic equation system:

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p-\mathbf{g}, \tag{1}$$

$$\frac{\partial \rho}{\partial t}+(\mathbf{u}\nabla)\rho+\rho\,\mathrm{div}\mathbf{u}=0, \tag{2}$$

$$\frac{\partial p}{\partial t}+(\mathbf{u}\nabla)p=c^2\left(\frac{\partial \rho}{\partial t}+(\mathbf{u}\nabla)\rho\right), \tag{3}$$

where $\mathbf{u}$ — particle velocity, ρ — medium density, p — pressure, $\mathbf{g}$ — free fall acceleration, c — local sound speed. The equation (3) is equation of the adiabatic process $s = const$ or $dp/dt = 0$, written with taking into account change of equilibrium state with height. The local sound speed depends on temperature and in the case of intensive waves includes the nonlinear terms due to nonlinear of state equation (Poisson adiabat).

Assume that wave beam propagates vertically upward along axis z. The equilibrium state is defined by the following equations:

$$\frac{\partial p_0}{\partial z}=-\rho_0 g, \quad p_0=\rho_0 RT. \tag{4}$$

For the isothermal atmosphere one can obtain the equilibrium density distribution $\rho_0 = \rho_{00}\exp(-z/H)$, where ρ_{00} — density near Earth surface and H — the width of standard atmosphere.

The following nonlinear equation for the vertical component of particle velocity can be obtained

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial t^2}-c^2\Delta w-\frac{1}{\rho}\frac{\partial c^2\rho}{\partial z}\frac{\partial w}{\partial z}\right)+c_0^2\omega_{BW}^2\Delta_\perp w = \left(\frac{\partial^2}{\partial t^2}-c^2\Delta_\perp\right)\left(Q-\frac{\partial W}{\partial t}\right)+c^2\frac{\partial R}{\partial z}. \tag{5}$$

Here the left side contains all linear terms and nonlinear term due to physical nonlinearity ($\rho=\rho_0+\rho'$, ρ' is acoustic perturbation of density), the right side contains only nonlinear terms Q, W, R of the complicated form.

It is reasonable to simplify equation (5) to construct some analytical solutions. The most appropriative approach is the method of slowly changing wave profile. This method demands the wave profile changes weak at scale of wave length λ. This condition is satisfied in the problem under discussion because the characteristic scale $H \gg \lambda$. In accordance with this method we look for the solution of this form $p' = p(\tau = t - z/c, \mu z, \sqrt{\mu}x, \sqrt{\mu}y)$,

$\mu \ll 1$ — small parameter. After simplifying one can obtain

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial^2 w}{\partial\tau\,\partial z} + \frac{1}{2\rho_0 c^2}\frac{\partial c_0^2\rho_0}{\partial z}\frac{\partial w}{\partial\tau} - \frac{c}{2}\Delta_\perp w\right) + c_0^2\omega_{BW}^2\Delta_\perp w = \frac{\varepsilon}{2c^2}\frac{\partial^4 w^2}{\partial\tau^4}. \quad (6)$$

At usual conditions the Brent frequency ω_{BW} is small and for high frequency acoustic waves one can use the following equation

$$\frac{\partial^2 w}{\partial\tau\,\partial z} + \frac{1}{2\rho_0 c^2}\frac{\partial c_0^2\rho_0}{\partial z}\frac{\partial w}{\partial\tau} - \frac{c}{2}\Delta_\perp w = \frac{\varepsilon}{2c^2}\frac{\partial^2 w^2}{\partial\tau^2}. \quad (7)$$

Equations (5)–(7) can be used for description of intensive acoustic beams in the stratified atmosphere.

3 The model nonlinear equation for the acoustic field at the axis of the gaussian wave beam

Now let us consider the linearized equation (7). For isothermal atmosphere we obtain equation

$$\frac{\partial^2 w}{\partial\tau\,\partial z} - \frac{1}{2H}\frac{\partial w}{\partial\tau} = \frac{c}{2}\Delta_\perp w. \quad (8)$$

After introducing new variable $u = w\exp(-z/2H)$ equation (8) has the form of linearized Khokhlov–Zabolotskaya equation

$$\frac{\partial^2 u}{\partial\tau\,\partial z} = \frac{c}{2}\Delta_\perp w. \quad (9)$$

Let consider the gaussian wave beam $u = \exp(-r^2/a^2)u_0(t)$ as the initial condition. Then the general solution for arbitrary function u_0 is

$$u = \frac{1}{2\pi}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty} u_0(t)\,dt \int_{-\infty}^{\infty}\frac{d\omega}{2cz/a^2 + i\omega} \times \exp\left(i\omega\left[\tau - t - \frac{r^2/a^2}{2cz/a^2 + i\omega}\right]\right). \quad (10)$$

Solution at the beam axis

$$u = u_0(\tau) - \frac{2cz}{a^2}\int_{-\infty}^{\tau} u_0(t)\exp\left(\frac{2cz}{a^2}(t-\tau)\right)dt. \quad (11)$$

Now one can write the expression for the Laplacian at the axis

$$\frac{c}{2}\Delta_\perp u\Big|_{r=0} = \frac{\partial^2 u}{\partial\tau\partial z}\Big|_{r=0} = -\frac{2c}{a^2}\frac{\partial}{\partial z}(zu)\Big|_{r=0}. \quad (12)$$

So we obtain the exact equation for acoustic field along the wave beam axis

$$\frac{\partial^2 u}{\partial\tau\,\partial z} = -\frac{2c}{a^2}\frac{\partial}{\partial z}(zu) \quad (13)$$

and the simplified model equation for field along the axis

$$\frac{\partial^2 u}{\partial\tau\,\partial z} = -\frac{2c}{a^2}u. \quad (14)$$

Now let return to the nonlinear equation (7) and use here expression (12). Then for the isothermal atmosphere we can write the model nonlinear equation

$$\frac{\partial}{\partial\tau}\left(\frac{\partial w}{\partial z} - \frac{w}{2H} - \frac{w}{c^2}\frac{\partial w}{\partial\tau}\right) = -\frac{2c}{a^2}\frac{\partial}{\partial z}(zu) \quad (15)$$

and simplified model nonlinear equation

$$\frac{\partial}{\partial\tau}\left(\frac{\partial w}{\partial z} - \frac{w}{2H} - \frac{w}{c^2}\frac{\partial w}{\partial\tau}\right) = -\frac{2c}{a^2}u. \quad (16)$$

Equation (16) can be written in new variables taking into account the exponential increasing with heght

$$w(z,\tau) = u(z,\tau)\exp(z/2H),$$

$$x_1 = \int_0^z \exp(z/2H)dz = 2H(\exp(z/2H) - 1)$$

in following form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial x_1} - \frac{u}{c^2}\frac{\partial u}{\partial\tau}\right) = -\frac{2c}{a^2}\frac{u}{1 + x_1/2H}. \quad (17)$$

The dimensionless variables can be introduced

$$z_{nl} = c_0^2\tau_0/\varepsilon u_0, \quad V = U/u_0, \quad x = x_1/z_{nl},$$

$$\theta = \tau/\tau_0, \quad x_0 = 2H/z_{nl}.$$

Equation (16) becomes

$$\frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial\theta}\right) = -\frac{NV}{1 + x/x_0}. \quad (18)$$

Give for comparison the exact equation for stratified atmosphere

$$\frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial\theta}\right) = -\frac{N\Delta_\perp V}{1 + x/x_0} \quad (19)$$

and the standard Khokhlov–Zabolotskaya equation

$$\frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial\theta}\right) = -N\Delta_\perp V. \quad (20)$$

It is obviously that for case $N \gg 1$ diffraction effects prevail and for case $N \ll 1$ nonlinear effects prevail. For case $x_0 \to \infty$ strafication effects can be neglected. The most interesting is that stratification leads to relatively decreasing diffraction effects in comparison with nonlinear effects. At heights $x/x_0 \gg 1$ diffraction effects can be neglected. It means at large heights the nonlinear geometrical acoustics will give good description.

4 Equation for nonlinear diffraction phase

The main goal is to construct analytical solutions for diffracting nonlinear wave. This case corresponds to small but finite N. If $N \gg 1$ the perturbation method will give good results where the nonlinear terms are consider as small value. If $N \to 0$ wave will be close to Riemann type wave.

Some additional information gives the following approach. Let consider variable θ as dependent on all other variables $V = V(\theta, x) \;\to\; \theta = S(V, x)$. The equation for phase S is

$$\frac{\partial}{\partial V}\left(\frac{\partial S/\partial x + V}{\partial S/\partial V}\right) = \frac{NV}{1 + x/x_0}\frac{\partial S}{\partial V} \tag{21}$$

or, after integrating

$$\frac{\partial S}{\partial x} + V = \frac{N}{1 + x/x_0}\frac{\partial S}{\partial V} \cdot \int V \frac{\partial S}{\partial V} dV. \tag{22}$$

For case $N = 0$ one can obtain the solution $S = -xV$ that is the phase of plane nondiffractive Riemann wave. On the other hand the linear problem correspond to neglecting term V in the left side of (22). This solution S_0 can be obtain from linearized Khokhlov–Zabolotskaya equation.

5 Solution for nonlinear diffraction phase

The idea of constructing analytical solution is as follows. Let expand function S as a series $S = S_0 + \tilde{S}$, $\tilde{S} = S_1 + NS_2$ on small parameter N, where S_0 is the solution of linear equation. Solution S_0 has the form $S_0 = \Phi(V, x) + f(x)$, $f(0) = 0$. Function $\Phi(V, x)$ is connected with initial condition. Function $\tilde{S}$ describes phase shift due to nonlinear effects and interactions of nonlinear and diffractive effects. Function $\tilde{S}$ satisfies to equation

$$\frac{\partial \tilde{S}}{\partial x} + V = \frac{N}{1 + x/x_0}\left(\frac{\partial S_0}{\partial V} \cdot \int V \frac{\partial \tilde{S}}{\partial V} dV + \frac{\partial \tilde{S}}{\partial V} \cdot \int V \frac{\partial S_0}{\partial V} dV + \frac{\partial \tilde{S}}{\partial V} \cdot \int V \frac{\partial \tilde{S}}{\partial V} dV\right). \tag{23}$$

It is interesting that function $f(x)$ from S_0 doesn't influence on nonlinear phase. Only distortion of initial function $\Phi(V, x)$ can influence on this phase. It is obviously that this part of full nonlinear phase S is connected with change of wave amplitude. And other part is connected with the "true phase". Of course there are also terms responsible for interaction.

First, let expand $\tilde{S}$ as a series $\tilde{S} = S_1 + NS_2$, so

$$\frac{\partial S_1}{\partial x} = -V, \quad S_1 = -xV \tag{24}$$

and

$$\frac{\partial S_2}{\partial x} = \frac{1}{1 + x/x_0}\left(\frac{\partial S_0}{\partial V} \cdot \int V \frac{\partial S_1}{\partial V} dV + \frac{\partial S_1}{\partial V} \cdot \int V \frac{\partial S_0}{\partial V} dV + \frac{\partial S_1}{\partial V} \cdot \int V \frac{\partial S_1}{\partial V} dV\right). \tag{25}$$

For initial condition $V(x = 0) = \sin\theta$ solution $S_0 = \arcsin V - Nx$. Therefore

$$\frac{\partial S_2}{\partial x} = \frac{1}{1 + x/x_0}\left(\frac{x^2 V^2}{2} - x\frac{3V^2 - 2}{\sqrt{1 - V^2}}\right). \tag{26}$$

The main contribution is given by diffraction-nonlinear interaction and is quadratic at small x. It is important that with diffraction taking into account phase of nonlinear wave contains not only the first power of V but all other powers.

Another approach to obtain analytical solution is the nonlinear geometrical acoustics. This method leads to equation which can be solved exactly. But nonlinear part of phase of wave in this approach coincide with phase of plane Riemann wave. Using the improved expression (24)–(26) for phase in the nonlinear geometrical acoustics solution can give more accurate solution for diffractive intensive wave beam at its axis.

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