Theory of selfrefraction effect of intensive focused acoustical beams

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The theory of selfrefraction of nonlinear acoustical beams is developed based on some exact and approximate analytical equations and solutions. The system of base equations in geometrical acoustics approximation is sequentially derived from Khokhlov–Zabolotskaya equation for nonlinear focused acoustical beams. The generalized method of extended characteristics allows to set up the simplified closed equation for ray convergence on the beam axis for the most interesting case of small diffraction, when large amplitudes in the focal area are observed. The exact solution is derived in particular case. For the common case of wave parameters there are suggested some analytical approximations and numerical solution. The amplitude dependencies on longitudinal and transversal distances and other wave parameters are obtained. It is shown that at the axis of gaussian beam in the focal area the local minimum of amplitude can be formed. Some initial transversal beamforms, such as gaussian, and initial phase modulation as parabolic or sinusoidal are analyzed.

1 Introduction

The problem of discontinuous focused acoustical beam propagation and calculation their parameters near focal area is under consideration in this paper. Discontinuous waves and waves with shock fronts are special objects of nonlinear acoustics. They are the general asymptotic solution at large distances for arbitrary initial wave profile and have some special features. In particular the speed of shock front propagation depends on its peak amplitude. This fact leads to such effect as selfrefraction (or nonlinear refraction) of intensive acoustical beams. Let us consider the bounded beam of discontinuous waves for example gaussian beam. The shock front amplitude and consequently its speed depends on the transversal coordinate so the wave front will be distorted and cause wave defocusing. Moreover the propagation speed of all nonsymmetrical acoustical pulses differs from the local sound speed and consequently all these pulses are also influenced by selfrefraction.

2 Model equations and previous results

The first self-consistent method describing the beam propagation with selfrefraction taken into account was suggested in paper [1]. The system of modified nonlinear geometrical acoustics approximation equations was written, where the new term responsible for the selfrefraction was added in the eikonal equation's right hand by analogy with plane waves:

$$\frac{\partial\alpha}{\partial z}+\alpha\frac{\partial\alpha}{\partial r}=-\frac{\mu}{2}\frac{\partial A}{\partial r}, \tag{1}$$

$$\frac{\partial p}{\partial z}-\gamma\left(p-\frac{A}{2}\right)\frac{\partial p}{\partial T}+\alpha\frac{\partial p}{\partial r}+\frac{p}{2}\left(\frac{\partial\alpha}{\partial r}+\frac{\alpha}{r}\right)=0. \tag{2}$$

These equations are already written in dimensionless variables, convenient to further calculations. Here $\alpha=\partial\psi/\partial r$ — ray inclination function, ψ — eikonal, z and r — dimensionless longitudinal and transversal coordinates, p — acoustical pressure, $A(z,r)$ — dimensionless beam amplitude, $T=\tau-\psi/c_0$, $\tau=t-z/c_0$ — retarded time. There are two dimensionless parameters: $\gamma=F/x_s$ defines the relative contribution of focusing (F — focal length) and nonlinearity (x_s — nonlinear length),

$$\mu=\frac{\varepsilon A_0}{\rho c_0^2}\frac{F^2}{r_0^2}=\frac{\lambda}{x_s}\frac{F^2}{r_0^2}=\frac{F}{x_s}\frac{F}{x_{\rm diff}}=\frac{A_0}{2p_{\rm int}\theta^2}$$

defines the selfrefraction "strength". Here $x_{\rm diff}=r_0^2/\lambda$, $p_{\rm int}=\rho c_0^2/2\varepsilon$, $\theta=r_0/F$, ε — nonlinearity parameter, A_0, r_0 — amplitude and beam radius.

However these equations were not derived from any more exact equations so the question about boundaries of their applicability is still open. Besides in paper [1] some suggestions such as paraxial approximation were used and only numerical solutions were obtained. The main result of [1] is the empirically obtained from numerical calculations expression for the pressure limit in the focal area $p_{\rm lim}=(1{,}3-1{,}7)p_{\rm int}\theta^2$, which depends only on parameters of medium and beam geometry and very weak on other parameters.

Another approach based on numerical solutions of more exact Khokhlov–Zabolotskaya equation was developed in [2] to calculate the pressure limit. It have been shown that the pressure limit changes weakly for the wide interval of parameters and is in accordance with results of the previous paper [1].

But some questions have not answered yet:

1. Could the simplified equation be derived more correctly and for arbitrary transversal form of beam?
2. Could some analytical estimations be obtained?
3. What is the main factor limiting the pressure in the focus — diffraction or selfrefraction?
4. Could the improved nonlinear geometrical approximation be used to describe focal area of discontinuous waves instead the Khokhlov–Zabolotskaya equation?

This work is an attempt to give answers to these questions.

3 Main equations for selfrefraction effect

First of all we proceed from the base equation of intensive acoustical beam theory — Khokhlov–Zabolotskaya equation

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial z} - \mu p \frac{\partial p}{\partial \tau}\right) = \Delta_\perp p. \tag{3}$$

It can be written for $\tau = \tau(p, z)$ as the function of independent variables p and z [3]

$$\frac{\partial}{\partial p}\left[\left(\frac{\partial \tau}{\partial p}\right)^{-1}\left(\frac{\partial \tau}{\partial z} + \frac{1}{2}\left(\frac{\partial \tau}{\partial r}\right)^2 + \mu p\right)\right] = \Delta_\perp \tau. \tag{4}$$

If we neglect diffraction ($\Delta_\perp \tau \to 0$) and use the "equal area" rule for the shock front area we derive the following equation for shock front movement

$$\frac{\partial \tau_s}{\partial z} + \frac{1}{2}\left(\frac{\partial \tau_s}{\partial r}\right)^2 + \mu \frac{A}{2} = 0, \tag{5}$$

which is coincide in sense with Eq. (1). So Eq. (1) is written in nondiffraction approximation and for shock front. So there is the principal possibility to calculate diffraction corrections. However we will consider further only the nondiffraction model.

Now let introduce in Eq. (1)–(2) new transversal coordinate ξ so called ray coordinate which physical meaning is the initial transversal coordinate of any ray. Thus current transversal coordinate $r = r(\xi, z)$. This allows to write the base system of equations describing arbitrary wave beam [4, 5]:

$$\frac{\partial \alpha}{\partial z} = -\frac{\mu}{2}\frac{1}{r_\xi}\frac{\partial A}{\partial \xi}, \qquad \frac{\partial r}{\partial z} = \alpha, \tag{6}$$

$$\frac{\partial p}{\partial z} - \gamma\left(p - \frac{A}{2}\right)\frac{\partial p}{\partial T} + \frac{p}{2}\left(\frac{1}{r_\xi}\frac{\partial \alpha}{\partial \xi} + \frac{\alpha}{r}\right) = 0. \tag{7}$$

The amplitude of initial N-wave with arbitrary transversal form ($p(z=0) = R(r)p_0(T)$, $p_0(T) = -T$ for $|T| < 1$ and $p_0(T) = 0$ for $|T| > 1$) is defined by expression

$$A = \frac{R(\xi)}{\sqrt{S}}\frac{1}{\sqrt{1 + \gamma R(\xi) s}}, \qquad S = \frac{r r_\xi}{\xi}, \qquad s = \int_0^z \frac{dz'}{\sqrt{S}}.$$

So equations for ray trajectory r and ray convergence r_ξ are

$$\frac{\partial^2 r}{\partial z^2} = -\frac{\mu}{2}\frac{1}{r_\xi}\frac{\partial A}{\partial \xi}, \quad \frac{\partial^2 r_\xi}{\partial z^2} = -\frac{\mu}{2}\frac{\partial}{\partial \xi}\left(\frac{1}{r_\xi}\frac{\partial A}{\partial \xi}\right). \tag{8}$$

The formal implicit solution for ray convergence at small μ can be written

$$r_\xi = 1 + \alpha_0'(\xi) z - \frac{\mu}{2}\int_0^z d\eta'' \int_0^{\eta''} \frac{\partial}{\partial \xi}\left(\frac{\partial A(\xi, \eta')}{\partial \xi}\frac{d\eta'}{r_\xi}\right).$$

Now one can conclude that selfrefraction 1) limits the peak amplitude in the focal area, 2) moves the maximum amplitude position along beam axis far from geometrical focus.

4 Analytical solutions for pressure along beam axis. Pressure limit

To obtain analytical solutions we make some suggestions based on physical sense. It can be shown that parameter μ, which describes the selfrefraction "strength", is small for more interesting cases, including relatively small Mach number, case of strong focusing etc. Besides, large μ corresponds to strong defocusing due to diffraction, nonlinear absorption and selfrefraction so this situation looks not very useful.

Now consider the classical case of gaussian beam. Simple equation can be written for field along the gaussian beam axis in the first order approximation on μ ($Q \equiv r_\xi(\xi = 0) = r(\xi = 0)$):

$$\frac{d^2 Q}{dz^2} = \frac{\mu}{2Q^2}\frac{2 + \gamma s}{(1 + \gamma s)^{3/2}}, \qquad s = \int_0^z \frac{dz'}{Q} \tag{9}$$

with conditions $Q(z = 0) = 1$, $dQ/dz(z = 0) = -1$.

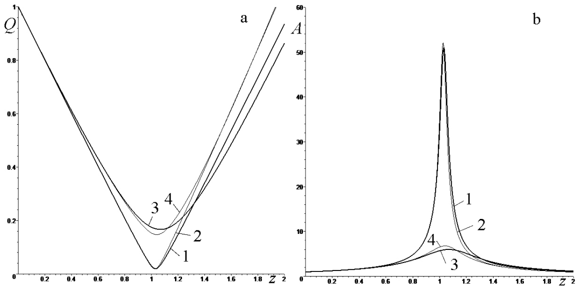

Figure 1: Ray convergence (a) and peak amplitude (b)

For qualitative estimation of the pressure limit we consider the case of gamma tending to zero when the exact solution can be obtained:

$$z_1 = (1+2\mu)^{-1}\left[1 - \sqrt{Q}\sqrt{(1+2\mu)Q - 2\mu}\right] + \frac{2\mu}{(1+2\mu)^{3/2}} \ln \frac{1 + 1/\sqrt{1+2\mu}}{\sqrt{Q} + \sqrt{Q - 2\mu/(1+2\mu)}} \quad (10)$$

before the turning point and $z_2 = 2z_0 - z_1$ after the turning point. The turning point z_0 is determined by the condition $z_0 = z_1\,(Q_{\min})$. This solution is shown on Fig. 1,a for parameters $\mu = 0{,}01$ (curves 1, 2) and $\mu = 0{,}1$ (curves 3, 4). There is also shown comparison between first (curves 1, 3) and second (curves 2, 4) order approximations on parameter μ. One can see that corresponding curves are in a good agreement with each other. At the Fig. 1,b the peak amplitude for the same values of parameters are shown. Main results are as follows. We take into account only selfrefraction without any diffraction and obtain the finite pressure limit. Maximum position moves beyond "linear" geometrical focus.

Solution (10) allows to find minimum value of ray convergence $Q_{\min} = 2\mu/(1+2\mu)$ and consequently the pressure limit ($A_{\max} = Q_{\min}^{-1}$) in focus:

$$A_{\max} = \frac{1+2\mu}{2\mu} \quad \Rightarrow \quad \frac{\tilde{A}_{\max}}{p_{\text{int}}\theta^2} = 1 + 2\mu \quad (11)$$

(here $\tilde{A}_{\max} = A_0 A_{\max}$ — physical (dimensional) amplitude). This theoretical expression is in a good accordance with empiric estimation from [1, 2]. And we should remember that expression (11) concerns only special case.

For nonzero gamma the most effective method to obtain analytical solution is the straight expansion on small μ: $Q = Q_0 + \mu Q_1 + \mu^2 Q_2$. The solution can be obtained in common case as quadratures. The comparison of this solution with numerical one is

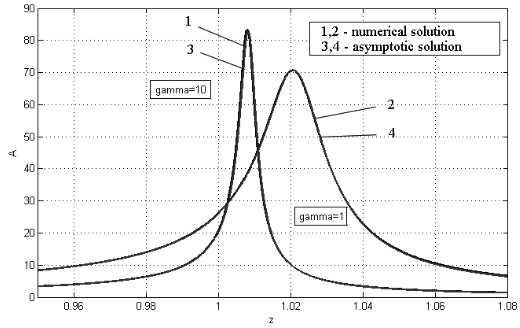

Figure 2: Comparison of numerical and asymptotic solutions

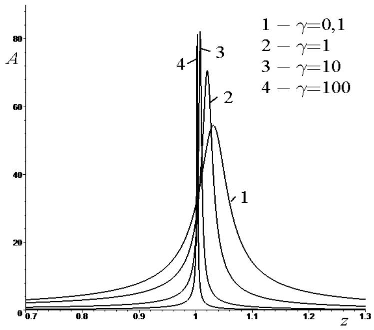

Figure 3: Peak amplitude for different γ

shown at Fig. 2 and one can see good agreement between them. At Fig. 3 the peak pressure along beam axis are shown for different γ. One can conclude that the pressure limit increases with gamma increasing for small gamma and reaches saturation for large gamma. Longitudinal focal area on the contrary decreases with gamma increasing.

5 Transversal structure

The peak pressure along different rays ξ are shown for small $\gamma = 0{,}1$ at Fig. 4 and $\gamma = 1$ at Fig. 5. Note that the local minimum of the peak pressure forms near beam axis for small gamma even for initial gaussian beam. It can be obtained that focal area defined by selfrefraction depends weakly on parameter gamma. On the other hand focal area defined by diffraction is $\gamma/\sqrt{1+\gamma^2}$. So for small gamma diffraction is not significant and for large gamma selfrefraction and diffraction works jointly.

The developed model allows to calculate and other spatial-modulated beams, not only gaussian beams. At Fig. 6 the peak amplitude for the beam with periodic modulated wave front $\alpha_0(\xi) = -\sin\xi$ are shown as example.

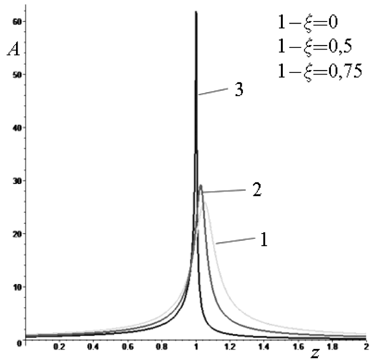

Figure 4: Peak amplitude along different rays, $\gamma = 0{,}1$

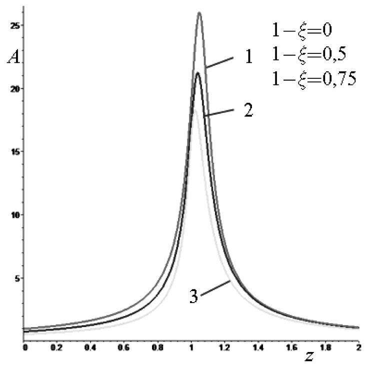

Figure 5: Peak amplitude along different rays, $\gamma = 1$

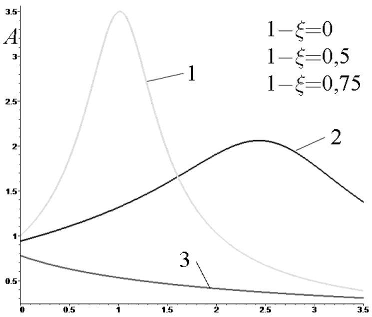

Figure 6: Peak amplitude for periodic-modulated beam

6 Conclusions

1. The correct derivation of the equation for the shock front propagation is suggested so there is the principal possibility to take into account diffraction effects.
2. Theoretical expressions for the pressure limit are obtained. The pressure limit does not change significantly for wide interval of parameters.
3. At small gamma diffraction does not influence significantly on the spatial structure of the wave beam and is determined mainly by selfrefraction. At large gamma diffraction can cause some corrections to transversal structure.
4. The developed theory can be used for describing discontinuous waves with arbitrary initial wave front and transversal form.
5. All these results allows to conclude that improved nonlinear geometrical approximation is able to describe propagation of discontinuous waves and waves with shock fronts even in focal area, at least, qualitatively.

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