Derivation of modified Smyshlyaev's formulae using integral transform of Kontorovich-Lebedev type

Valery Yu. Valyaev, Andrey V. Shanin
Physics Department, Moscow State University, Russia; e-mail: valery-valyaev@yandex.ru

The aim of this work is to fill the gap between the embedding formulae for cones and the modified Smyshlyaev's formulae. Embedding formulae for cones represent the directivity of the scattered field as multiple integrals over spatial variables. Modified Smyshlyaev's formulae represent the same directivity as a single contour integral over parameter ν. This situation resembles the convolution theorem for Fourier transform: multiple convolutions can be represented as a single integral over frequency.
Originally, modified Smyshlyaev's formulae have been "guessed" and then proved by study of the poles of the integrands instead of being regularly derived. Extension of the analogy with Fourier transform allows to obtain a regular method for deriving the modified Smyshlyaev's formulae.
To perform this extension we introduce integral transform of Kontorovich - Lebedev type and prove for it the analogues of Plancherel's and convolution formulae. Using the developed technique we demonstrate the possibility to derive the modified Smyshlyaev's formulae.

1 Introduction

1.1 Motivation

As an example we are considering the scalar problem of plane wave diffraction by a quarter plane. As usual, our main goal is to find the diffraction coefficient of the scattered field. General approach to this kind of diffraction problems is separation of radial variable and studying the Laplace-Beltrami problem on the unit sphere. This approach has been significantly developed by Smyshlyaev and coworkers [1]. He has obtained the following formula for the diffraction coefficient:

$$f(\omega, \omega_0) = \frac{i}{\pi} \int_\gamma e^{-i\pi\nu} g(\omega, \omega_0, \nu) \nu d\nu, \qquad (1)$$

where ω_0 and ω are directions of incidence and scattering, ν is the separation parameter and g is the Green's function of the of the spherical problem.

Further extension of this approach has been achieved in work [2] in which the formulae of the same type as (1) were obtained. In these formulae integrand is constructed from so called spherical edge Green's functions, which can be treated as a limiting case of Green's function g as the source location approaches edge of the scatterer. Integrals over separation parameter in these modified Smyshlyaev's formulae have better convergence properties than one in (1).

Derivation of these formulae consists of three steps. At first an embedding operator is applied to the total field and the result is expressed in terms of edge Green's functions in 3D space. This expression is called embedding formula. Then the directivities of edge Green's functions in 3D space are represented as the series over the eigenvalues of Laplace-Beltrami operator. By using the embedding formula one obtains the series for the diffraction coefficient of the scattered field. Final step is transformation of the series to the contour integral. In fact the resulting formula had to be guessed and then checked by the study of the poles of the integrands instead of being regularly derived.

This approach can also be applied to more complicated problems, for example to the problem of diffraction by the wedge of the cube [3]. Guessing of the formulae here becomes more difficult and potentially lead to errors.

This paper fills the mentioned gap between the embedding formulae for cones and the modified Smyshlyaev formulae, giving a regular way of deriving the later from the former.

1.2 Basic ideas

There are embedding formulae of two sorts:

$$f(\omega, \omega_0) = \int_0^\infty f_1(\omega, r) f_2(\omega_0, r) \frac{dr}{r} \qquad (2)$$

and

$$f(\omega, \omega_0) = \\ = \int_0^\infty \frac{dr}{r} \int_0^\infty f_1(\omega, r) g(r, r') f_2(\omega_0, r') \frac{dr'}{r'} \qquad (3)$$

with $f_{1,2}$ of the form (15) (possibly multiplied by r^{-n} for some integer n), and $g(r, r')$ of the form (17). Note that $f_{1,2}$ and g are expressed through contour integrals over parameter ν. Formally, a direct implementation of (2) or (3) leads to calculation of three (in case of (2)) or five (in case of (3)) nested integrals over ν and r. However, fortunately, these integrals can be converted into modified Smyshlyaev's formulae expressing $f(\omega, \omega_0)$ as a single integral over the parameter ν.

The possibility of reducing several integrals to a single one reminds the well-known properties of Fourier transform. Let us explain this analogy. Let $\hat{F}_{1,2}(\xi)$ be transforms of the functions $F_{1,2}(x)$. Let also $\hat{G}(\xi)$ be transform of function $G(x)$, and we introduce the function $G(x, y) = G(x - y)$. Then

$$\int_{-\infty}^{\infty} F_1(x) F_2^*(x) dx = \int_{-\infty}^{\infty} \hat{F}_1(\xi) \hat{F}_2^*(\xi) d\xi \qquad (4)$$

which is the Plancherel's theorem, and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(x) G(x, y) F_2^*(y) dx dy = \\ = \int_{-\infty}^{\infty} \hat{F}_1(\xi) \hat{G}(\xi) \hat{F}_2^*(\xi) d\xi, \qquad (5)$$

which is a combination of the Plancherel's theorem and the convolution theorem. Here superscript star stands for complex conjugation. If $F_{1,2}(x)$ and $G(x, y)$ are expressed as integrals containing $\hat{F}_{1,2}(\xi)$ and $\hat{G}(\xi)$ then the left-hand side of (4) contains three integrals, and the left-hand side of (5) contains five integrals. In both cases the right-hand side contains only one integral.

The most straightforward way to extend this analogy to the conical case is to use the Kontorovich-Lebedev transform. However, we cannot use it directly because of convergence problems. In particular, for the classical Kontorovich-Lebedev procedure it is necessary for the parameter k_0 of the Helmholtz equation to be purely imaginary, which is hardly interesting from the practical point of view.

That is why, we develop a slightly different approach. Instead of the Kontorovich-Lebedev transform we use another representation that differs by the choice of the cylindrical function (Bessel instead of Hankel), and, more important, by the contour of integration. As the result, the functions participating in the representation are no longer orthogonal. However, for our needs the orthogonality (and even the uniqueness and invertibility of the representation) is not relevant, we need only the analogs of Plancherel formula and convolution formula. That is why, we prove only these important formulae without using orthogonality and demonstrate the possibility of deriving the modified Smyshlyaev's formulae.

2 Basic relations

2.1 *Problem statement*

We are considering the scalar Dirichlet problem of diffraction by quarter plane $Q = \{(x, y, z) | x \geq 0, y \geq 0, z = 0\}$ (see Fig. 1).

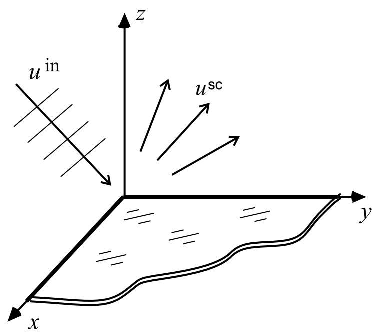

Figure 1: Geometry of the problem.

Let the Helmholtz equation

$$\Delta u + k_0^2 u = 0 \qquad (6)$$

be valid in the 3D space (x, y, z). The time dependence of all variables is of the form $e^{-i\Omega t}$ and is omitted henceforth.

The Dirichlet boundary conditions fulfilled on both surfaces of the quarter plane is of the form:

$$u|_Q = 0. \qquad (7)$$

Let the incident field have the form of a plane wave coming from direction ω_0:

$$u^{inc} = e^{-ik_0(\omega_{0x} x + \omega_{0y} y + \omega_{0z} z)}. \qquad (8)$$

Beside the governing equation and boundary conditions, the radiation, edge and vertex conditions should be imposed to make a proper problem formulation. We do not discuss these matters here and refer reader to [2].

Our main interest is to find the diffraction coefficient $f(\omega, \omega_0)$ of the scattered field $u^{sc} = u - u^{inc}$, which we define as the amplitude of the spherical wave diffracted by the tip of the quarter plane:

$$u^{sc}(r, \omega) = 2\pi \frac{e^{ik_0 r}}{k_0 r} f(\omega, \omega_0) + O(r^{-2}),\ r \to \infty. \quad (9)$$

It depends not only on the scattering direction ω but also on the direction ω_0 from which the incident wave comes.

2.2 Edge Green's functions in 3D space

Let us consider Greens function $G(x, y, z; x_0, y_0, z_0)$ of our problem, i.e. the function which obeys the equation

$$\Delta G + k_0^2 G = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad (10)$$

and the same boundary, edge, vertex and radiation conditions as field u does.

We define the pair of edge Green's functions in 3D space $G_x(x, y, z; X)$ and $G_y(x, y, z; Y)$ as following limits:

$$G_x(x, y, z; X) = \lim_{\varepsilon \to 0} \sqrt{\frac{\pi}{\varepsilon}} G(x, y, z; X, -\varepsilon, 0)$$

and

$$G_y(x, y, z; Y) = \lim_{\varepsilon \to 0} \sqrt{\frac{\pi}{\varepsilon}} G(x, y, z; -\varepsilon, Y, 0), \quad (11)$$

i.e. as fields produced by sources lying on the edges of the scatterer (see Fig. 2).

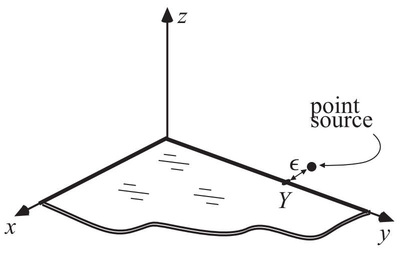

Figure 2: To the definition of the edge Green's function G_y.

Hereafter we consider the diffraction coefficients $f_x(\omega, X)$ and $f_y(\omega, Y)$ of the edge Green's functions G_x and G_y correspondingly, defined in the same way as f in (9).

In [2] the embedding formulae which connects directivity f with directivities f_x and f_y are derived. Here we present one of them, namely

$$f = \frac{4\pi^2 i}{k_0^2(\omega_y + \omega_{0y})} \int_0^\infty f_x(\omega; X) f_x(\omega_0; X) dX \quad (12)$$

which is of the form (2). Embedding formula of type (3) is also presented in [2], but we do not consider it here for the sake of brevity.

2.3 Edge Green's functions on the unit sphere

Solving the problem (6)-(8), after the separation of the radial variable one comes to the Laplace-Beltrami problem on the unit sphere S with a cut S_q produced by the quarter plane $S_q = S \cap Q$ (see Fig. 3).

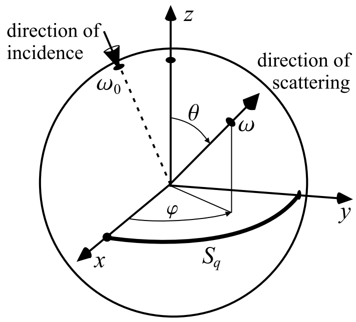

Figure 3: Geometry of the problem on the sphere.

We introduce the Green's function $g(\omega, \omega_0, \nu)$ of this sphere as the solution of the following problem

$$\left[\tilde{\Delta} + \left(\nu^2 - \frac{1}{4}\right)\right] g = \delta(\omega - \omega_0), \quad (13)$$

which obeys Dirichlet conditions on the cut: $g|_{S_q} = 0$ and Meixner conditions at the ends of the cut (see [2]). It is the function participating in (1).

Let us define the edge Green's functions on the sphere $v_x(\omega, \nu)$ and $v_y(\omega, \nu)$ as the following limits:

$$v_x(\omega, \nu) = \lim_{\kappa \to 0} \sqrt{\frac{\pi}{\kappa}} g(\omega, \omega_{\kappa x}, \nu) \text{ and}$$
$$v_y(\omega, \nu) = \lim_{\kappa \to 0} \sqrt{\frac{\pi}{\kappa}} g(\omega, \omega_{\kappa y}, \nu), \quad (14)$$

where $\omega_{\kappa x}$ is the point with conventional spherical coordinates $\theta = \pi/2$ and $\phi = -\kappa$ (see Fig. 4) and similarly for $\omega_{\kappa y}$.

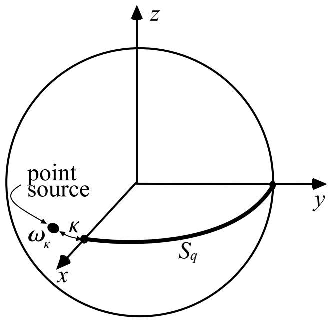

Figure 4: To the definition of the edge Green's function on the sphere.

Our aim now is to transform (12) into the modified Smyshlyaev's formula, i.e. the contour integral over parameter ν from the function involving $v_x(\omega,\nu)$. In order to do it in the regular way we introduce the integral transform of Kontorovich-Lebedev type.

3 The representation of Kontorovich-Lebedev type

3.1 Definition of the representation

We introduce the *representation* for two types of functions. For a function of a single variable $h(r)$, $r > 0$ the representation is as follows:

$$h(r) = \frac{1}{2}\int_\gamma e^{-\mathrm{i}\pi\nu/2} J_\nu(k_0 r)\phi(\nu)\nu\, d\nu. \qquad (15)$$

Here $\phi(\nu)$ is the *transformant* of $h(r)$. Contour γ is shown in Fig. 5.

We assume that

1. function $\phi(\nu)$ is even

$$\phi(-\nu) = \phi(\nu); \qquad (16)$$

2. singularities of $\phi(\nu)$ are only isolated poles on the real axis, and $\phi(\nu)$ is regular at $\nu = 0$;

3. function $\phi(\nu)$ decays exponentially as $|\mathrm{Im}[\nu]| \to \infty$.

Let function $g(r,r')$, $r,r' > 0$ of two variables admit the following representation:

$$g(r,r') = \frac{1}{2}\int_\gamma J_\nu(k_0 r_<) H_\nu^{(1)}(k_0 r_>)\psi(\nu)\nu\, d\nu, \qquad (17)$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$ If $\psi(\nu)$ obeys conditions 1–3 listed above, this function is called the transformant of g.

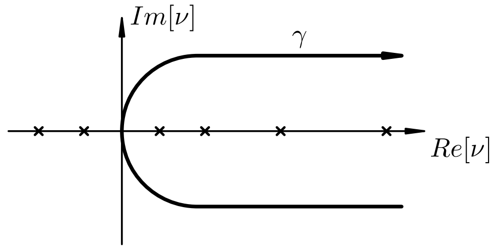

Figure 5: Contour γ.

Note that there is a considerable difference between representations (15) and (17). In (15) a function of one variable is represented through another function of one variable, while in (17) a function of two variables is represented through a function of one variable. Thus, representation (17) exists for a very restricted class of functions.

We do not need the transformation converting $h(r)$ into $\phi(\nu)$. We also do not need uniqueness of the transformants in (15) and (17). The functions that have these representations emerge naturally from solving the Helmholtz equation in conical coordinates [2].

Let us now prove some important for our goals properties of the representation.

4 Properties of the representation

Here we study two types of integrals emerging in embedding formulae. The first one is an analog of a convolution property.

Theorem 1 *Let $h(r)$ and $g(r,r')$ be functions having representations (15) and (17) with transformants ϕ and ψ, respectively. Then*

$$\int_0^\infty g(r,r')h(r')\frac{dr'}{r'} = \frac{1}{2}\int_\gamma e^{-\mathrm{i}\pi\nu/2} J_\nu(k_0 r)\phi(\nu)\psi(\nu)\nu\, d\nu, \qquad (18)$$

i.e. the integral in the l.-h.s. of (18) has the representation of the form (15) with the transformant $\phi(\nu)\psi(\nu)$.

The proof is as follows. Transform the contours of integration in representations of ϕ and ψ to γ_μ and γ_ν correspondingly (see Fig. 6). Then convert the product $g(r,r')h(r')$ to double integral over cartesian product $\gamma_\nu \times \gamma_\mu$.

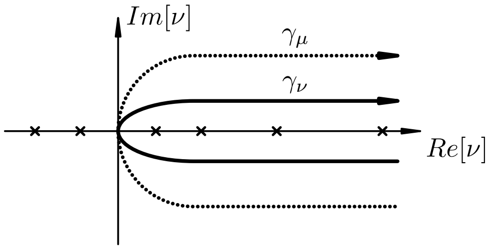

Figure 6: Contours γ_μ and γ_ν.

Denote the integral in the l.-h.s. of (18) by $K(r)$.

$$K(r) = \frac{1}{4}\int_0^\infty \frac{dr'}{r'} \iint_{\gamma_\nu\times\gamma_\mu} e^{-\mathrm{i}\frac{\pi}{2}\mu}\mu\,\nu\,\phi(\mu)\psi(\nu)\times \\ \times J_\mu(k_0 r')J_\nu(k_0 r_<)H_\nu^{(1)}(k_0 r_>)\,d\mu\,d\nu \quad (19)$$

Change the integration order and calculate the integral over r first. Namely, find

$$I = \int_0^\infty J_\mu(k_0 r')J_\nu(k_0 r_<)H_\nu^{(1)}(k_0 r_>)\frac{dr'}{r'}. \quad (20)$$

Represent I as a sum of two integrals

$$I = H_\nu^{(1)}(k_0 r)\int_0^r J_\mu(k_0 r')J_\nu(k_0 r')\frac{dr'}{r'} + \\ + J_\nu(k_0 r)\int_r^\infty J_\mu(k_0 r')H_\nu^{(1)}(k_0 r')\frac{dr'}{r'}. \quad (21)$$

Use a well-known formula

$$\int^r Z_\mu^{(1)}(k_0 r')Z_\nu^{(2)}(k_0 r')\frac{dr'}{r'} = \\ = \frac{1}{\mu+\nu}Z_\mu^{(1)}(k_0 r)Z_\nu^{(2)}(k_0 r) - \\ - \frac{k_0 r}{\mu^2-\nu^2}\left[Z_{\mu+1}^{(1)}(k_0 r)Z_\nu^{(2)}(k_0 r) - \right. \\ \left. - Z_\mu^{(1)}(k_0 r)Z_{\nu+1}^{(2)}(k_0 r)\right], \quad (22)$$

where $Z^{(1)}$ and $Z^{(2)}$ stand for general cylindrical functions (i.e. they can be replaced by J or $H^{(1)}$ in our formulae). Performing all computations, get

$$I = \frac{2\mathrm{i}}{\pi}\frac{J_\mu(k_0 r) - e^{\mathrm{i}\pi(\mu-\nu)/2}J_\nu(k_0 r)}{\mu^2-\nu^2}. \quad (23)$$

Note that I is regular at $\mu = \nu$.

Substitute (23) into (19). Split the double integral into sum of two terms and convert them to iterated integrals.

$$K(r) = \\ = -\frac{\mathrm{i}}{2\pi}\left[\int_{\gamma_\mu}\int_{\gamma_\nu}\frac{e^{-\mathrm{i}\frac{\pi}{2}\mu}}{\nu^2-\mu^2}J_\mu\mu\,\nu\,\phi(\mu)\psi(\nu)\,d\nu\,d\mu + \right. \\ \left. + \int_{\gamma_\nu}\int_{\gamma_\mu}\frac{e^{-\mathrm{i}\frac{\pi}{2}\nu}}{\mu^2-\nu^2}J_\nu\mu\,\nu\,\phi(\mu)\psi(\nu)\,d\nu\,d\mu\right]. \quad (24)$$

In the first term transform the contour of integration over ν into γ'_ν shown at Fig. 7 (the part with $\mathrm{Im}[\nu] < 0$ is symmetrically reflected with respect to zero). Due to relation (16) this change does not affect the integral. For each non-zero μ the integral over ν can be then taken by residue method after closing the integration path at $+\mathrm{i}\infty$.

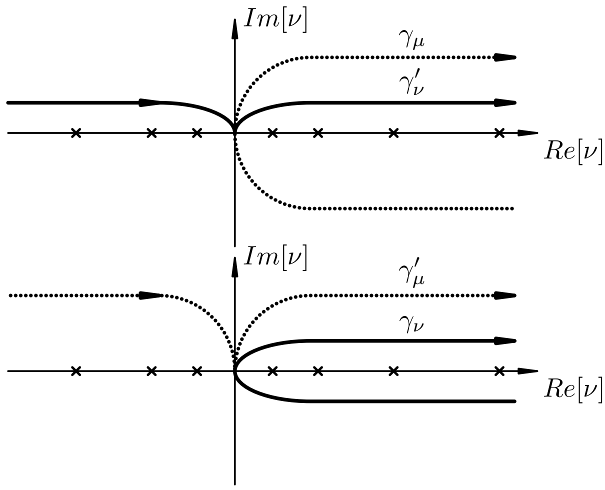

Figure 7: Contours γ'_μ and γ'_ν.

In the second term the contour γ_μ is deformed into γ'_μ. It is also closed in the upper half-plane.

The first term gives non-trivial poles at $\nu = \pm\mu$, and the second term compensates the singularity of the first term at $\mu = \nu = 0$. The result is (18).

The second result reminds of the Plancherel's equation.

Theorem 2 *Let $h_1(r)$ and $h_2(r)$ be functions that have the representation (15) with transformants ϕ_1*

and ϕ_2 respectively. Then

$$\int\limits_0^\infty h_1(r)h_2(r)\frac{dr}{r} = \\ = \frac{1}{4}\int_\gamma e^{-\mathrm{i}\pi\nu}\phi_1(\nu)\phi_2(\nu)\,\nu\,d\nu. \quad (25)$$

To prove it let us follow the procedure used for proving the previous theorem, i.e. deform the contours into γ_μ and γ_ν, change the order of integration, then use well-known formula

$$\int\limits_0^\infty J_\mu(k_0r)J_\nu(k_0r)\frac{dr}{r} = \frac{2}{\pi}\frac{\sin[\pi(\mu-\nu)/2]}{\mu^2-\nu^2}. \quad (26)$$

After that we split the integral into two, using the symmetry deform the contour of inner integration into γ'_μ or γ'_ν and calculate the inner integral by residues. As the result, we get (25).

Concluding this section let us prove the theorem describing how multiplication by $1/r$ affects the representation of the function.

Theorem 3 *Let $h(r)$ have representation (15) with the transformant ϕ. Let ν_* be the only pole of $\phi(\nu)$ on the segment* $(0,1]$.
Then

$$\frac{h(r)}{r} = \frac{1}{2}\int_{\gamma+\Gamma} e^{-\mathrm{i}\pi\nu/2}J_\nu(k_0r)\tilde{\phi}(\nu)\nu d\nu \qquad (27)$$

where $\tilde{\phi}(\nu) = \frac{\mathrm{i}k_0}{2\nu}[\phi(\nu-1)-\phi(\nu+1)]$ and additional contour Γ shown at Fig. 8 consists of two loops encircling points ν_*-1 and $1-\nu_*$.

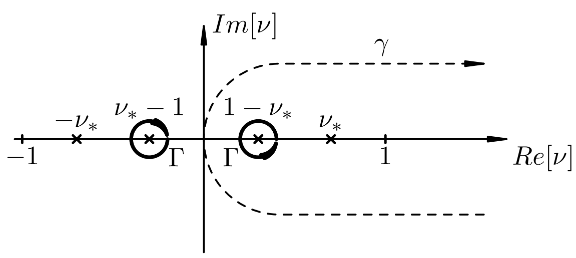

Figure 8: Contour Γ.

The proof is as follows. Let us take into account the well-known formula

$$\frac{J_\nu(z)}{z} = \frac{J_{\nu+1}(z)+J_{\nu-1}(z)}{2\nu} \qquad (28)$$

Thus,

$$\frac{h(r)}{r} = \frac{\mathrm{i}k_0}{4}\left[\int_{\gamma+1} e^{-\mathrm{i}\pi\nu/2}J_\nu(k_0r)\phi(\nu-1)d\nu - \right. \\ \left. -\int_{\gamma-1} e^{-\mathrm{i}\pi\nu/2}J_\nu(k_0r)\phi(\nu+1)d\nu\right]. \quad (29)$$

Now note that the only pole of $\phi(\nu-1)$ on $(0,1]$ is $1-\nu_*$ while $\phi(\nu+1)$ is regular at it, and vice versa for the point ν_*-1. This allows us to deform contours $\gamma\pm1$ to γ taking care of singularities and write (27). Note that $\tilde{\phi}(\nu)$ obeys all the conditions imposed on proper transformant.

Now let us show how the described above methods help in derivation of modified Smyshlyaev's formulae.

5 Example of derivation of a modified Smyshlyaev's formula

We will be working with embedding formula (12):

$$f = \frac{4\pi^2\mathrm{i}}{k_0^2(\omega_y+\omega_{0y})}\int\limits_0^\infty f_x(\omega;X)f_x(\omega_0;X)dX. \quad (30)$$

In [2] the following formulae are proven for f_x and v_x:

$$f_x(\omega,X) = \\ = \sqrt{\frac{k_0}{2\pi}}\frac{e^{-i\frac{3\pi}{4}}}{X}\sum_{j=1}^\infty C_j\Phi_j(\omega)J_{\nu_j}(k_0X)e^{-i\frac{\pi}{2}\nu_j}, \quad (31)$$

$$v_x(\omega,\nu) = 2\sum_{j=1}^\infty\frac{C_j\Phi_j(\omega)}{\nu^2-\nu_j^2}, \qquad (32)$$

where ν_j and $\Phi_j(\omega)$ are eigenvalues and eigenfunctions of Laplace-Beltrami operator on sphere with a Dirichlet cut and C_j are some constants describing the behavior of $\Phi_j(\omega)$ near the ends of the cut.

From (31) and (32) it obviously follows, that

$$f_x(\omega;X) = \\ = \sqrt{\frac{k_0}{2\pi}}\frac{e^{-\mathrm{i}\frac{\pi}{4}}}{2\pi X}\int_\gamma e^{-\mathrm{i}\frac{\pi}{2}\nu}J_\nu(k_0X)v_x(\omega,\nu)\nu\,d\nu \quad (33)$$

i.e.

$$f_x(\omega;X) = \frac{A}{X}h(\omega;X), \quad A = \sqrt{\frac{k_0}{2\pi}}\frac{e^{-\mathrm{i}\pi/4}}{\pi} \quad (34)$$

where $h(\omega; X)$ for each fixed ω has representation (15) with the transformant $v_x(\omega, \nu)$. Thus,

$$f(\omega, \omega_0) = \frac{2/\pi k_0}{\xi + \xi_0} \int_0^\infty h(\omega; X) \frac{h(\omega_0; X)}{X} \frac{dr}{r} \quad (35)$$

i.e. the integral has the form of (25).

To proceed transform $h(\omega_0; X)/X$ according to (27). Here we use the empirical fact that the only pole of v_x on $(0, 1]$ is $\nu_1 \equiv \nu_*$. Then deform the contour $\gamma + \Gamma$ in its representation into contour γ_μ (see Fig. 9). In order to assure convergence of forthcoming integrals over X let us shift the contour γ in representation of $h(\omega; X)$ to $\gamma' = \gamma + 1/2$. It is possible, since $1/2 < \nu_* < 1$ and thus $v_x(\omega, \nu)$ has no poles on $(0, 1/2]$.

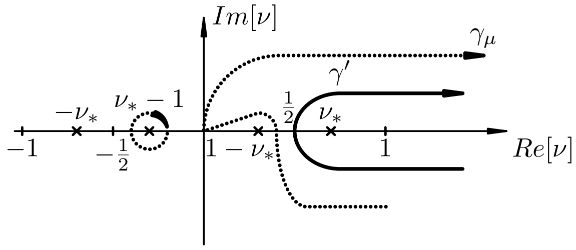

Figure 9: Deformation of the contour $\gamma + \Gamma$

Now let us follow the procedure of proof of "Plancherel's" theorem. We convert the product $h(\omega; X)h(\omega_0; X)/X$ to the double integral over $\gamma_\mu \times \gamma'$, substitute it into the (35) and do the integration over X first. As a result one obtains the following:

$$f(\omega, \omega_0) = \frac{1}{2\pi^2(\omega_x + \omega_{0x})} \times$$
$$\times \iint_{\gamma_\mu \times \gamma'} v_x(\omega, \nu)\phi(\mu)\mu\nu \frac{e^{-i\pi\nu} - e^{-i\pi\mu}}{\mu^2 - \nu^2} d\nu d\mu, \quad (36)$$

where

$$\phi(\mu) = \frac{v_x(\omega_0, \mu - 1) - v_x(\omega_0, \mu + 1)}{2\mu}.$$

Now let us transform $\gamma + 1/2$ to γ_ν (Fig. 10) and follow the rest of the procedure. As a result we obtain the sought modified Smyshlyaev's formula:

$$f(\omega, \omega_0) = \frac{i/4\pi}{\omega_x + \omega_{0x}} \int_{\gamma+\Gamma} e^{-\mathrm{i}\pi\mu} v_x(\omega, \mu) \times$$
$$\times [v_x(\omega_0, \mu - 1) - v_x(\omega_0, \mu + 1)] d\nu. \quad (37)$$

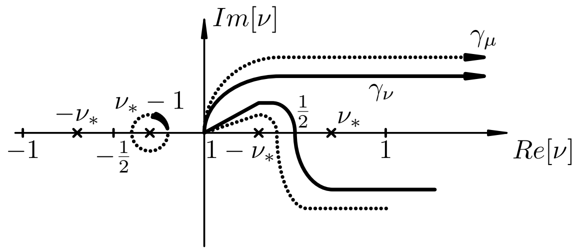

Figure 10: Deformation of the contour γ'

6 Conclusion

Let us briefly summarize the work. A new integral transform of Kontorovich-Lebedev type was introduced. Analogues of convolution and Plancherel's theorems were proven for it without demands for orthogonality, uniqueness and invertibility. Developed technique gives a neat method of transformation of spatial integrals emerging in embedding formulae for conical problems into contour integrals of Smyshlyaev's type. As an example of usage of this technique a modified Smyshlyaev's formula for the problem of diffraction of plane wave by a Dirichlet quarter plane was derived in a way different from the original work [2].

References

[1] Smyshlyaev, V.P., 1990, Diffraction by conical surfaces at high frequencies, *Wave motion,* **Vol. 12**, pp. 329–339.

[2] Shanin, A.V., 2005, Modified Smyshlyaev's formulae for the problem of diffraction of a plane wave by an ideal quarter-plane, *Wave motion,* **Vol. 41**, pp. 79–93.

[3] Skelton E.A. et al., 2010, Embedding formulae for scattering by three-dimensional structures, *Wave motion,* **Vol. 47**, pp. 299–317.