Embedding formulae for Laplace-Beltrami problems on the sphere with a cut.

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Abstract

We consider the problem of diffraction of a plane by a quarter-plane with Dirichlet boundary conditions. For this problem exist various expressions for diffraction coefficient. These expressions have the form of contour integrals over separation parameter. Integrands are constructed from the solutions of Laplace-Beltrami problems on the unit sphere with a cut produced by the quarter- plane. In this paper we derive embedding formulae which connect these solutions and show the possibility to derive expressions for diffraction coefficient from one another.

Keywords: Diffraction by cones, Embedding formulae

1. Introduction

We are considering the scalar problem of plane wave diffraction by a quarter-plane. Our main goal is to find the diffraction coefficient of the scattered field.

Since a plane sector is a degenerated case of an elliptic cone, this problem has an explicit solution in sphero-conal coordinates [1, 2]. This solution is a series of Lamé functions. Computations with this series are quite ineffective and it is difficult to extract from it the structure of the diffracted field.

Another approach to problems of diffraction by cones is separation of radial variable and studying the Laplace-Beltrami problem on the unit sphere

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for each value of separation parameter. This approach has been significantly developed by Smyshlyaev and co-workers [3]. Applying the Bessel-Sommerfeld technique he has obtained the following formula for the diffraction coefficient:

$$f(\omega, \omega_0) = \frac{i}{\pi} \int\limits_{\gamma} e^{-i\pi\nu} g(\omega, \omega_0, \nu) \nu d\nu, \tag{1}$$

where ω_0 and ω are directions of incidence and scattering, ν is the separation parameter and g is the Green's function of the of the spherical problem.

To compute with this formula one has to solve an integral equation for Green's function for each ν [4]. Integral over ν is rapidly convergent only in the domain of directions in which propagates only the spherical wave diffracted by the tip of the cone. It diverges in domain of directions where geometrically reflected wave or the waves diffracted by the edges of the quarter-plane exist. It is still possible to use Smyshlyaev's formula in the domain of divergence [5] but in this case the integral should be understood in sense of Abel-Poisson limit, which is difficult for numerical computations.

In works [6, 7] the formulae of the same type as (1) were obtained. In these formulae integrand is constructed from so called spherical edge Green's functions, which are the fields of singular sources lying at the edges of the scatterer. These formulae are based on application of embedding operators to the field in 3D space. These operators cancel the geometrically reflected wave and one of the waves diffracted by the edges (or both of them). Thus the domain of convergence of contour integral in these formulae is wider than than one in (1). When both waves diffracted by the edges are canceled the domain of divergence consists of directions in which propagate waves consequently scattered by both edges. For some directions of incidence this domain does not even exist. Computation of spherical edge Green's functions can be performed by using integral equations [8], but there exist much more effective way to compute them based on the equations with multidimensional time [9]. So these formulae are more effective than Smyshlyaev's formula.

Basic steps of derivation of modified Smyshlyaev's formulae in [7] are as follows. Application of an embedding operator to the field in 3D space allows to express the diffraction coefficient in terms of directivities of edge Green's functions in 3D space. This expression is called embedding formula. The directivities of edge Green's functions in 3D space can be represented as contour integrals over the separation parameter. Final step is substitution of these representations into the embedding formula and transformation of resulting multiple integral to a single contour integral over the separation parameter.

In this paper we use a different technique of derivation of these formulae. We apply embedding operators directly to solutions of Laplace-Beltrami problems on the unit sphere. This approach allows to obtain non-trivial relations between these solutions and consequently derive all the formulae from Smyshlyaev's formula (1).

The paper is organized as follows. Section 2 contains the problem formulation and introduces the notions used throughout the paper. In section 3 we introduce embedding operators on the unit sphere and, their properties and derive the embedding formulae. In section 4 we use these formulae for derivation of MSF from Smyshlyaev's formula (1).

2. Basic relations

2.1. Problem formulation

We seek the scalar field u which satisfies the Helmholtz equation

$$\Delta u + k_0^2 u = 0 \tag{2}$$

in the 3D space (x, y, z). The time dependence of all variables is of the form $e^{-i\Omega t}$ and is omitted henceforth. The scatterer is the quarter-plane $Q = \{(x, y, z)|x \geq 0, y \geq 0, z = 0\}$ (see Fig. 1). The field u satisfies the

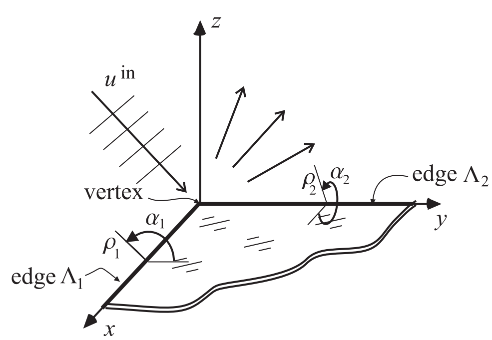

Figure 1: Geometry of the problem.

Dirichlet boundary conditions on the quarter-plane:

$$u|_Q = 0. \tag{3}$$

The incident field u^{in} is the plane wave coming from direction defined by the unit vector $\boldsymbol{\omega_0}$:

$$u^{in}(\boldsymbol{r}) = e^{-ik_0(\boldsymbol{\omega_0 r})}. \tag{4}$$

This problem can be symmetrized in a standard way. The field u is represented as a sum of even and odd functions of z. Solution for the odd part contains only the incident wave and geometrically reflected wave. In what follows we denote as u only the even part which obeys the homogeneous Neumann conditions on the complement $\tilde{Q}$ of the quarter-plane Q to the whole xy plane:

$$\left.\frac{\partial u}{\partial n}\right|_{\tilde{Q}} = 0. \tag{5}$$

Beside the governing equation and boundary conditions, the radiation, edge and vertex conditions should be imposed to make a proper problem formulation. We do not discuss these matters here for the sake of brevity and refer the reader to [6, 7].

The most important feature of the field u is the diffraction coefficient $f(\boldsymbol{\omega}, \boldsymbol{\omega_0})$ of its scattered part $u^{sc} = u - u^{in}$. It can be defined as the amplitude of the spherical wave diffracted by the tip of the quarter plane:

$$u^{sc}(r, \boldsymbol{\omega}) = 2\pi \frac{e^{ik_0 r}}{k_0 r} f(\boldsymbol{\omega}, \boldsymbol{\omega_0}) + O(r^{-2}), \quad \text{as } r \to \infty. \tag{6}$$

This definition is valid for directions $\boldsymbol{\omega}$ in which propagate only the spherical wave. It can be analytically continued in the domain of directions in which the geometrically reflected wave or waves diffracted by the edges of the quarter-plane exist.

2.2. Spectrum of the spherical problem

A natural way to solve our problem is to separate the radial variable and to study the spherical problem for each value of separation constant. This leads to the following eigenvalue problem on the unit sphere S with the cut

$S_q = S \cap Q$ "produced" by the quarter plane Q (see Fig. 2):

$$\tilde{\Delta}_\nu \Phi(\boldsymbol{\omega}) = 0, \tag{7}$$

$$\Phi|_{S_q} = 0, \tag{8}$$

$$\left.\frac{\partial \Phi}{\partial n}\right|_{\tilde{S}_q} = 0, \tag{9}$$

$$\Phi \sim \zeta_1^{1/2} \sin\frac{\phi_1}{2} + O(\zeta_1^{3/2}), \quad \text{as } \zeta_1 \to 0, \tag{10}$$

$$\Phi \sim \zeta_2^{1/2} \sin\frac{\phi_2}{2} + O(\zeta_2^{3/2}), \quad \text{as } \zeta_2 \to 0. \tag{11}$$

Here $\tilde{S}_q$ is the complement of S_q to the whole equator; (ζ_1, ϕ_1) and (ζ_2, ϕ_2) are the spherical coordinates shown on Fig. 2; $\tilde{\Delta}_\nu = \tilde{\Delta} + \nu^2 - 1/4$, and $\tilde{\Delta}$ is the Laplace-Beltrami operator on the unit sphere, which in conventional spherical coordinates (ζ, ϕ) has the form

$$\tilde{\Delta} = \frac{1}{\sin\zeta}\frac{\partial}{\partial\zeta}\left(\sin\zeta\frac{\partial}{\partial\zeta}\right) + \frac{1}{\sin^2\zeta}\frac{\partial^2}{\partial\phi^2}. \tag{12}$$

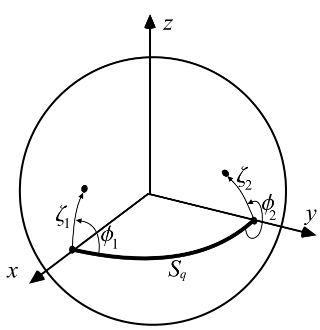

Figure 2: Geometry of the problem on the sphere.

This problem has a solution only for a discrete set of real values of $\nu = \pm\nu_n$, $n = 1, 2...$ which form the *spectrum* of the problem. Since the self-adjoint operator $-\tilde{\Delta}$ has a positive discrete spectrum [3] we can write that $1/2 \le \nu_1 \le \nu_2 \le \ldots \nu_n \to \infty$.

2.3. Edge Green's functions on the unit sphere

Besides the eigenfunctions an important role in solution of our problem play the Green's function and spherical edge Green's functions. Let us introduce them. The definitions below exactly repeat the ones in [3] and [6].

The Green's function $g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu)$ is the solution of the following Laplace-Beltrami problem

$$\tilde{\Delta}_\nu g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu) = \delta(\boldsymbol{\omega} - \boldsymbol{\omega_0}), \tag{13}$$

which obeys Dirichlet conditions on the cut: $g|_{S_q} = 0$, Neumann conditions on the rest of the equator $\frac{\partial g}{\partial n}|_{S_q} = 0$, and Meixner conditions at the ends of the cut (see [6]). It is the function participating in (1). Note that $g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu)$ is even function of ν and points $\nu = \pm\nu_n$ are its poles [3].

We define the spherical edge Green's functions $v_1(\boldsymbol{\omega}, \nu)$ and $v_2(\boldsymbol{\omega}, \nu)$ as the following limits:

$$v^1(\boldsymbol{\omega}, \nu) = \lim_{\kappa \to 0} \sqrt{\frac{\pi}{\kappa}} g(\boldsymbol{\omega}, \boldsymbol{\omega}_{\kappa x}, \nu) \quad \text{and} \tag{14}$$

$$v^2(\boldsymbol{\omega}, \nu) = \lim_{\kappa \to 0} \sqrt{\frac{\pi}{\kappa}} g(\boldsymbol{\omega}, \boldsymbol{\omega}_{\kappa y}, \nu), \tag{15}$$

where $\boldsymbol{\omega}_{\kappa x}$ is the point with spherical coordinates $\zeta_1 = \kappa$ and $\phi_1 = \pi$ (see Fig. 3) and similarly for $\boldsymbol{\omega}_{\kappa y}$. One can prove [9, 6] that these func-

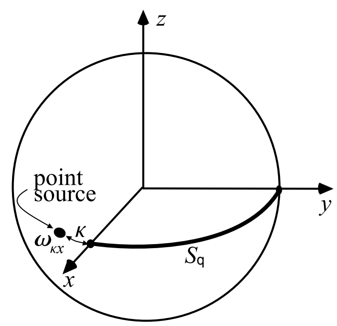

Figure 3: To the definition of the edge Green's function on the sphere.

tions have the following asymptotics near the edges of the cut:

$$v^1(\zeta_1, \phi_1, \nu) = -\frac{1}{\sqrt{\pi}} \zeta_1^{-1/2} \sin\frac{\phi_1}{2} + O(\zeta_1^{1/2}), \quad \text{as } \zeta_1 \to 0, \tag{16}$$

$$v^2(\zeta_1, \phi_1, \nu) = \frac{2C_2^1(\nu)}{\sqrt{\pi}} \zeta_1^{1/2} \sin\frac{\phi_1}{2} + O(\zeta_1^{3/2}), \quad \text{as } \zeta_1 \to 0, \tag{17}$$

$$v^2(\zeta_2, \phi_2, \nu) = -\frac{1}{\sqrt{\pi}} \zeta_2^{-1/2} \sin\frac{\phi_2}{2} + O(\zeta_2^{1/2}), \quad \text{as } \zeta_2 \to 0, \tag{18}$$

$$v^1(\zeta_2, \phi_2, \nu) = \frac{2C_2^1(\nu)}{\sqrt{\pi}} \zeta_2^{1/2} \sin\frac{\phi_2}{2} + O(\zeta_2^{3/2}), \quad \text{as } \zeta_2 \to 0. \tag{19}$$

Here $C_2^1(\nu)$ is an unknown coefficient. Note that $v^{1,2}(\boldsymbol{\omega},\nu)$ and $C_2^1(\nu)$ are even functions of ν and points $\nu=\pm\nu_n$ are their poles. From definition of edge Green's functions and reciprocity principle follow the asymptotics of Green's function at the edges of the cut:

$$g(\zeta_1,\phi_1;\boldsymbol{\omega_0},\nu)=\frac{v^1(\boldsymbol{\omega_0},\nu)}{\sqrt{\pi}}\zeta_1^{1/2}\sin\frac{\phi_1}{2}+O(\zeta_1^{3/2}),\quad \text{as } \zeta_1\to 0, \tag{20}$$

$$g(\zeta_2,\phi_2;\boldsymbol{\omega_0},\nu)=\frac{v^2(\boldsymbol{\omega_0},\nu)}{\sqrt{\pi}}\zeta_2^{1/2}\sin\frac{\phi_2}{2}+O(\zeta_2^{3/2}),\quad \text{as } \zeta_2\to 0. \tag{21}$$

Edge Green's functions participate in the following formulae for diffraction coefficient, which we call modified Smyshlyaev's formulae [6, 7].

$$f(\boldsymbol{\omega},\boldsymbol{\omega_0})=\frac{i/4\pi}{\omega_x+\omega_{0x}}\int\limits_{\gamma+\Gamma}e^{-i\pi\mu}v^2(\boldsymbol{\omega},\mu)\phi^2(\boldsymbol{\omega_0},\mu)\mu d\mu; \tag{22}$$

$$f(\boldsymbol{\omega},\boldsymbol{\omega_0})=\frac{i/4\pi}{\omega_y+\omega_{0y}}\int\limits_{\gamma+\Gamma}e^{-i\pi\mu}v^1(\boldsymbol{\omega},\mu)\phi^1(\boldsymbol{\omega_0},\mu)\mu d\mu; \tag{23}$$

$$\begin{aligned}f(\boldsymbol{\omega},\boldsymbol{\omega_0})=\frac{i/8\pi}{(\omega_x+\omega_{0x})(\omega_y+\omega_{0y})}\int\limits_{\gamma+\Gamma}e^{-i\pi\nu}[V^1(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)+V^2(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)+\\+2\nu\omega_{0x}v^1(\boldsymbol{\omega},\nu)\phi^1(\boldsymbol{\omega_0},\nu)+2\nu\omega_{0y}v^2(\boldsymbol{\omega},\nu)\phi^2(\boldsymbol{\omega_0},\nu)]d\nu,\end{aligned} \tag{24}$$

$$\begin{aligned}f(\boldsymbol{\omega},\boldsymbol{\omega_0})=\frac{i/8\pi}{(\omega_x+\omega_{0x})(\omega_y+\omega_{0y})}\int\limits_{\gamma+\Gamma}e^{-i\pi\mu}C_2^1(\mu)[\phi^1(\boldsymbol{\omega},\mu)\phi^2(\boldsymbol{\omega_0},\mu)-\\-\phi^1(\boldsymbol{\omega_0},\mu)\phi^2(\boldsymbol{\omega},\mu)]\mu d\mu.\end{aligned} \tag{25}$$

Here we use the following notation:

$$\phi^k(\boldsymbol{\omega},\mu)=\frac{v^k(\boldsymbol{\omega},\mu-1)-v^k(\boldsymbol{\omega},\mu+1)}{\mu},\quad k=1,2. \tag{26}$$

$$\begin{aligned}V^k(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)=v^k(\boldsymbol{\omega},\nu+1)v^k(\boldsymbol{\omega_0},\nu-1)-\\-v^k(\boldsymbol{\omega},\nu-1)v^k(\boldsymbol{\omega_0},\nu+1),\quad k=1,2.\end{aligned} \tag{27}$$

Contour of integration $\gamma+\Gamma$ is shown on Fig. 4. Contour Γ consists of two loops encircling points $1-\nu_1$ and ν_1-1.

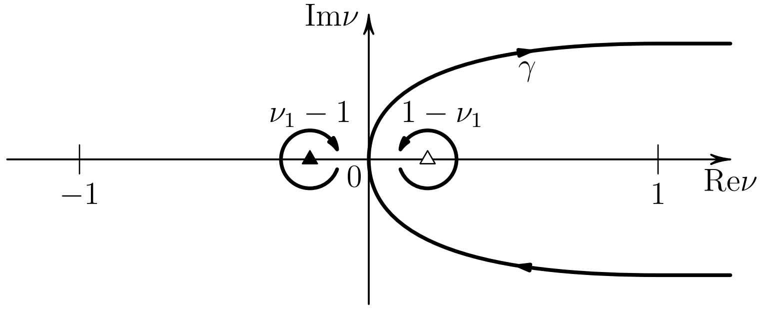

Figure 4: Contour $\gamma + \Gamma$.

3. Embedding formulae on the unit sphere

3.1. Embedding operators

We introduce operators on the unit sphere X_ν and Y_ν by the following identities.

$$\frac{\partial}{\partial x} r^{\nu-1/2}\Omega(\boldsymbol{\omega}) = r^{\nu-3/2} X_\nu[\Omega(\boldsymbol{\omega})], \tag{28}$$

$$\frac{\partial}{\partial y} r^{\nu-1/2}\Omega(\boldsymbol{\omega}) = r^{\nu-3/2} Y_\nu[\Omega(\boldsymbol{\omega})], \tag{29}$$

We call X_ν and Y_ν embedding operators. These operators have the following representations in spherical coordinates (ζ_1, ϕ_1) and (ζ_2, ϕ_2) shown on Fig. 2.

$$X_\nu = \left(\nu - \frac{1}{2}\right) \cos\zeta_1 - \sin\zeta_1 \frac{\partial}{\partial \zeta_1}, \tag{30}$$

$$Y_\nu = \left(\nu - \frac{1}{2}\right) \sin\zeta_1 \cos\phi_1 + \cos\zeta_1 \cos\phi_1 \frac{\partial}{\partial \zeta_1} - \frac{\sin\phi_1}{\sin\zeta_1}\frac{\partial}{\partial \phi_1}, \tag{31}$$

$$X_\nu = \left(\nu - \frac{1}{2}\right) \sin\zeta_2 \cos\phi_2 + \cos\zeta_2 \cos\phi_2 \frac{\partial}{\partial \zeta_2} - \frac{\sin\phi_2}{\sin\zeta_2}\frac{\partial}{\partial \phi_2}, \tag{32}$$

$$Y_\nu = \left(\nu - \frac{1}{2}\right) \cos\zeta_2 - \sin\zeta_2 \frac{\partial}{\partial \zeta_2}. \tag{33}$$

Let us formulate two important properties of these operators.

Lemma 1. *If some function $\Omega(\boldsymbol{\omega})$ satisfies the boundary conditions*

$$\Omega(\boldsymbol{\omega})|_{S_q} = 0, \quad \textit{and} \quad \frac{\partial}{\partial n}\Omega(\boldsymbol{\omega})|_{\tilde{S}_q} = 0, \tag{34}$$

then $X_\nu[\Omega(\boldsymbol{\omega})]$ and $Y_\nu[\Omega(\boldsymbol{\omega})]$ also satisfy these conditions.

Proof. From conditions of lemma it follows that combination $r^{\nu-1/2}\Omega(\boldsymbol{\omega})$ satisfies homogeneous Dirichlet conditions on Q and homogeneous Neumann conditions on $\tilde{Q}$. Since differentiation with respect to x and y preserve these conditions, from definition of X_ν and Y_ν we obtain the statement of the lemma. □

Lemma 2. *If some function $\Omega(\boldsymbol{\omega})$ satisfies the equation*

$$\tilde{\Delta}_\nu \Omega(\boldsymbol{\omega}) = h(\boldsymbol{\omega}), \tag{35}$$

then $X_\nu[\Omega(\boldsymbol{\omega})]$ and $Y_\nu[\Omega(\boldsymbol{\omega})]$ satisfy the following equations

$$\tilde{\Delta}_{\nu-1} X_\nu[\Omega(\boldsymbol{\omega})] = X_{\nu-2}[h(\boldsymbol{\omega})], \tag{36}$$

$$\tilde{\Delta}_{\nu-1} Y_\nu[\Omega(\boldsymbol{\omega})] = Y_{\nu-2}[h(\boldsymbol{\omega})], \tag{37}$$

Proof. We will prove the property only for operator X_ν. Proof for Y_ν is literally the same. From conditions of lemma it follows that combination $r^{\nu-1/2}\Omega(\boldsymbol{\omega})$ satisfies the equation

$$\Delta[r^{\nu-1/2}\Omega(\boldsymbol{\omega})] = r^{\nu-5/2} h(\boldsymbol{\omega}). \tag{38}$$

Since Laplacian commutes with differentiation with respect to x, we can write that

$$\frac{\partial}{\partial x} r^{\nu-5/2} h(\boldsymbol{\omega}) = \Delta[\frac{\partial}{\partial x} r^{\nu-1/2}\Omega(\boldsymbol{\omega})], \tag{39}$$

Applying the definition of X_ν we transform this equation into the following

$$r^{\nu-7/2} X_{\nu-2}[h(\boldsymbol{\omega})] = \Delta[r^{\nu-3/2} X_\nu[\Omega(\boldsymbol{\omega})]]. \tag{40}$$

Statement of the lemma directly follows from the last equation. □

3.2. Embedding formulae

Properties of the operators X_ν and Y_ν formulated above allow us to prove two theorems which are main results of this paper.

Theorem 1. *If ν and $\nu \pm 1$ do not belong to the spectrum, then the following formulae are valid.*

$$\begin{aligned} 2\nu\omega_x v^1(\boldsymbol{\omega},\nu) = \nu[v^1(\boldsymbol{\omega},\nu-1) + v^1(\boldsymbol{\omega},\nu+1)] + \\ + C_2^1(\nu)[v^2(\boldsymbol{\omega},\nu-1) - v^2(\boldsymbol{\omega},\nu+1)], \end{aligned} \tag{41}$$

$$\begin{aligned} 2\nu\omega_y v^2(\boldsymbol{\omega},\nu) = \nu[v^2(\boldsymbol{\omega},\nu-1) + v^2(\boldsymbol{\omega},\nu+1)] + \\ + C_2^1(\nu)[v^1(\boldsymbol{\omega},\nu-1) - v^1(\boldsymbol{\omega},\nu+1)]. \end{aligned} \tag{42}$$

Proof. We will prove only the first formula. Proof of the second is the same. Let us consider the function $X_\nu[v^1(\boldsymbol{\omega},\nu)]$. Applying the representations (30) and (32) to asymptotics (16) and (19) we obtain the following asymptotics of $X_\nu[v^1(\boldsymbol{\omega},\nu)]$ at the edges of the cut:

$$X_\nu[v^1(\zeta_1,\phi_1,\nu)] = -\frac{\nu}{\sqrt{\pi}}\zeta_1^{-1/2}\sin\frac{\phi_1}{2} + O(\zeta_1^{1/2}), \qquad \text{as } \zeta_1 \to 0, \tag{43}$$

$$X_\nu[v^1(\zeta_2,\phi_2,\nu)] = -\frac{C_2^1(\nu)}{\sqrt{\pi}}\zeta_2^{-1/2}\sin\frac{\phi_2}{2} + O(\zeta_2^{1/2}), \qquad \text{as } \zeta_2 \to 0. \tag{44}$$

Thus combination $X_\nu[v^1(\boldsymbol{\omega},\nu)] - \nu v^1(\boldsymbol{\omega},\nu-1) - C_2^1(\nu)v^2(\boldsymbol{\omega},\nu-1)$ satisfies Meixner conditions at at the edges of the cut. From lemmas 1 and 2 and definition of edge Green's functions it follows that it also satisfies homogeneous Dirichlet conditions on S_q, homogeneous Neumann conditions on $\tilde{S}_q$ and equation

$$\tilde{\Delta}_{\nu-1}[X_\nu[v^1(\boldsymbol{\omega},\nu)] - \nu v^1(\boldsymbol{\omega},\nu-1) - C_2^1(\nu)v^2(\boldsymbol{\omega},\nu-1)] = 0. \tag{45}$$

Appealing to the uniqueness theorem we conclude that this combination is identically zero. Thus, taking into account representation (30), we can write

$$\begin{aligned}\left(\nu-\frac{1}{2}\right)\cos\zeta_1 v^1(\boldsymbol{\omega},\nu) - \sin\zeta_1\frac{\partial}{\partial\zeta_1}v^1(\boldsymbol{\omega},\nu) = \\ = \nu v^1(\boldsymbol{\omega},\nu-1) + C_2^1(\nu)v^2(\boldsymbol{\omega},\nu-1).\end{aligned} \tag{46}$$

Substituting in this equation $-\nu$ instead of ν and taking into account evenness of edge Green's functions and $C_2^1(\nu)$ we obtain

$$\begin{aligned}\left(-\nu-\frac{1}{2}\right)\cos\zeta_1 v^1(\boldsymbol{\omega},\nu) - \sin\zeta_1\frac{\partial}{\partial\zeta_1}v^1(\boldsymbol{\omega},\nu) = \\ = -\nu v^1(\boldsymbol{\omega},\nu+1) + C_2^1(\nu)v^2(\boldsymbol{\omega},\nu+1).\end{aligned} \tag{47}$$

Subtracting these equations and taking into account that $\cos\zeta_1 = \omega_x$ we obtain (41). $\square$

In what follows we will omit arguments $\boldsymbol{\omega}$ and $\boldsymbol{\omega_0}$ of Green's function g where it doesn't lead to a confusion. We denote spherical coordinates $(\zeta_{1,2},\phi_{1,2})$ of $\boldsymbol{\omega_0}$ as $(\zeta_{1,2}^0,\phi_{1,2}^0)$.

Theorem 2. *If ν and $\nu \pm 1$ do not belong to the spectrum, then the following formulae are valid.*

$$\begin{aligned} 2\nu\omega_x g(\nu) = \nu\omega_{0x}[g(\nu-1)+g(\nu+1)] + \frac{\nu}{2}v^2(\boldsymbol{\omega_0},\nu)\phi^2(\boldsymbol{\omega},\nu) - \\ - \left(\frac{3\omega_{0x}}{2} + \sin\zeta_1^0 \frac{\partial}{\partial\zeta_1^0}\right)[g(\nu-1)-g(\nu+1)]. \end{aligned} \tag{48}$$

$$\begin{aligned} 2\nu\omega_y g(\nu) = \nu\omega_{0y}[g(\nu-1)+g(\nu+1)] + \frac{\nu}{2}v^1(\boldsymbol{\omega_0},\nu)\phi^1(\boldsymbol{\omega},\nu) - \\ - \left(\frac{3\omega_{0y}}{2} + \sin\zeta_2^0 \frac{\partial}{\partial\zeta_2^0}\right)[g(\nu-1)-g(\nu+1)]. \end{aligned} \tag{49}$$

Proof. We will prove only the first formula. Proof of the second is the same. Let us consider the function $X_\nu[g(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)]$. Applying the representations (30) and (32) to asymptotics (20) and (21) we obtain the following asymptotics of $X_\nu[g(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)]$ at the edges of the cut:

$$X_\nu[g(\zeta_1,\phi_1;\boldsymbol{\omega_0},\nu)] = O(\zeta_1^{1/2}), \qquad \text{as } \zeta_1 \to 0, \tag{50}$$

$$X_\nu[g(\zeta_2,\phi_2;\boldsymbol{\omega_0},\nu)] = -\frac{v^2(\boldsymbol{\omega_0},\nu)}{2\sqrt{\pi}}\zeta_2^{-1/2}\sin\frac{\phi_2}{2} + O(\zeta_2^{1/2}), \qquad \text{as } \zeta_2 \to 0. \tag{51}$$

Thus combination

$$g_*(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu) := X_\nu[g(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu)] - v^2(\boldsymbol{\omega_0},\nu)v^2(\boldsymbol{\omega},\nu-1)/2 \tag{52}$$

satisfies Meixner conditions at at the edges of the cut. From lemmas 1 and 2 and definition of Green's function it follows that it also satisfies homogeneous Dirichlet conditions on S_q, homogeneous Neumann conditions on $\tilde{S}_q$ and equation

$$\tilde{\Delta}_{\nu-1} g_*(\boldsymbol{\omega},\boldsymbol{\omega_0},\nu) = X_{\nu-2}[\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0})]. \tag{53}$$

Taking into account representation (30) we can write

$$X_{\nu-2}[\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0})] = \left(\nu - \frac{5}{2}\right)\cos\zeta_1\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0}) - \sin\zeta_1\frac{\partial}{\partial\zeta_1}\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0}). \tag{54}$$

Using the properties of delta function this expression can be transformed as follows

$$X_{\nu-2}[\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0})] = \left(\nu - \frac{3}{2}\right)\cos\zeta_1^0\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0}) - \sin\zeta_1^0\frac{\partial}{\partial\zeta_1^0}\delta(\boldsymbol{\omega}-\boldsymbol{\omega_0}). \tag{55}$$

Thus for $g_*(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu)$ we obtain

$$g_*(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu) = \left[\left(\nu - \frac{3}{2}\right)\cos\zeta_1^0 - \sin\zeta_1^0 \frac{\partial}{\partial \zeta_1^0}\right] g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu - 1). \tag{56}$$

From representation (30) it follows, that

$$\left[\left(\nu - \frac{1}{2}\right)\cos\zeta_1 - \sin\zeta_1 \frac{\partial}{\partial \zeta_1}\right] g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu) = v^2(\boldsymbol{\omega_0}, \nu) v^2(\boldsymbol{\omega}, \nu - 1)/2 + \\ + \left[\left(\nu - \frac{3}{2}\right)\cos\zeta_1^0 - \sin\zeta_1^0 \frac{\partial}{\partial \zeta_1^0}\right] g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu - 1). \tag{57}$$

Substituting in this equation $-\nu$ instead of ν and taking into account evenness of Green's functions we obtain

$$\left[\left(-\nu - \frac{1}{2}\right)\cos\zeta_1 - \sin\zeta_1 \frac{\partial}{\partial \zeta_1}\right] g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu) = v^2(\boldsymbol{\omega_0}, \nu) v^2(\boldsymbol{\omega}, \nu + 1)/2 + \\ + \left[\left(-\nu - \frac{3}{2}\right)\cos\zeta_1^0 - \sin\zeta_1^0 \frac{\partial}{\partial \zeta_1^0}\right] g(\boldsymbol{\omega}, \boldsymbol{\omega_0}, \nu + 1). \tag{58}$$

Subtracting these equations and taking into account that $\cos\zeta_1 = \omega_x$ and $\cos\zeta_1^0 = \omega_{0x}$ we obtain (41). □

4. Derivation of modified Smyshlyaev's formulae

Now let us apply embedding formulae obtained above to derivation of modified Smyshlyaev's formulae (22) – (25).

4.1. From formula (1) *to* (22)

Let us multiply formula (48) by $e^{-i\pi\nu}$ and integrate the result over the contour $\gamma + \Gamma$. Since $g(\nu)$ is regular at points $\nu_1 - 1$ and $1 - \nu_1$ we can write

$$\int\limits_{\gamma+\Gamma} e^{-i\pi\nu} g(\nu)\nu d\nu = \int\limits_{\gamma} e^{-i\pi\nu} g(\nu)\nu d\nu. \tag{59}$$

Let us consider integral of the first term on the right-hand side of (48). Changing the variable of integration to $\mu = \nu \pm 1$ we obtain

$$\int\limits_{\gamma+\Gamma} e^{-i\pi\nu}[g(\nu - 1) + g(\nu + 1)]\nu d\nu = \\ = -\int\limits_{\gamma+\Gamma+1} e^{-i\pi\mu} g(\mu)(\mu - 1) d\mu - \int\limits_{\gamma+\Gamma-1} e^{-i\pi\mu} g(\mu)(\mu + 1) d\mu. \tag{60}$$

Contour of integration in the first term, $\gamma+\Gamma+1$, is shown on Fig. 5. Since

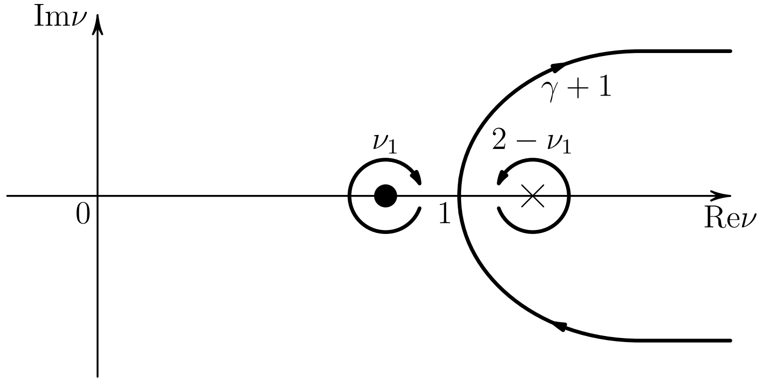

Figure 5: Contour $\gamma+\Gamma+1$.

integrand is regular at point $2-\nu_1$ and ν_1 is its only pole on $[0,1]$ we can deform this contour into γ. Performing the same procedure with contour $\gamma+\Gamma-1$ we get

$$\int\limits_{\gamma+\Gamma} e^{-i\pi\nu}[g(\nu-1)+g(\nu+1)]\nu d\nu = -2\int\limits_{\gamma} e^{-i\pi\mu} g(\mu)\mu d\mu. \tag{61}$$

Consideration of the third term on the right-hand side of (48) is essentially the same. As a result we get

$$\int\limits_{\gamma+\Gamma} e^{-i\pi\nu}[g(\nu-1)-g(\nu+1)]d\nu = 0. \tag{62}$$

Combining all these results we obtain

$$2\omega_x \int\limits_{\gamma} e^{-i\pi\nu} g(\nu)\nu d\nu = -2\omega_{0x}\int\limits_{\gamma} e^{-i\pi\nu} g(\nu)\nu d\nu + \\ + \int\limits_{\gamma+\Gamma} e^{-i\pi\nu}\frac{\nu}{2} v^2(\boldsymbol{\omega_0},\nu)\phi^2(\boldsymbol{\omega},\nu)d\nu. \tag{63}$$

Using (1) we get (22). Formula (23) can be obtained in the same way.

4.2. From formula (22) *to formula* (25)

Using the reciprocity principle let us write formula (22) in two equivalent forms:

$$f(\boldsymbol{\omega},\boldsymbol{\omega_0}) = \frac{i/4\pi}{\omega_x+\omega_{0x}} \int\limits_{\gamma+\Gamma} e^{-i\pi\mu} v^2(\boldsymbol{\omega},\mu)\phi^2(\boldsymbol{\omega_0},\mu)\mu d\mu, \tag{64}$$

$$f(\boldsymbol{\omega},\boldsymbol{\omega_0}) = \frac{i/4\pi}{\omega_x+\omega_{0x}} \int\limits_{\gamma+\Gamma} e^{-i\pi\mu} v^2(\boldsymbol{\omega_0},\mu)\phi^2(\boldsymbol{\omega},\mu)\mu d\mu. \tag{65}$$

Multiplying the first formula by $2\omega_x$ and the second by $2\omega_{0x}$ and using embedding formula (42) we obtain

$$2\omega_x f(\boldsymbol{\omega},\boldsymbol{\omega_0}) = \frac{i/4\pi}{\omega_x+\omega_{0x}} \int\limits_{\gamma+\Gamma} e^{-i\pi\mu}[v^2(\boldsymbol{\omega},\mu-1)+v^2(\boldsymbol{\omega},\mu+1)+ \\ +C_2^1(\mu)\phi^1(\boldsymbol{\omega},\mu)]\phi^2(\boldsymbol{\omega_0},\mu)\mu d\mu, \tag{66}$$

$$2\omega_{0x} f(\boldsymbol{\omega},\boldsymbol{\omega_0}) = \frac{i/4\pi}{\omega_x+\omega_{0x}} \int\limits_{\gamma+\Gamma} e^{-i\pi\mu}[v^2(\boldsymbol{\omega_0},\mu-1)+v^2(\boldsymbol{\omega_0},\mu+1)+ \\ +C_2^1(\mu)\phi^1(\boldsymbol{\omega_0},\mu)]\phi^2(\boldsymbol{\omega},\mu)\mu d\mu. \tag{67}$$

Adding up these formulae and using the expression for ϕ^2 we get

$$2(\omega_x+\omega_{0x})f(\boldsymbol{\omega},\boldsymbol{\omega_0}) = \frac{i/4\pi}{\omega_x+\omega_{0x}} \times \\ \times \left(\int\limits_{\gamma+\Gamma} e^{-i\pi\mu}\, C_2^1(\mu) \left[\phi^1(\boldsymbol{\omega},\mu)\phi^2(\boldsymbol{\omega_0},\mu)+\phi^1(\boldsymbol{\omega_0},\mu)\phi^2(\boldsymbol{\omega},\mu)\right] \mu d\mu + \right. \\ + \int\limits_{\gamma+\Gamma} e^{-i\pi\mu} \left[v^2(\boldsymbol{\omega},\mu-1)v^2(\boldsymbol{\omega_0},\mu-1)- \right. \\ \left. \left. -\; v^2(\boldsymbol{\omega},\mu+1)v^2(\boldsymbol{\omega_0},\mu+1)\right] d\mu \right). \tag{68}$$

The second integral is equal to zero. This can be obtained in the same way as (61). Thus, we get (25). Formula (25) can be obtained from (24) by using (41) and (42) in the same way.

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