Embedding formulae for diffraction by wedge and angular geometries

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Embedding is the process of taking the far field directivity from a diffraction problem (or problems) involving line sources or multipoles placed at a sharp edge, and then constructing the far field, for the same geometry, for more general incidence using only this canonical problem(s). Thus far, embedding has been limited to planar, parallel scattering surfaces, for instance, collections of parallel cracks or slits; it had appeared that there was a fundamental limitation to embedding disallowing its use for angular structures. In this article we overcome this limitation and demonstrate the use of embedding upon wedge diffraction problems and upon a simple polygonal shape.

Keywords: Embedding, integral equations, acoustics, electromagnetism, elasticity, reciprocity

1. Introduction

Embedding is a relatively new, and under used, idea in diffraction theory and is relevant to the scattering of electromagnetic, acoustic and elastic waves by structures. The applications are widespread and diffraction is important in, for instance, radar, underwater acoustics, seismology and the non-destructive testing of elastic media.

The fundamental idea of embedding is that one only ever solves a single master, canonical, problem (or set of problems). Thereafter to extract the far-field behaviour for any plane wave incidence one only manipulates results from this master problem. The result is that quantities (directivities) previously dependent upon two parameters, become factorized into products of a function of a single variable, together with a simple trigonometric term; this facilitates rapid numerical evaluation. In principle, this should revolutionize many scattering calculations in every area where diffraction occurs as one need only evaluate the directivities from the set of canonical problems once. Given these directivities one manipulates them to generate solutions for more general problems, rather than continually recalculating and re-evaluating.

To date there have been few applications of the embedding idea. The large majority of previous work has involved integral equation formulations (Williams 1982; Martin & Wickham 1983; Gautesen 1983; Biggs *et al* 2000; Biggs & Porter 2001,2002) and primarily involves scattering by (possibly multiple parallel) thin slits or cracks. Valuable though this integral equation approach certainly is, it does not easily allow for physical interpretation, nor do the ideas transplant readily into, say, finite element calculations commonly used in the acoustics/ engineering community. With this in mind Craster *et al* (2003) developed a physical viewpoint of embedding based upon canonical solutions (edge Green's functions) i.e. line or point sources at the edges of diffracting cracks or slits. The ideas then transplant easily into elasticity, electromagnetism and waveguiding geometries; embedding then becomes a general feature of diffraction theories unfettered by any specific solution technique.

We now turn our attention to another facet of embedding. Notably all previous applications have involved parallel cracks or slits (bar Biggs & Porter (2001) who deal with thick barriers, and thus have two right-angles at the end of each barrier) and thus it is natural to inquire, how and whether, the theory can be advanced to deal with non-parallel angled surfaces; the classical wedge geometries we consider in this article provide an ideal, non-trivial, testing ground. We have however other, more general reasons for wishing to extend the embedding idea further: Embedding has recently been employed to good effect for scattering by a three-dimensional structure, the quarter plane by Shanin (2004a, 2004b), which is a flat cone and provides a three dimensional extension of the crack or slit theory of Craster *et al.* (2003); although it is once again a planar geometry. Understanding embedding for the two-dimensional angled wedge casts light upon how to tackle the more general cone geometry and will eventually allow efficient treatment of it.

Moreover, recently Norris & Osipov (1999) in their study of an impedance wedge noticed that their solution for the directivity had an intriguing and interesting structure, it was the product of two simple functions, dependent upon a single variable and a purely trigonometric term. The impedance problem introduces further difficulties and ultimately we aim to treat this too. Their result is strongly indicative of the existence of an embedding formula for such geometries.

Here we consider plane wave scattering by wedge geometries (and later also a polygonal shape), with either Dirichlet or Neumann boundary conditions on the wedge faces, as a vehicle to advance our thesis that embedding is *the* fundamental way of tackling diffraction problems. We begin, in section 2, by briefly describing the solution to scattering of a plane wave by a wedge geometry and giving the conventional directivity $D(\theta, \theta_0)$; a function of two variables. The embedding formula for $D(\theta, \theta_0)$ is derived in section 3(f) in terms of products of directivities, $\hat{D}_n(\theta)$, functions of a single variable, deduced from edge Green's functions. Section 3 summarizes the embedding ideas using in Craster *et al* (2003) for line cracks, discussing how those ideas fail for wedge geometries, and then describes in detail how the embedding formula is constructed for angular domains. In particular, in section 3(d), an operator is introduced that overcomes the previous restriction that appeared to disallow angular structures from the embedding theory. Importantly, we can move beyond the classical wedge geometry to consider polygonal structures and in section 4 we consider scattering by an equilateral triangle.

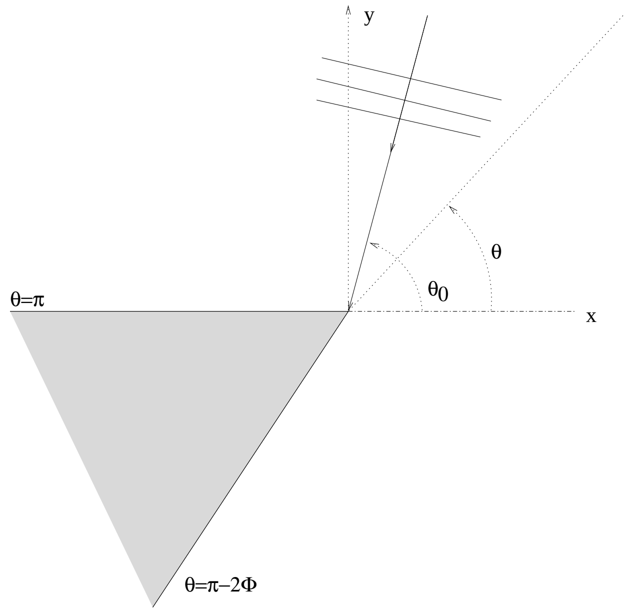

Figure 1. The wedge geometry, showing the angles and notation used.

2. The classical wedge solution

(*a*) *Problem formulation*

We consider acoustic material occupying a fluid wedge of angle $\pi - 2\Phi \leq \theta \leq \pi$; the fluid wedge subtends an angle of 2Φ (see figure 1). The Helmholtz equation

$$\nabla^2 u + k_0^2 u = 0 \tag{2.1}$$

holds for a velocity potential, u, in this wedge.

The boundary conditions are either Dirichlet i.e. $u^D = 0$ or Neumann, $\partial u^N/\partial\theta = 0$ on $\theta = \pi - 2\Phi, \pi$; the superscripts D, N throughout this article will distinguish the Dirichlet and Neumann cases respectively. We are interested in the scattering of an incident field consisting of an incoming plane wave

$$u^{in} = \exp\left[-i(k_* x + \sqrt{k_0^2 - k_*^2 y})\right] = \exp[-ik_0 r\cos(\theta - \theta_0)] \tag{2.2}$$

where $k_* = k_0 \cos\theta_0$; θ_0 being the angle of incidence (cf figure 1).

We also require Meixner's edge condition that states that the field near the edge should behave as $u \sim r^\delta$, for $\delta > 0$ in the Dirichlet case or $A + Br^\delta$, for $\delta > 0$ and A, B constant, in the Neumann case; the precise values of δ are given later.

Additionally, the Sommerfeld radiation condition is also satisfied by the scattered field; it does not contain waves incoming from infinity or fields growing at infinity. In the wedge case the total field satisfying the radiation condition can be additively decomposed into the incident and reflected fields (according to geometric optics) and the scattered cylindrical field.

(b) *Local behaviour of the field*

The wedge scattering problem is classical and several forms of solution are possible, using the Sommerfeld integral as reviewed in say, Bowman and Senior (1969) or Felsen & Marcuvitz (1973), see also Rawlins (1987), or using an eigenfunction expansion as in MacDonald (1902). For later reference we summarize those pieces that we shall use later for comparative purposes. For instance, an eigenfunction expansion gives

$$u^D(r,\theta) = 4\nu_1 \sum_{n=1}^{\infty} \sin[\nu_n(\theta - \pi + 2\Phi)] \sin[\nu_n(\theta_0 - \pi + 2\Phi)] e^{-i\pi\nu_n/2} J_{\nu_n}(kr) \quad (2.3)$$

where $\nu_n = n\pi/2\Phi$, and likewise

$$u^N(r,\theta) = 2\nu_1 \sum_{n=0}^{\infty} \epsilon_n \cos[\nu_n(\theta - \pi + 2\Phi)] \cos[\nu_n(\theta_0 - \pi + 2\Phi)] e^{-i\pi\nu_n/2} J_{\nu_n}(kr).$$

Where $\epsilon_n = 1$ for $n = 0$ and is 2 otherwise.

The full wedge solution can be found using Malhyuzhinet's method (see the review by Osipov & Norris 1999a) or, after more effort, using Fourier transform and Wiener-Hopf techniques (Shanin 1997; Daniele 2003).

For this physically relevant problem, scattering by an incoming plane wave, the solutions, $u^{D,N}$, are constrained to be non-singular at the wedge tip. By using separation of variables on Laplace's equation Meixner-like conditions emerge, that is, $u^D(r,0) \sim r^{\nu_n}$ for all integer n such that $\nu_n = n\pi/2\Phi$. Local to the edge we then use this separation of variables and edge condition to specify the following form of the total field:

$$u^{D,N} = \sum_{n=1}^{\infty} (2/k_0)^{\nu_n}\, \Gamma(1+\nu_n)\, K_n^{D,N}(\theta_0)\, u_n^{D,N}, \quad (2.4)$$

where

$$u_n^{D,N}(r,\theta) = J_{\nu_n}(k_0 r) \begin{cases} \sin[\nu_n(\theta - \pi + 2\Phi)] & \text{for Dirichlet,} \\ \cos[\nu_n(\theta - \pi + 2\Phi)] & \text{for Neumann,} \end{cases} \quad (2.5)$$

for some functions $K_n^{D,N}(\theta_0)$ independent of r, θ and

$$\nu_n = n\pi/2\Phi.$$

The coefficients in (2.4) are chosen in a such way that the nth term of the sum has the following asymptotic expansion:

$$(2/k_0)^{\nu_n}\, \Gamma(1+\nu_n)\, K_n^{D,N}\, J_{\nu_n}(k_0 r) = K_n^{D,N}(r^{\nu_n} + O(r^{\nu_n+2})).$$

From equation (2.3), the $K_n^D(\theta_0)$ introduced in (2.5) are directly evaluated as

$$K_n^D(\theta_0) = \frac{4e^{-i\pi\nu_n/2}}{n\Gamma(\nu_n)} \sin[\nu_n(\theta_0 - \pi + 2\Phi)] \left(\frac{k_0}{2}\right)^{\nu_n}.$$

We shall not actually need this information, in fact the eigenfunction solution is not required at all for the embedding idea. We just require the knowledge of the form of the local behaviour contained in (2.5) that emerges from separation of variables. To within a multiplicative function of θ_0, the $K_n^{D,N}$, the local edge behaviour in terms of r and θ is assumed to be known.

(c) *The directivity*

In the opposite range of r, the far field, we define the far-field directivity, $D^{D,N}(\theta, \theta_0)$, via

$$u^{D,N}(r,\theta) \sim \frac{D^{D,N}(\theta,\theta_0)}{(2\pi k_0 r)^{\frac{1}{2}}} \exp[i(k_0 r + \pi/4)] \quad \text{as} \quad k_0 r \to \infty \tag{2.6}$$

then it is well-known as in, say, Bowman and Senior (1969), that for wedge problems

$$D^{D,N}(\theta,\theta_0) = \nu \sin \pi\nu \left(\frac{1}{\cos \pi\nu - \cos \nu(\theta - \theta_0)} \pm \frac{1}{\cos \pi\nu + \cos \nu(2\pi - 2\Phi - \theta - \theta_0)} \right) \tag{2.7}$$

where we take the positive sign for Neumann and negative for Dirichlet, and $\nu = \pi/2\Phi$. This far-field directivity is well-known and elegant, one aim of this paper is derive an equivalent formula for this directivity, but by using embedding rather than directly from some integral transform.

By extracting the far field behaviour of this scattering problem solely using embedding techniques demonstrates that, to extract the far-field directivity, all one ever needs to do is to solve simpler edge Green's function problems. Thereby "embedding" the directivity of this simpler problem in the directivity of the physical problem. The value of this is that one can then apply embedding to more complex angular structures, see section 4 for an illustration using an equilateral triangle.

3. Derivation of the embedding formulae

(a) *The outline of the embedding procedure*

Let us recall the sequence of operations, and physical ideas, required to deduce the embedding formulae for thin cracks ($\Phi = \pi$ in the notation here) developed in Craster *et al* (2003); the task is to relate the far fields of edge Green's functions to that of the physical field:

- First, one applies an operator, H, to the physical field u which is that corresponding to an incoming plane wave incident upon the crack. For the crack lying along $y = 0$ with $x < 0$ one applies

  $$H = \frac{\partial}{\partial x} + ik_0 \cos \theta_0. \tag{3.1}$$

  Notably, the field given by $H[u]$ is now an eigensolution (i.e. it does not contain any incoming waves and satisfies the necessary boundary conditions), and furthermore the physical field which has local edge behaviour $u \sim Kr^{1/2}$ transforms after differentiation such that $H[u] \propto Kr^{-1/2}$.

- Second, one defines the edge Green's function, $\hat{u}$, for a line source placed at the tip of a crack having the same homogeneous boundary conditions as used in the problem defining the physical field; this has local behaviour such that $\hat{u} \sim r^{-1/2}$ as $r \to 0$. This edge Green's function is also an eigensolution all be it an overly singular one.

- Next, one applies the reciprocity theorem to both the edge Green's function and the physical field with plane-wave incidence. The field due to a line source at the crack tip when observed at infinity is, by interchanging the positions of the source and observer, related to the near field due to a line source suitably displaced to infinity (plane wave incidence). Thus reciprocity relates the edge Green's function directivity, $\hat{D}(\theta)$, to the near field of the physical problem via K.

- Invoking uniqueness is the final, important, step. Here one notes that $H[u]$ is actually proportional to $K\hat{u}$; the constant of proportionality depends upon the boundary conditions on the crack and is known. Both $H[u]$ and the appropriate multiple of $K\hat{u}$ share the same far field, near field and boundary conditions and thus by uniqueness are the same. One is then free to take the far field limit of $H[u]$ and of $K\hat{u}$ (recall from the previous point that K and $\hat{D}$ are related) and this then provides $D(\theta, \theta_0)$ in terms of $\hat{D}(\theta)$ and $\hat{D}(\theta_0)$ and that then furnishes the embedding formula.

Similar arguments follow for many parallel, possibly, finite length cracks.

We shall follow the same procedure here, but there are several important differences. Most urgently the operator H, (3.1), if applied to the wedge geometry, no longer produces eigensolutions along the edge $\theta = \pi - 2\Phi$. Not only that, but neither does the field $H[u]$ have the singular behaviour at the wedge tip that corresponds to that of an edge Green's function. Thus we, apparently, have no way of generating the appropriate overly singular eigensolution using a differential operator; this lies at the root of why it is not thought to be possible to use embedding for non-parallel geometries.

(*b*) *Edge Green's functions for a wedge*

We begin with the second of the points above, that is, by considering the apparently abstract problem of edge Green's functions created by multipole excitation at the tip of a wedge of the same angle, a schematic of which is shown in figure 2.

Let us construct the mth multipole source near the edge; we introduce a new (more convenient) angular coordinate

$$\varphi = \theta + 2\Phi - \pi.$$

Furthermore, we place m line sources at the points

$$r = \epsilon, \qquad \varphi = \frac{\Phi(2j-1)}{m}, \qquad j = 1 \ldots m,$$

and let the amplitudes of the sources be equal to $-\pi\epsilon^{-\nu_m}$ for the odd values of j, and to $\pi\epsilon^{-\nu_m}$ for the even values of j (we recall that $\nu_m = \pi m/(2\Phi)$) and we assume that $\epsilon \ll 1$. The field generated by this configuration of sources, in the limit as $\epsilon \to 0$, is the mth edge Green's function for our problem, and it is denoted by $\hat{u}^D_m$. The hat decoration is used to distinguish these edge Green function solutions from those generated by incoming plane waves.

Let us now focus upon the local behaviour of this mth edge Green's function, $\hat{u}^D_m$. In the vicinity of the edge one solves Laplace's equation, instead of Helmholtz's

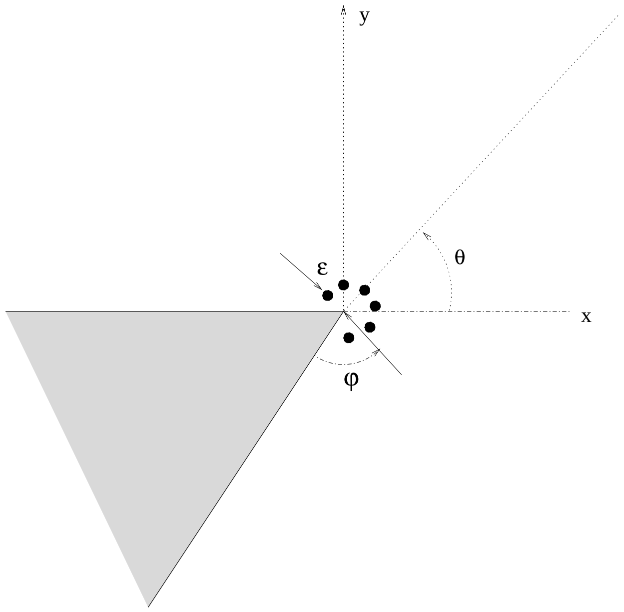

Figure 2. The wedge geometry, for the edge Green's function showing the angles and notation used, together with line sources near the edge (here $m = 6$).

equation, and the solution is obtained using complex variables, for finite ϵ, as:

$$\hat{u}^{inner} \approx -\frac{\epsilon^{-\nu_m}}{2} \operatorname{Re}\left[\frac{z^{\nu_m} - \epsilon^{\nu_m}}{z^{\nu_m} + \epsilon^{\nu_m}}\right],$$

where $z = re^{i\varphi}$; one could view this as the inner limit of an asymptotic expansion.

The outer asymptotic limit, as $\epsilon \to 0$, of this then gives

$$\hat{u}_m^D = r^{-\nu_m} \sin(\nu_m \varphi)(1 + o(1)). \tag{3.2}$$

The edge Green's function $\hat{u}_m^N$ for the Neumann problem is obtained as the result of a similar limiting procedure, but the sources are then placed at the points

$$r = \epsilon, \qquad \varphi = \frac{2\Phi j}{m}, \qquad j = 1 \ldots m.$$

The equivalent result to (3.2) is that

$$\hat{u}_m^N = r^{-\nu_m} \cos(\nu_m \varphi)(1 + o(1)).$$

The arguments above are valid near any polygonal vertex, in the special case of a wedge the edge Green's functions emerge explicitly in terms of Hankel functions, namely,

$$\hat{u}_m^{D,N} = \frac{\pi i}{\Gamma(\nu_m)} \left(\frac{k_0}{2}\right)^{\nu_m} H_{\nu_m}^{(1)}(k_0 r) \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} (\nu_m \varphi). \tag{3.3}$$

(c) Applying the reciprocity theorem to the edge Green's functions

As in the physical problem we consider the far field and define the directivities, $\hat{D}_n^{D,N}(\theta)$, of the nth edge Green's function $\hat{u}_n^{D,N}$ from

$$\hat{u}_n^{D,N}(r,\theta) \sim \frac{\hat{D}_n^{D,N}(\theta)}{(2\pi k_0 r)^{\frac{1}{2}}} e^{i(k_0 r + \pi/4)} \tag{3.4}$$

and for the wedge this directivity, from (3.3), is

$$\hat{D}_n^{D,N}(\theta) = \frac{2\pi}{\Gamma(\nu_n)} \left(\frac{k_0}{2}\right)^{\nu_n} e^{-i\nu_n\pi/2} \begin{cases} \sin[\nu_n(\theta - \pi + 2\Phi)] \text{ for Dirichlet} \\ \cos[\nu_n(\theta - \pi + 2\Phi)] \text{ for Neumann} \end{cases} .$$

These directivities are only a function of a single angular variable.

We now apply the reciprocal theorem to a pair of solutions for the Helmholtz equation with the wedge geometry described above, namely to the "physical" solution $u^{D,N}$ describing the field generated by the plane wave incidence and to one of the edge Green's functions $\hat{u}_n^{D,N}$. The physical solution can be considered as being generated by a line source located far enough from the wedge tip and having an appropriate amplitude. By using the integral form of the reciprocal theorem and performing an integral around the wedge tip, one can obtain the following relation

$$\hat{D}_n^{D,N}(\theta_0) = \frac{n\pi}{2} K_n^{D,N}. \tag{3.5}$$

This relation establishes the connection between the edge behaviour of $u^{D,N}$ and the far-field directivity of one of the edge Green's functions $\hat{u}_n^{D,N}$. In our case these directivities are known to be simple functions, but it is also the case that the relation (3.5) remains valid in significantly more complicated cases, e.g. for vertices of polygons, in which case the directivities of the edge Green's functions are not known explicitly.

Indeed, this relation can also be verified directly for the special wedge geometry used here, using the eigenfunction expansion (2.3) and the large argument asymptotics for Hankel functions.

This relation has explicitly connected an edge Green's function solution to that from the physical problem. In actual fact we require, for our embedding formula, a relation between their far fields; so far this relation connects the near field of one to the far field of the other.

(d) The operator

This is probably the most crucial ingredient of the embedding recipe, and the piece that has hitherto hindered further development of embedding into more general geometries. The differential operator H that produces the embedding formula for the wedge case must have the following properties:

- It should map a solution of the Helmholtz equation into a solution of the same equation,
- It should maintain homogeneous boundary conditions on the faces of the wedge (we shall find an operator preserving the common types of boundary conditions: Dirichlet, Neumann, impedance),

- It should have the property $H[u^{\rm in}] = 0$ in order to generate an eigensolution satisfying the radiation condition.

The first condition is satisfied by all operators having the form

$$H = c + \sum_{n=1}^{N} \left(a_n \frac{\partial^n}{\partial x^n} + b_n \frac{\partial^n}{\partial y^n} \right) \tag{3.6}$$

for some constants a_n, b_n and c. Since the incident field, $u^{\rm in}$, is an exponential in x and y the third condition can be satisfied by an appropriate choice of the constant c.

The most interesting question is how to construct the operator that *a-priori* preserves the boundary conditions. First we note that any operator having all the b_n equal to zero, i.e.

$$H = c + \sum_{n=1}^{N} a_n \frac{\partial^n}{\partial x^n} \tag{3.7}$$

preserves the boundary conditions (Dirichlet, Neumann or impedance) on the face $\theta = \pi$. If we were to rotate the coordinate system in such a way such that the new x-coordinate axis corresponds to the wedge face $\theta = \pi - 2\Phi$, then the operator H would again be of the form (3.7) but now in this new coordinate system.

At first glance it is not clear how to find such an operator, however we shall use an important additional property of the field, namely that it satisfies Helmholtz's equation (2.1). This equation is valid inside the fluid wedge, but it can also be continued onto the faces of the wedge (except, maybe, at the wedge tip itself). So, we obtain the following rule to transform the operators:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_0^2 = 0. \tag{3.8}$$

This means that the operators H_x, H_y defined as

$$H_x = \frac{i}{k_0}\frac{\partial}{\partial x} \quad \text{and} \quad H_y = \frac{i}{k_0}\frac{\partial}{\partial y}$$

obey the same algebraic rules as do $\cos\theta$ and $\sin\theta$, respectively.

We settle upon the case of "rational" wedge angles

$$2\Phi = \frac{q\pi}{p} \tag{3.9}$$

for integer p and q. Notably, at present, it is only for rational wedge angles that we can find an appropriate operator H.

We introduce the operator H as

$$H = (-ik_0)^p \left[T_p\left(H_x\right) - T_p(\cos\theta_0) \right] \tag{3.10}$$

where $T_p(x)$ denotes the Tchebyshev polynomial. For instance for $p = 3$ the operator is

$$H = 4\frac{\partial^3}{\partial x^3} + 3k_0^2 \frac{\partial}{\partial x} - 4(-ik_0\cos\theta_0)^3 - 3k_0^2(-ik_0\cos\theta_0). \tag{3.11}$$

Let us now prove that the operator (3.10) obeys all of the conditions mentioned above, and therefore that it is ideal for producing embedding formulae.

First, this operator has the form shown in (3.7); it maps the Helmholtz equation to itself and $H[u^{\rm in}] = 0$. It directly preserves the boundary condition on the wedge face $\theta = \pi$. Our remaining task is to show that this operator preserves the boundary condition along the wedge face $\theta = \pi - 2\Phi$.

We rotate the coordinate system clockwise through an angle of 2Φ and consider the form of H in the new coordinate system x' and y' (x' is along the wedge face $\theta = \pi - 2\Phi$ and y' perpendicular to it). This is performed as follows: The H_x operator can be formally substituted by $\cos\theta$ and then θ is replaced by $\theta' - 2\Phi$, some trigonometric identities are applied, and then finally $\cos\theta'$ and $\sin\theta'$ are substituted by H'_x and H'_y, respectively.

In more detail, we note that the definition of the Tchebyshev polynomial is

$$T_p(\cos\theta) = \cos(p\theta).$$

Consequently,

$$T_p(\cos(\theta' - q\pi/p)) = \cos(p(\theta' - q\pi/p)) = (-1)^q \cos(p\theta'),$$

and then

$$H = (-ik_0)^p \left[(-1)^p T_p\left(H'_x\right) - T_p(\cos\theta_0)\right].$$

The result of these calculations is that the operator H in the transformed coordinate system is again in the form (3.7), but now with derivatives taken with respect to the new coordinate x' directed along the second face of the wedge; therefore it preserves the boundary conditions along this wedge face too.

(*e*) *The operator H and uniqueness*

The operator H as defined by (3.10) is now applied to the terms $u_n^{D,N}$ of the representation (2.4); we shall find $H[u_n^{D,N}]$, and then consider its local asymptotics.

We do the Dirichlet case in detail, and just provide the Neumann result at the end. The function $H[u_n^D]$ necessarily satisfies the Helmholtz equation and boundary conditions on both faces and outgoing waves at infinity, therefore it can be written as

$$H[u_n^{D,N}] = \sum_m [c_{n,m} J_{\nu_m}(k_0 r) + d_{n,m} H^{(1)}_{\nu_m}(k_0 r)] \sin(\nu_m(\theta - \pi + 2\Phi)), \qquad (3.12)$$

where the sum is taken over a set of integer indices m; we identify this finite set in this section.

At this point, although ultimately interested in the *near* field, it is useful to study the *far* field of the expression $H[u_n^{D,N}]$, where $u_n^{D,N}$ is given by (2.5). This enables us to find the coefficients c and d rapidly.

Consider the leading term of the far-field asymptotics of $H[u_n^{D,N}]$, i.e. the term behaving as $r^{-1/2}$ at infinity. Note that $u_n^{D,N}$, has the following asymptotic expansion:

$$u_n^{D,N} = \left(\frac{2}{\pi k_0 r}\right)^{1/2} \cos\left(k_0 r - \frac{\nu_n \pi}{2} - \frac{\pi}{2}\right) \sin(\nu_n(\theta - \pi + 2\Phi))(1 + O(1/r)).$$

To find the leading term of $H[u_n^{D,N}]$ we can restrict ourselves to differentiation only of the radial part of $u_n^{D,N}$ which leads to

$$H[u_n^{D,N}] = (-1)^p k_0^p \left(\frac{2}{\pi k_0 r}\right)^{1/2} \{(-1)^q \cos\left(k_0 r - \frac{(\nu_n - p)\pi}{2} - \frac{\pi}{2}\right) \times$$

$$\times \sin(\nu_n\varphi)\cos(p\,\varphi) - i^p\, T_p(\cos\theta_0) u_n^D\}(1 + O(1/r)). \tag{3.13}$$

We recall that $\varphi = \theta - \pi + 2\Phi$.

This asymptotic expansion enables us to say what terms appear in the sum (3.12). Considering the angular part of (3.13) it becomes apparent that it is a sum of harmonics with indices ν_n, $\nu_n - p$ and $\nu_n + p$. Note that

$$\nu_n \pm p = \nu_{n\pm q}.$$

Thus, we find that m in equation (3.12) can take only three values:

$$m = n,\ |n-q|,\ n+q.$$

To find the values of the coefficients $c_{m,n}$ and $d_{m,n}$ we inspect the radial piece of equation (3.13). There are two different cases: the first is when $n \geq q$, and the second is when $n < q$. In the first case only Bessel functions appear in the expansion (3.12):

$$H[u_n^{D,N}] = \frac{(-1)^q k_0^p}{2} u_{n+q}^D + \frac{(-1)^{q+p} k_0^p}{2} u_{n-q}^D - (-ik_0)^p\, T_p(\cos\theta_0) u_n^D \qquad \text{for} \quad n \geq q.$$

In the second case the situation is more complicated:

$$H[u_n^{D,N}] = \frac{(-1)^q k_0^p}{2} u_{n+q}^D - (-ik_0)^p\, T_p(\cos\theta_0) u_n^D +$$

$$\frac{(-1)^{q+p} k_0^p}{2}(-\exp\{-i\nu_{q-n}\pi\} u_{q-n} - i\sin(\nu_{q-n}\pi) H^{(1)}_{\nu_{q-n}}(k_0 r)\sin(\nu_{q-n}\varphi)) \text{ for } n < q.$$

Note that the Hankel function appears only for the index $m = q - n$. For our further consideration only the oversingular part is important as that is related to the edge Green's function. The study of the local edge behaviour enables us to find the following edge asymptotics of the total field:

$$H[u^D] = 2^p(-1)^{q-p+1} \sum_{n=1}^{q-1} K_n^D(\theta_0)[\nu_n(\nu_n - 1)...(\nu_n - p + 1)] r^{-\nu_{q-n}} \sin(\nu_{q-n}\varphi) +$$

$$+ \text{Meixner terms.} \tag{3.14}$$

The oversingular terms in the sum are precisely the oversingular asymptotics of edge Green's functions $\hat{u}^D_{\nu_{q-n}}$.

To utilize uniqueness we now construct the following function:

$$w = H[u^D] - 2^p(-1)^{q-p+1} \sum_{n=1}^{q-1} K_n^D(\theta_0)[\nu_n(\nu_n - 1)...(\nu_n - p + 1)]\, \hat{u}^D_{\nu_{q-n}}.$$

This function obeys the Helmholtz equation, Dirichlet boundary conditions, the radiation condition and the edge condition; by uniqueness, it is identically zero. So, we obtain the following (exact) representation for the field $H[u^D]$:

$$H[u^D] = 2^p(-1)^{q-p+1}\sum_{n=1}^{q-1} K_n^D(\theta_0)[\nu_n(\nu_n-1)...(\nu_n-p+1)]\,\hat{u}^D_{\nu_{q-n}}. \tag{3.15}$$

Hence the differential operator we use, (3.10), relates the physical solution to a collection of edge Green's functions generated by multipoles at the wedge tip. Additionally, from reciprocity, we know the relation, (3.5), between the near field coefficients K_n^D and the far-field directivity $\hat{D}_n^D$.

(f) *The embedding formulae*

We are fundamentally interested in the far field behaviour, and specifically in the directivities; to obtain these we take the large r limit of equation (3.15). We know the far field behaviour of u^D is characterized by a directivity $D^D(\theta,\theta_0)$ and that of each edge Green's function, u_n^D, by $\hat{D}_n^D(\theta)$ (notably a function only of a single variable). These directivities are modulated by known radial dependence, and thus the derivatives in the differential operator, $H[u^D]$, are easily done and we find that:

$$D^D(\theta,\theta_0) = \frac{\sum_{n=1}^{q-1}(-1)^{q-p+1}2^p[\nu_n(\nu_n-1)...(\nu_n-p+1)]\hat{D}_n^D(\theta_0)\hat{D}_{q-n}^D(\theta)/(n\pi)}{k_0^p e^{\frac{i\pi p}{2}}[\cos p\theta-(-1)^p\cos p\theta_0]}. \tag{3.16}$$

In many ways this is a remarkably simple formula; it is in the form of a sum of products of the edge function directivities divided by a trigonometric term involving both θ and θ_0. The Neumann result is identical with the superscript N replacing D, and the right-hand side multiplied by minus one.

We have the directivities for the edge Green's functions $\hat{D}^D$ to hand and thus

$$D^D(\theta,\theta_0) = \sum_{n=1}^{q-1}\frac{(-1)^{q+1}(2\pi)^2[\nu_n(\nu_n-1)...(\nu_n-p+1)}{n\pi\Gamma(\nu_n)\Gamma(\nu_{q-n})}$$
$$\times\frac{\sin[\nu_n(\theta_0-\pi+2\Phi)\sin[\nu_{q-n}(\theta-\pi+2\Phi)]}{[\cos p\theta-(-1)^p\cos p\theta_0]}. \tag{3.17}$$

There are many existing formulae for the directivity using various trigonometric identities and the usual formulae (2.7), and for rational wedges some simplifications occur, see Rawlins (1987), but this embedded formula has, apparently, not been found before. Although this formula is not in reciprocal form, due to the coefficient n, it can be shown that it does satisfy reciprocity.

(g) *Illustrative case: right-angled wedge*

The right-angled wedge is an important special case, and the operator we utilize is simpler than the general form and is:

$$H = \frac{\partial^2}{\partial x^2} - (-ik_*)^2$$

(note we have removed a factor of 2 from the general operator used earlier). Clearly, this nullifies the incident field. It also evidently generates an eigensolution along the negative x axis. To see that it also does along the negative y axis one uses the fact that the physical field satisfies the Helmholtz equation to replace x derivatives with y derivatives and the generation of an eigensolution is almost immediate. Thus we use this operator to connect the edge Green's functions with the physical field. The intervening steps are the same as for the general formula, so we now turn to the final formulae.

The well-known directivity is

$$D^D(\theta,\theta_0) = \frac{2}{3}\sin\left[\frac{2}{3}(\theta_0-\pi)\right]\left(\frac{1}{\cos\frac{2}{3}\theta-\cos\frac{2}{3}(\theta_0-\pi)} - \frac{1}{\cos\frac{2}{3}(\theta-2\pi)-\cos\frac{2}{3}(\theta_0-\pi)}\right)$$

after some reductions, the embedding formula:

$$D^D(\theta,\theta_0) = -\frac{2\nu_1(\nu_1-1)\hat{D}_1^D(\theta_0)\hat{D}_2^D(\theta)+\nu_2(\nu_2-1)\hat{D}_2^D(\theta_0)D_1^D(\theta)}{\pi k_0^2(\cos^2\theta-\cos^2\theta_0)}$$

emerges from (3.16) and further reduces to

$$D^D(\theta,\theta_0) = \frac{4}{3}\frac{\sin\pi/3}{(\cos^2\theta-\cos^2\theta_0)}[\sin[\frac{4}{3}(\theta_0+\pi/2)]\sin[\frac{2}{3}(\theta+\pi/2)]-\sin[\frac{2}{3}(\theta_0+\pi/2)]\sin[\frac{4}{3}(\theta+\pi/2)]].$$

It is not obvious that the first and last of these directivity formulae are actually identical - but, after some algebra, they are. A notable point is that one requires two edge Green's functions for the right-angled wedge, one corresponding to a line source and the other to a dipole at the wedge tip; for the line cracks in Craster *et al.* (2003) only line sources were required. As we further alter the wedge angle more multipole edge Green's functions are required.

4. Scattering by an equilateral triangle

As an illustration of how embedding works in a less trivial case, namely in the particular case of diffraction by a polygon, we consider the problem of diffraction of a plane wave by an equilateral triangle (see figure 3); the boundary conditions along the triangle faces are taken to be Dirichlet. Now we introduce three sets of local coordinates (r_j,φ_j), $j=1,2,3$ near the vertices of the triangle to study the edge behaviour of the field and the global coordinates (r,θ) for describing the far-field. The incident wave is again a plane wave of the form (2.2).

For each edge of the scatterer the parameter 2Φ is equal to $5\pi/3$, and so, in our earlier notation, we set $p=3$, $q=5$, and $\nu_n=3n/5$.

For each of the edges we must define $q-1(=4)$ multipole edge Green's functions, namely $\hat{u}_n^j$, for $j=1\ldots3$, $n=1\ldots4$ according to the limiting procedure introduced above. Each edge Green's function certainly belongs to the Dirichlet type. For each of the edge Green's functions we define the directivity $\hat{D}_n^j(\theta)$, and we shall find the directivity $D(\theta,\theta_0)$ of the physical field u in terms of them. Finding the edge Green's functions and their directivities is a non-trivial problem at least as complicated as solving the initial physical problem; but let us assume that they are known, or obtainable.

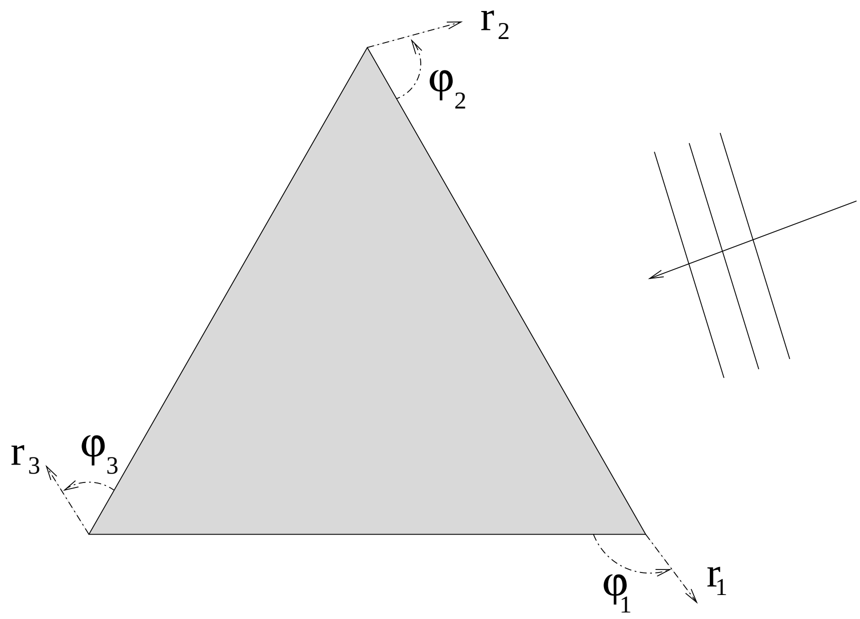

Figure 3. The triangle, showing the angles and notation.

We introduce the coefficients $K_n^j(\theta_0)$ describing the local behaviour of the physical field through the expansions

$$u(r_j,\varphi_j) = \sum_{n=1}^{\infty} (2/k_0)^{\nu_n} \Gamma(1+\nu_n) K_n^j(\theta_0) J_{\nu_n}(k_0 r_j) \sin(\nu_n \varphi_j) \tag{4.1}$$

valid near the edges. By applying the reciprocity argument we again establish that

$$K_n^j(\theta_0) = \frac{2}{n\pi} \hat{D}_n^j(\theta_0). \tag{4.2}$$

Now we can apply the operator (3.11) to the physical field u. By studying the local behaviour of the field at the edges and applying the theorem of uniqueness, we conclude that the result has the form

$$H[u] = \sum_{j=1}^{3} \sum_{n=1}^{4} h_n K_n^j \hat{u}_n^j, \tag{4.3}$$

where

$$h_n = -8[\nu_n(\nu_n - 1)(\nu_n - 2)].$$

Now we are in a position to study the far-field asymptotics of the left- and right-hand sides of (4.3), to obtain the embedding formula in its final form:

$$D(\theta,\theta_0) = \frac{i}{\pi k_0^3} \sum_{j=1}^{3} \sum_{n=1}^{4} \frac{h_n \hat{D}_n^j(\theta_0) \hat{D}_{5-n}^j(\theta)}{n(\cos 3\theta + \cos 3\theta_0)}. \tag{4.4}$$

On a practical level one would have to find the directivities for the multipole edge Green's functions, only once, and thereafter as the angle of incidence or observation varied one manipulates them to generate to full directivity; this avoids continually resolving the scattering problem.

5. Concluding remarks

Evidently, as demonstrated in detail here, embedding formulae can be constructed for angular domains. This now opens the way to developing embedding for, say, inclined cracks to surfaces and other finite geometries, and also to extending the quarter plane (three-dimensional) results of Shanin (2004a,b) to polyhedral domains and vertices. Other technical issues such as generalizing the governing equations to, say, elasticity have been discussed for planar geometries in Craster *et al* (2003), the wedge geometry for elasticity is much more challenging, but there is no fundamental reason why it too could not be embedded.

Notably, here, we were limited to rational wedge angles, which raises the question of whether we can generalize this to irrational, and thus completely general, wedge angles. It is unclear whether this is so, and further work is underway to clarify this. Nonetheless, it is now clear that embedding is not fundamentally limited to parallel planar geometries.

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