Embedding formulae for planar cracks

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Abstract.
Embedding formulae allow one to decompose scattering problems apparently dependant upon several angular variables (angles of incidence and observation) into those dependant upon fewer angular variables. In terms of facilitating rapid computations across considerable parameter regimes this is a considerable advantage. In this short article we concentrate on embedding formulae for a typical problem from acoustics in three dimensions.

Keywords: Embedding, integral equations, acoustics, asymptotics, ray theory

1. Introduction

In three dimensions, the solution of a diffraction problem is usually represented as a function of four angular variables: two of them specify the direction of the wave vector of the incident plane wave illuminating the obstacle, and the other two are the direction of the scattered wave. The far-field diffraction pattern is a function of these directions. For numerical work, it can be time consuming to perform a parametric study - all angular variables must be independently varied and the numerical routine rerun for each value.

Fortunately, for many practically important cases there exists an elegant mathematical theory, unfortunately it is little known and not often utilized, that enables one to reduce the dimension of the problem. The essence of this theory is the following: instead of directly solving the main diffraction problem with the desired plane-wave incidence, a set of different auxiliary problems are solved. For example, if the obstacle is a planar crack in the medium, the auxiliary problems are associated with the excitation of the field by a point source located asymptotically close to the edge of the crack. We could also interpret these auxiliary solutions as unphysically singular eigensolutions of the problem, in the sense that they no longer have the usual local square root dependence of the acoustic potential at the edge (i.e. having the dependence of the form $\sim r^{1/2}$) but are square root singular there (i.e. have the form $\sim r^{-1/2}$).

The solution of the auxiliary diffraction problem (in 3D) depends on only three variables: the position of the crack edge, where the source is located and the two angles which determine the direction of the scattering. The solution of the original diffraction problem is represented as the integral of the solutions of the auxiliary problems. Such a representation is called an embedding formula.

Embedding formulae have been derived by several previous authors for various diffraction problems, these have used a different set of auxiliary problems, or have used theories based explicitly upon integral equations beginning with (Williams, 1980), and this triggered further applications to cracks in elastic solids, (Martin and Wickham, 1983), and recently the method has been embraced by (Biggs et al., 2000; Biggs and Porter, 2001; Biggs and Porter, 2002). However, except for the article by (Williams, 1980) who uses grazing incidence to generate the auxiliary solutions, the derivation of embedding formulae is typically through complicated manipulations of integral equations which obscures the final structure of the formula.

One purpose of this article is to demonstrate an easy way to derive embedding formulae, which has a physical interpretation and can be easily implemented, that is, we use a set of auxiliary solutions that have immediate interpretations. Here we shall consider incident fields that consist of plane waves and this is important for the success of the embedding technique as we utilize this in an operator we apply.

2. Formulation

We suppose that the Helmholtz equation

$$\nabla^2\phi + k_0^2\phi = 0 \tag{1}$$

holds in 3D where Cartesian coordinates (x, y, z) are utilized, so $\phi(x, y, z)$ and the crack occupies area S in the (x, y) plane.

For definiteness we take the Neumann boundary condition $\partial_z\phi = 0$ on the faces of the planar defect/crack; the approach remains valid for Dirichlet or in electromagnetic theory for impedance boundary conditions.

The total field ϕ is the sum of an incident field $\phi^{(\mathrm{in})}$ and a scattered field $\phi^{(\mathrm{sc})}$. The incident field is assumed to be a plane wave

$$\phi^{(\mathrm{in})} = \exp[-i(\mathbf{k}^{(\mathrm{in})} \cdot \mathbf{x} + \sqrt{k_0^2 - |\mathbf{k}^{(\mathrm{in})}|^2} z)] \tag{2}$$

where $\mathbf{k}^{(\mathrm{in})} = (k_x^{(\mathrm{in})}, k_y^{(\mathrm{in})})$ and $\mathbf{x} = (x, y)$.

We also require for physically meaningful solutions that suitable edge conditions (Meixner's) are fulfilled, which for our planar problem means that the field near the edge of the crack has the asymptotic behaviour that

$$\phi \sim K r^{1/2} \cos \varphi/2, \tag{3}$$

where r is the distance between the observation point and the edge of the crack/defect; φ is the angle in the local cylindrical coordinates taken such that φ lies along the crack face on $z = 0_+$.

We utilize uniqueness, that is, we consider only the scattered field, i.e. $\phi = \phi^{(\mathrm{sc})}$ and assume that the Helmholtz equation, boundary, radiation and edge conditions be fulfilled. Then $\phi = 0$ identically. We assume that the theorem of uniqueness is satisfied by all diffraction problems considered here.

Using the spectral language, we assume that the parameter k_0 of equation (1) does not belong to the spectrum of the problem, that is, we cannot have any trapped modes. Note that the method proposed below is applicable even if k_0 belongs to the spectrum and has finite degeneration. However, in this case the method should be modified.

2.1. Auxiliary solutions of the diffraction problems

We require the solutions of auxiliary problems, namely diffraction problems with point source incidence (or a line source for a 2D problem). The scatterer is assumed to have the same geometry and (homogeneous) boundary conditions as the scatterer of the initial diffraction problem, and the source is located near the edge of the crack. For our present purpose one cannot simply place the source near the edge of the crack, since the Neumann condition is fulfilled on the crack faces, and we still assume the physically meaningful, Meixner's condition, is fulfilled at the edge. It is also assumed that the radiation condition at infinity holds.

We now consider a limiting procedure, that is, we quantify how near the source is to the edge, in terms of which the auxiliary functions will be treated. We introduce a coordinate l along the edge of the crack, and take a point lying in the (x, y) plane a small distance, ϵ, from a position $l_0 = (x_0, y_0, 0)$ on the contour Γ (see Fig. 1). We consider a diffraction problem with a pair of point sources, strength $-\pi\epsilon^{-1/2}/2$, above and below the crack; we solve the inhomogeneous Helmholtz equation for the function $\hat{\phi}_\epsilon(x, y, z; l_0)$:

$$\nabla^2 \hat{\phi}_\epsilon + k_0^2 \hat{\phi}_\epsilon = -\frac{1}{2}\pi\epsilon^{-1/2}\delta(x - x')\delta(y - y')\delta(z - 0) +$$

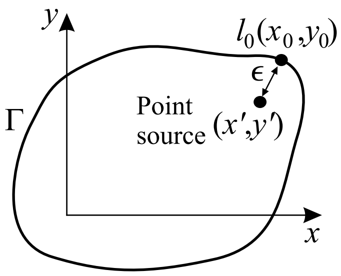

Figure 1. Location of a point source

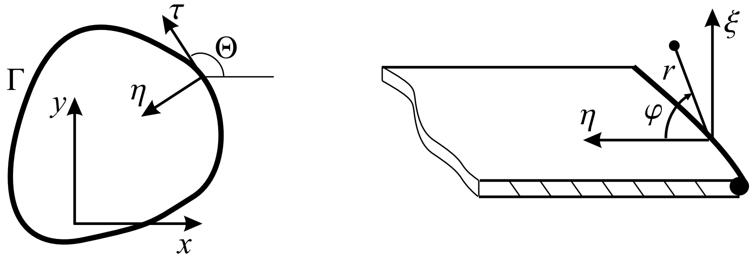

Figure 2. To the geometry of the problem

$$\frac{1}{2}\pi\epsilon^{-1/2}\delta(x-x')\delta(y-y')\delta(z+0), \tag{4}$$

where δ is the delta-function,

$$x' = x_0 - \epsilon\sin\Theta, \qquad y' = y_0 + \epsilon\cos\Theta.$$

Here Θ is the angle between the vector dl tangential to Γ and the x-axis (see Fig. 2).

A detailed study shows that for each point (x, y, z), with the exception of the point l_0 on Γ, there exists a finite limit

$$\hat{\phi}(x, y, z; l_0) = \lim_{\epsilon\longrightarrow 0}\hat{\phi}_\epsilon(x, y, z; l_0). \tag{5}$$

The function $\hat{\phi}(x, y, z; l)$ is the auxiliary solution.

The auxiliary problem has one important property: it depends on fewer variables than the physical diffraction problem. The function $\phi^{(\mathrm{sc})}$ depends explicitly on 3 variables (the spatial coordinates) and implicitly on 2 variables: the parameters k_x and k_y of the incident wave, i.e., the total number of variables is 5. The number of the arguments of $\hat{\phi}$ is 4. We assume that the function $\hat{\phi}(x, y, z; l)$ is known and express the solution of the initial diffraction problem, with plane-wave incidence, in terms of this function.

Below we shall use the asymptotics of the auxiliary solution at the edge. Consider the integral

$$\phi^*(x,y,z) = \int_\Gamma \rho(l)\hat{\phi}_\epsilon(x,y,z;l)dl, \tag{6}$$

where the sources of the field are concentrated along the contour Γ and have line density $\rho(l)$. We assume this density to be a continuous function having period equal to the length of the contour Γ. One can show that for fixed r

$$\lim_{\epsilon\to 0}\phi \sim \frac{\rho(l)\cos(\varphi/2)}{r^{1/2}} + O(r^{1/2}). \tag{7}$$

One can see that the source near the edge of the crack leads to a field with edge asymptotics stronger, than is allowed by the usual Meixner's conditions. This property will be used below.

We see that there are two equivalent ways to introduce the auxiliary solution. The first one is to introduce a point source near the the edge and the corresponding limiting procedure. The other is to formally introduce a solution having the edge asymptotics that is stronger than it is allowed by the edge conditions. However, the second way is a bit cumbersome in the 3D case, where it is necessary to provide the oversingular behaviour at a single point of the edge.

2.2. Directivity of the field

In the far field zone the leading term of the scattered field can be written as the modulated spherical wave:

$$\phi^{(\mathrm{sc})}(x,y,z) \sim -\frac{e^{\mathrm{i}k_0 R}}{2\pi R} D(\theta_x,\theta_y;\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})}), \tag{8}$$

where $R = \sqrt{x^2+y^2+y^2}$; $\theta_x = \arccos(x/R)$; $\theta_y = \arccos(y/R)$; $\theta_x^{(\mathrm{in})} = \arccos(k_x^{(\mathrm{in})}/k_0)$; $\theta_y^{(\mathrm{in})} = \arccos(k_y^{(\mathrm{in})}/k_0)$; D is the directivity of the field. Utilizing the Green's formula, one can express the directivity as the Fourier-transform of normal derivative of the scattered field on the crack:

$$D(\theta_x,\theta_y;\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})}) = \mathrm{i}k_0\sqrt{1-(\cos^2\theta_x+\cos^2\theta_y)}\times$$
$$\iint_S \phi^{(\mathrm{sc})}(x,y,+0)\, e^{-\mathrm{i}k_0(x\cos\theta_x + y\cos\theta_y)}dx\, dy. \tag{9}$$

Analogously, the auxiliary solution can also be represented using its directivity:

$$\hat{\phi}^{(\mathrm{sc})}(x,y,z;l) \sim -\frac{e^{\mathrm{i}k_0 R}}{2\pi R}\hat{D}(\theta_x,\theta_y;l), \tag{10}$$

and the directivity can be calculated as

$$\hat{D}(\theta_x, \theta_y; l) = \mathrm{i}k_0 \sqrt{1 - (\cos^2\theta_x + \cos^2\theta_y)} \times$$

$$\iint_S \hat{\phi}^{(\mathrm{sc})}(x, y, +0; l)\, e^{-\mathrm{i}k_0(x\cos\theta_x + y\cos\theta_y)} dx\, dy. \tag{11}$$

The embedding formula, which will be derived below, expresses the function $D(\theta_x, \theta_y; \theta_x^{(\mathrm{in})}, \theta_y^{(\mathrm{in})})$ in terms of $\hat{D}(\theta_x, \theta_y; l)$.

2.3. Derivation of the embedding formula

We are going to derive the embedding formula in three steps: beginning with applying operators to the total field, followed by an application of the uniqueness theorem and the reciprocity principle.

Consider

$$\mathbf{H} = (H_x, H_y) = [\nabla + \mathrm{i}\mathbf{k}^{(\mathrm{in})}]\phi = (\partial_x + \mathrm{i}k_x^{(\mathrm{in})}, \partial_y + \mathrm{i}k_y^{(\mathrm{in})}) \tag{12}$$

Apply one of this operators (say, H_x) to the total field ϕ related to the initial diffraction problem with a plane wave incidence. The function

$$\overline{\phi}(x, y, z) = H_x[\phi(x, y, z)]$$

has the following properties: it satisfies the Helmholtz equation (1), contains no incoming waves from infinity or growth at infinity (note that $H_x[\phi^{in}] \equiv 0$), and $\overline{\phi} = 0$ on the crack surfaces. The conditions of the uniqueness theorem are satisfied, except for the edge condition. If the local asymptotics of the field, ϕ, near the edge are

$$\phi \sim K(l) r^{\frac{1}{2}} \cos\left(\frac{\varphi}{2}\right) + O(r^{\frac{3}{2}}),$$

then

$$\overline{\phi} \sim -\frac{1}{2} K(l) r^{-\frac{1}{2}} \sin\Theta \cos\left(\frac{\varphi}{2}\right) + O(r^{\frac{3}{2}}), \tag{13}$$

where θ is the angle between the x-axis and the unit vector dl tangential to the contour Γ. That is, $\overline{\phi}$ has an overly singular behaviour at the edge.

Comparing the edge asymptotics of the function $\overline{\phi}$ in (13) and the integral of the auxiliary functions (7), one finds that the combination

$$w(x, y, z) = \overline{\phi}(x, y, z) + \frac{1}{2} \int_\Gamma K(l) \sin\Theta(l)\, \hat{\phi}(x, y, z; l) dl \tag{14}$$

obeys the usual Meixner's condition at the edge. Furthermore this function obeys the Helmholtz equation, the radiation condition and the Dirichlet boundary condition. Therefore, we apply uniqueness to this combination and thus $w(x,y,z) \equiv 0$, and

$$H_x[\phi] = -\frac{1}{2}\int_\Gamma K(l)\sin\Theta(l)\,\hat{\phi}(x,y,z;l)dl. \tag{15}$$

This is a *weak form* of the embedding formula.

The function $K(l)$ in (15) remains unknown, to generate the complete embedding formula we must express $K(l)$ in terms of $\hat{\phi}(x,y,z;l)$. Instead of having an incident plane wave let us take a point source of the unit strength located at a point (X,Y,Z), such that

$$X = R\frac{k_x}{k_0}, \qquad Y = R\frac{k_y}{k_0}, \qquad Z = \sqrt{R^2 - X^2 - Y^2}$$

and the lengthscale R is much greater than both the size of the scattering region and the wavelength (being more accurate, we assume that the point (X,Y,Z) is located in the far field zone). The incident field from the source is asymptotically a plane wave having the form (2) multiplied by the factor $-(4\pi R)^{-1}e^{\mathrm{i}k_0R}$. To find $K(l)$ we take the observation point, in the (x,y) plane, to be at a small distance ϵ from the point l on the edge contour Γ. We multiply the value of the field at the observation point by $\epsilon^{-1/2}$ and take the simultaneous limits that $R \to \infty$ and $\epsilon \to 0$. The result is $K(l)$ from the formula (15) multiplied by $-(4\pi R)^{-1}e^{\mathrm{i}k_0R}$.

We now use the reciprocity principle (Junger and Feit, 1986) and interchange the source and observation point in the limit procedure described above, that is, the source is now near the edge, and the observation point is at (X,Y,Z). From the Helmholtz reciprocity principle, the value of the field for this interchanged problem is the same as that of the original problem. The diffraction problem with the point source located near the edge is the auxiliary problem, the solution under the appropriate limit is $\hat{\phi}(x,y,z;l)$. Hence

$$K(l) = 4\lim_{R\to\infty}[Re^{-\mathrm{i}k_0R}\hat{\phi}(X,Y,Z;l)]. \tag{16}$$

Using the formula (10), we obtain that

$$K(l) = -\frac{2}{\pi}\hat{D}(\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})};l), \tag{17}$$

That is, the edge behaviour of the physical problem is represented in terms of the far field of the auxiliary solution.

Next we substitute the relation (17) into the embedding formula (15), differentiate with respect to z and perform the Fourier transformation in the (x,y)-plane. The result is

$$D(\theta_x,\theta_y;\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})}) =$$

$$-\frac{\mathrm{i}}{\pi k_0\left(\cos\theta_x+\cos\theta_x^{(\mathrm{in})}\right)}\int_\Gamma \hat{D}(\theta_x,\theta_y;l)\hat{D}(\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})};l)\sin\Theta(l)dl. \tag{18}$$

It is interesting to note that another embedding formula emerges by applying the operator H_y and repeating the arguments above:

$$D(\theta_x,\theta_y;\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})}) =$$

$$\frac{\mathrm{i}}{\pi k_0\left(\cos\theta_x+\cos\theta_x^{(\mathrm{in})}\right)}\int_\Gamma \hat{D}(\theta_x,\theta_y;l)\hat{D}(\theta_x^{(\mathrm{in})},\theta_y^{(\mathrm{in})};l)\cos\Theta(l)dl. \tag{19}$$

Note that the arguments remain the same when the boundary conditions on the crack are chosen to be the Dirichlet or impedance ones.

2.4. High frequency asymptotics

One is not limited to dealing with exact or numerical solutions to the eigenstates, it is perfectly viable to adopt an asymptotic approach for, say, high frequencies and utilize this within the embedding framework. So although the embedding formulae themselves are valid for arbitrary ratios of wavelength to the size of the scatterer, it is interesting to construct the short wave/ high frequency approximation and apply it to the embedding formulae. For high frequencies an explicit approximation for the auxiliary function/ eigenstate $\hat{\phi}$ is easy to find and, thus, to write down a complete approximate solution for the diffraction problem. By eigenstate we mean the situation where we take the source to lie precisely at the crack edge.

The embedding formula is in the same form as that arising through the geometric theory of diffraction Keller (1962), Achenbach *et al* (1982); this therefore provides another mathematical route to these asymptotic solutions, and provides justification for their efficiency and accuracy even at mid to low frequencies when they might be supposed to be poor (Keller, 1962; Achenbach et al., 1982).

The calculation of the function $\hat{D}(\theta_x,\theta_y;l)$ is a very complicated problem. If the wavelength is much smaller than the characteristic size of the scatterer, the crack near the edge is locally a half-plane, and the remote parts of the crack do not play an important role in diffraction. So, it is natural to approximate $\hat{D}$ by the corresponding function for a half-plane crack. Using the exact solution of the half-plane problem with point source incidence,

$$\hat{D}(\theta_x,\theta_y;l)\approx-\sqrt{-\pi\mathrm{i}}\left(\sqrt{k_0^2-k_\tau^2}-k_\eta\right)^{1/2}e^{\mathrm{i}(k_x x_0+k_y y_0)} \tag{20}$$

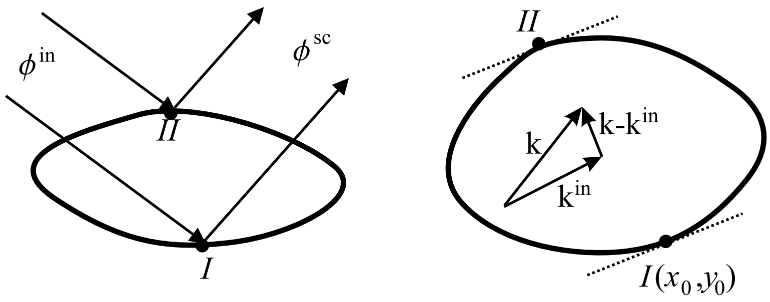

Figure 3. The diffracted rays

where (x_0, y_0) are the coordinates of the point of the edge, k_η and k_τ are the projections of the wavenumber k on the directions normal to Γ and tangential to it, respectively. These values can be calculated using the relations

$$\begin{aligned} k_\eta &= -k_0 \cos\theta_x \sin\Theta + k_0 \cos\theta_y \cos\Theta, \\ k_\tau &= k_0 \cos\theta_x \cos\Theta + k_0 \cos\theta_y \sin\Theta. \end{aligned} \tag{21}$$

Substituting the function (20) into the embedding formula (18) and consider the exponential factor, the integrand oscillates rapidly everywhere except at stationary points of Γ, that is, where the vector dl is orthogonal to the difference of the vectors $\mathbf{k} = (k_0 \cos\theta_x, k_0 \cos\theta_y)$ and $\mathbf{k}^{(\text{in})} = -(k_0 \cos\theta_x^{(\text{in})}, k_0 \cos\theta_y^{(\text{in})})$. These stationary points provide the main terms of the asymptotics of the field.

If we consider the case $\mathbf{k} \neq \mathbf{k}^{(\text{in})}$. There are two stationary points, I and II, at which Γ is orthogonal to $\mathbf{k} - \mathbf{k}^{(\text{in})}$. (there are 2 such points, namely, the points I and II in Fig. 3). At each point we (first, say, for the point I) use the local coordinates η and τ, and calculate the components of the vectors (k_τ, k_η) and $(k_\tau^{(\text{in})}, k_\eta^{(\text{in})})$ using the transformation formulae (21). Note that $k_\tau = k_\tau^{(\text{in})}$.

Using the method of stationary phase we obtain

$$D_I \approx D_I^{\text{e}} \times D_I^{\text{a}} \times D_I^{\text{c}}, \tag{22}$$

where D_I^{e}, D_I^{a} and D_I^{c} are the exponential, angular and curvature factors, respectively:

$$\begin{aligned} D_I^{\text{e}} &= \exp\{-\mathrm{i}k_0[x_0(\cos\theta_x + \cos\theta_x^{(\text{in})}) + y_0(\cos\theta_y + \cos\theta_y^{(\text{in})})]\}, \\ D_I^{\text{a}} &= \frac{(\sqrt{k_0^2 - k_\tau^2} + k_\eta^{(\text{in})})^{1/2}(\sqrt{k_0^2 - k_\tau^2} - k_\eta)^{1/2}}{k_\eta - k_\eta^{(\text{in})}}, \\ D_I^{\text{c}} &= \left(\frac{\pi\mathrm{i}}{(k_\eta - k_\eta^{(\text{in})})d\Theta/dl}\right)^{1/2}. \end{aligned}$$

Analogously, the term corresponding to the stationary point II (and all other stationary points, if there are any others) should be estimated, and the sum over all of them should be taken.

One can see that the expression (22) has the structure peculiar to the classical ray asymptotics of the Geometrical Theory of Diffraction (Keller, 1962).

3. Concluding remarks

Overly singular eigenstates/ auxiliary functions are clearly a useful device for extracting directivities using embedding and allows for a physical interpretation in terms of line source incidence. We have demonstrated that embedding is related to high frequency asymptotic techniques. The approach is also useful in combination with Wiener-Hopf techniques and embedding is clearly applicable to elasticity and surface waves, (Craster and Shanin, 2002). Thus embedding should become a method of choice when solving integral equations in diffraction theory.

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