Embedding formulae in diffraction theory

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Embedding formulae are remarkable as they allow one to decompose scattering problems apparently dependent upon several angular variables (angles of incidence and observation) into those dependent upon fewer angular variables. In terms of facilitating rapid computations across considerable parameter regimes this is a considerable advantage. Our aim is to derive embedding formulae for scattering and diffraction problems in acoustics, electromagnetism, and elasticity.

Here we construct a general approach to formulating and using embedding formulae, we do this using complementary approaches: overly singular states, and a physical interpretation in terms of sources. The crucial point we identify is the form of the auxiliary state used in the reciprocal theorem, this is unphysically singular at the edge and is reminiscent of weight function methods utilized in fracture mechanics. Illustrative implementations of our approach are given using Wiener-Hopf techniques for semi-infinite model problems in both elasticity and acoustics. We also demonstrate our approach using a numerical example from acoustics and we make connections with high frequency asymptotic methods.

Keywords: Embedding, integral equations, acoustics, electromagnetism, elasticity, reciprocity

1. Introduction

In three dimensions the solution to a diffraction problem is usually represented as a function of four angular variables: two of them specify the direction of the wave vector of the incident plane wave illuminating the obstacle, and the other two are the direction of the scattered wave. For two dimensional problems we have two, rather than four, angular variables. The far-field diffraction pattern is a function of these directions. If the diffraction problem is solved numerically then it is a time consuming procedure to perform a parametric study — all of the angular variables must be independently varied, and the numerical routine rerun for each value.

For many practically important cases there exists an elegant mathematical theory, little known and not often utilized, that enables one to reduce the dimension of the problem. The essence of this theory is the following: instead of directly solving the main diffraction problem with the desired plane-wave incidence, one solves a set of different auxiliary problems. For example, if the obstacle is a planar crack in the medium, then the auxiliary problems are associated with the excitation of the field by a point source located asymptotically close to the edge of the crack. We could also interpret these auxiliary solutions as unphysically singular eigensolutions of the problem, in the sense that they no longer have the usual local square root dependence on radial distance for the acoustic potential ($\phi(r,\theta) \sim r^{\frac{1}{2}}$) at the edge, but are instead square root singular there ($\phi(r,\theta) \sim r^{-\frac{1}{2}}$). In this interpretation, the source and the edge have conjoined and the material, exterior to the cracks, is source-free.

The solution of the auxiliary diffraction problem (in three dimensions) depends on only three variables: the position along the crack edge at which the source is located, and the two angles that determine the direction of the scattering. The solution of the original diffraction problem is represented as the integral of the solutions of the auxiliary problems. Such a representation is called an embedding formula. The practical benefits of using an embedding formula are potentially huge; the numerical procedure now no longer relies upon continually resolving the same set of equations, and therefore the numerical effort is reduced dramatically. Moreover, an exact analytical relation is now known for these complicated diffraction problems. This enables us to justify the numerical procedures for some cases.

Embedding formulae have previously been derived for several diffraction problems, these have used a different set of auxiliary problems, or have used theories based explicitly upon integral equations and cover: scattering by a rigid or absorbing strip in acoustics (or the analogous slit problem) by Williams (1982), penny-shaped cracks in elastic solids, Martin & Wickham (1983), and more recently for scattering by thin and thick breakwaters for surface waves (in acoustics these are diffraction gratings) the method has been embraced by Biggs *et al.* (2000); Biggs & Porter (2001, 2002). However, except for Williams (1982) who uses grazing incidence to generate the auxiliary solutions, the derivation of embedding formulae is typically through complicated manipulations of integral equations that can obscure the route to the final structure of the formula.

One purpose of this article is to demonstrate an easy way to derive embedding formulae that has a physical interpretation and can be easily implemented, that is, we use a set of auxiliary solutions that have immediate interpretations. We also establish a general framework for embedding formulae and describe the classes of problem for which embedding formulae can be determined. Here we shall consider incident fields that consist of plane waves and this is important for the success of the embedding technique, at least in the form in which we are presenting it.

We begin with an example demonstrating the basic ideas using a physical approach and valid in three dimensions. We then retreat to two dimensions and use a formalism that connects more closely to weight functions, and use the reciprocal theorem directly, together with a differential operator that we define later. The ideas are demonstrated in the context of scattering by semi-infinite cracks or strips, these are explicitly solvable using Wiener-Hopf techniques; any problem usually approached using Wiener-Hopf or some other analytic method for scattering by a plane wave can be interpreted and solved using embedding. An illustrative numerical implementation is given in section 5, asymptotic methods are also useful and we compare the embedding formulae with high frequency asymptotics. We close with some concluding remarks in section 7.

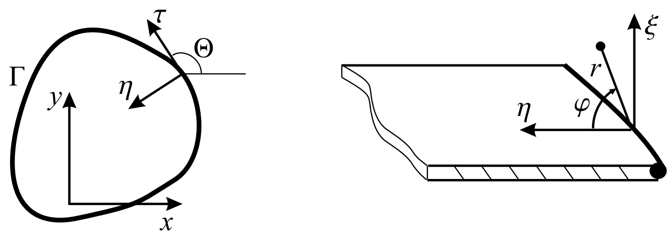

Figure 1. The planar crack geometry and local coordinates.

2. Embedding formulae for a planar crack

First, we illustrate our ideas through a simple three dimensional example, and we derive embedding formulae, valid for acoustic scattering of an incident plane wave, by a planar crack (or cracks).

(*a*) *Problem formulation*

We consider the acoustic potential $\phi(x, y, z)$ that satisfies the Helmholtz equation

$$\nabla^2\phi + k_0^2\phi = 0 \tag{2.1}$$

in the infinite domain, where Cartesian coordinates (x, y, z) are utilized, and the cracks/defects occupy an area S in the (x, y) plane. The edges of the crack/defect are the smooth, not necessarily simply connected, curve Γ. For definiteness, we take the Dirichlet boundary condition $\phi = 0$ to hold on the faces of the defect/crack; the approach remains valid for Neumann or, in electromagnetic theory, for impedance boundary conditions.

The total field ϕ is the sum of an incident field ϕ^{in} and a scattered field ϕ^{sc}. The incident field is assumed to be a plane wave

$$\phi^{\text{in}} = \exp\left[-\mathrm{i}(\mathbf{k}^{\text{in}} \cdot \mathbf{x} + \sqrt{k_0^2 - |\mathbf{k}^{\text{in}}|^2} z)\right]. \tag{2.2}$$

where $\mathbf{k}^{\text{in}} = (k_x^{\text{in}}, k_y^{\text{in}})$ and $\mathbf{x} = (x, y)$.

For physically meaningful solutions we require suitable edge conditions (Meixner's) to be satisfied, this means that the field near the edge of the crack has the asymptotic behaviour

$$\phi \sim K r^{1/2} \sin(\varphi/2), \tag{2.3}$$

as $r \to 0$. Here r is the distance from the edge of the crack/defect, and φ is the angle in the local cylindrical coordinates taken such that φ lies along the crack face on $\varphi = 0_+$ (see Fig. 1).

We utilize uniqueness (Jones 1986), that is, we consider only the scattered field, i.e. $\phi = \phi^{\text{sc}}$ and assume that the Helmholtz equation, the boundary conditions, radiation and edge conditions are all satisfied. Then $\phi = 0$ identically. We assume that the theorem of uniqueness is satisfied by all diffraction problems considered here.

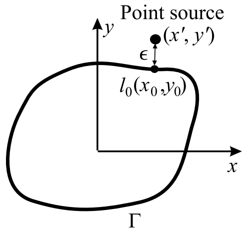

Figure 2. The coordinates for the source close to the edge.

(b) *Auxiliary solutions of the diffraction problems*

We now introduce the auxiliary problems. These are diffraction problems, but now with point source incidence. The scatterer is assumed to have exactly the same geometry, and (homogeneous) boundary conditions, as the scatterer of the initial diffraction problem, and the source is located close to the edge of the crack. It is also assumed that the radiation condition at infinity holds.

Since the Dirichlet condition is taken on the crack faces, and we still assume the physically meaningful, Meixner's, condition to be satisfied at the edge, we cannot simply place the source directly at the edge of the crack. We now consider a limiting procedure, that is, we quantify how near the source is to the edge, and the auxiliary functions are analysed in terms of this limiting procedure.

We introduce a coordinate l along the edge of the crack, and take a point lying in the (x, y) plane a small distance, ϵ, from a position $l_0 = (x_0, y_0, 0)$ lying on the contour Γ. We consider a diffraction problem with a point source, strength $-\pi\epsilon^{-1/2}$, located at this point and define $\hat{\phi}_\epsilon(x, y, z; l_0)$ to be the acoustic potential for this problem. To determine this we solve an inhomogeneous Helmholtz equation:

$$\nabla^2\hat{\phi}_\epsilon + k_0^2\hat{\phi}_\epsilon = -\pi\epsilon^{-1/2}\delta(x - x')\delta(y - y')\delta(z), \tag{2.4}$$

where δ is the delta-function, and the coordinates x', y' are

$$x' = x_0 + \epsilon\sin\Theta, \qquad y' = y_0 - \epsilon\cos\Theta.$$

Here Θ is the angle between the vector dl tangential to Γ and the x-axis (see Fig. 1).

A detailed study shows that for each point (x, y, z) in space, with the exception of the point l_0 on Γ, there exists a finite limit

$$\hat{\phi}(x, y, z; l_0) = \lim_{\epsilon\to 0}\hat{\phi}_\epsilon(x, y, z; l_0). \tag{2.5}$$

The function $\hat{\phi}(x, y, z; l)$ is taken to be the auxiliary solution; we use the hat decoration to distinguish quantities associated with the auxiliary problem.

The auxiliary problem has one important property: it depends on fewer variables than the physical diffraction problem. The function $\phi^{\rm sc}$ depends explicitly on three variables (the spatial coordinates) and implicitly on two variables: the parameters $k_x^{\rm in}$ and $k_y^{\rm in}$ of the incident wave, i.e., the total number of variables is five. The number of arguments for the auxiliary problem $\hat{\phi}$ is four, the two incident parameters are replaced by the position of the source along l. We assume that the function $\hat{\phi}(x, y, z; l)$ is known, and now aim to express the far field of the initial diffraction problem, (2.1,2.2,2.3), in terms of the far field behaviour of the auxiliary function.

To proceed, we shall require the asymptotic behaviour of the auxiliary solution at the edge. Consider the integral

$$\phi^*(x,y,z) = \int_\Gamma \rho(l)\hat{\phi}_\epsilon(x,y,z;l)dl, \tag{2.6}$$

where the sources of the field are concentrated along the contour Γ and have line density $\rho(l)$. We assume this density to be a continuous function having period equal to the length of the contour Γ. Consider a small cylinder with local coordinates τ, η, ξ. Let some intermediate state of the limit process is taken, i.e., the parameter ϵ takes some small, but non-zero value. We assume that the radius of the cylinder is larger than ϵ, but much smaller than k_0^{-1}; in terms of perturbation theory, this is the *inner* problem. The inner solution for ϕ^* is, to leading order,

$$\phi^* \approx -\frac{\rho(l)\epsilon^{-1/2}}{2}\,\mathrm{Re}[\mathrm{Log}(\sqrt{\eta+\mathrm{i}\xi}-\mathrm{i}\sqrt{\epsilon}) - \mathrm{Log}(\sqrt{\eta+\mathrm{i}\xi}+\mathrm{i}\sqrt{\epsilon})]. \tag{2.7}$$

We now take the limit as $\epsilon \to 0$, and introduce local cylindrical coordinates (r,φ) defined as $\eta = r\cos\varphi, \xi = r\sin\varphi$, to obtain the asymptotic behaviour at the edge, that is in terms of perturbation theory, the inner limit of the outer solution. In the outer coordinates ϕ^* behaves as

$$\lim_{\epsilon\to 0}\phi^* = \frac{\rho(l)\sin(\varphi/2)}{r^{1/2}} + O(r^{1/2}), \tag{2.8}$$

when $r \to 0$. Placing the source near the edge of the crack leads to an outer field with asymptotic edge behaviour stronger, that is more singular, than the usual conditions of Meixner ($\phi \sim r^{1/2}$ as $r \to 0$).

There are two equivalent ways to introduce the auxiliary solution. The first one is to introduce a point source near the edge, and use the limiting procedure above. The other is to formally introduce a solution having edge asymptotics stronger than that usually allowed by the physically relevant edge conditions, that is, an overly singular solution. This latter route is a bit cumbersome in the three dimensions, where it is necessary to provide the oversingular behaviour at a single point of the edge.

(*c*) *Directivities of scattered and auxiliary fields*

In the far field, the leading term of the scattered field is written as a modulated spherical wave:

$$\phi^{\mathrm{sc}}(x,y,z) \sim -\frac{e^{\mathrm{i}k_0R}}{2\pi R}D(\theta_x,\theta_y;\theta_x^{\mathrm{in}},\theta_y^{\mathrm{in}}), \tag{2.9}$$

where $R=\sqrt{x^2+y^2+z^2}$, $\theta_x = \arccos(x/R)$, $\theta_y = \arccos(y/R)$, $\theta_x^{\mathrm{in}} = \arccos(k_x^{\mathrm{in}}/k_0)$, $\theta_y^{\mathrm{in}} = \arccos(k_y^{\mathrm{in}}/k_0)$, and D is the directivity of the field.

Analogously, the far field of the auxiliary solution can be represented using its directivity, it is distinguished by the hat decoration:

$$\hat{\phi}(x,y,z;l) \sim -\frac{e^{\mathrm{i}k_0R}}{2\pi R}\hat{D}(\theta_x,\theta_y;l). \tag{2.10}$$

It is useful to note that Green's formula can be used to express the directivity as a Fourier-transform involving the scattered field. In this example it is the transform of the normal derivative of the scattered field, the integral is taken over the crack surface $z = 0_+$.

The embedding formulae, that will be derived below, express the function $D(\theta_x, \theta_y; \theta_x^{\rm in}, \theta_y^{\rm in})$ in terms of $\hat{D}(\theta_x, \theta_y; l)$.

(*d*) *Derivation of the embedding formula*

We are going to derive the embedding formula in three steps: first we apply differential operators to the total field, second we apply the uniqueness theorem, and finally we use the reciprocity principle.

Consider the differential operators defined as

$$\mathbf{H} = (H_x, H_y) = [\nabla + {\rm i}\mathbf{k}^{\rm in}] = \left(\frac{\partial}{\partial x} + {\rm i}k_x^{\rm in}, \frac{\partial}{\partial y} + {\rm i}k_y^{\rm in}\right). \tag{2.11}$$

We now apply either of these operators, for definiteness we apply H_x, to the total field ϕ (the solution to equations (2.1),(2.2),(2.3)). The function

$$\overline{\phi}(x, y, z) = H_x[\phi(x, y, z)] \tag{2.12}$$

has the following properties: it satisfies the Helmholtz equation (2.1), it contains no incoming waves from infinity nor allows growth at infinity (note that $H_x[\phi^{\rm in}] \equiv 0$), and furthermore $\overline{\phi} = 0$ on the crack surfaces. The conditions of the uniqueness theorem are satisfied, except for the edge condition. If the local asymptotic behaviour of the field, ϕ, near the edge is that

$$\phi \sim K(l) r^{\frac{1}{2}} \sin\left(\frac{\varphi}{2}\right) + O(r^{\frac{3}{2}}), \quad \text{then } \overline{\phi} \sim \frac{1}{2} K(l) r^{-\frac{1}{2}} \sin\Theta \sin\left(\frac{\varphi}{2}\right) + O(r^{\frac{1}{2}}). \tag{2.13}$$

Here Θ is the angle between the x-axis and the unit vector dl tangential to the contour Γ (see Figure 1) and r, φ are local polar coordinates at the edge. That is, $\overline{\phi}$ has overly singular behaviour at the edge.

Comparing the asymptotic behaviour at the edge for the function $\overline{\phi}$ in (2.13) with that of the integral of the auxiliary functions, ϕ^* in (2.8), one finds that the combination

$$w(x, y, z) = \overline{\phi}(x, y, z) - \frac{1}{2} \int_\Gamma K(l) \sin\Theta(l)\, \hat{\phi}(x, y, z; l) dl \tag{2.14}$$

obeys the usual Meixner's condition at the edge. Furthermore, this function obeys the Helmholtz equation, the radiation condition, and the Dirichlet boundary condition. Therefore, we apply uniqueness to this combination, and thus $w(x, y, z) \equiv 0$. From equation (2.12) we can now identify the function $\overline{\phi}$ in terms of $\hat{\phi}$ and $K(l)$ as

$$\overline{\phi} = H_x[\phi] = \frac{1}{2} \int_\Gamma K(l) \sin\Theta(l)\, \hat{\phi}(x, y, z; l) dl. \tag{2.15}$$

This is a *weak form* of the embedding formula.

The function $K(l)$ in (2.15) remains unknown, to generate the complete embedding formula we must also express $K(l)$ in terms of $\hat{\phi}(x, y, z; l)$. Instead of having an incident plane wave let us take a point source of unit strength located at a point (X, Y, Z), such that

$$X = R\frac{k_x}{k_0}, \qquad Y = R\frac{k_y}{k_0}, \quad \text{and} \quad Z = \sqrt{R^2 - X^2 - Y^2}.$$

We take the lengthscale R to be much greater than both the size of the scattering region and the wavelength (being more accurate, we assume that the point (X, Y, Z) is located in the far field). The incident field from the source is asymptotically a plane wave having the form (2.2) multiplied by the factor $-(4\pi R)^{-1}e^{\mathrm{i}k_0R}$. To find $K(l)$ we take the observation point, in the (x, y) plane, to be at a small distance ϵ from the point l on the edge contour Γ. We multiply the value of the field at the observation point by $\epsilon^{-1/2}$ and take the simultaneous limits that $R \to \infty$ and $\epsilon \to 0$. The result is $K(l)$ from the formula (2.15) multiplied by $-(4\pi R)^{-1}e^{\mathrm{i}k_0R}$.

We now use the reciprocity principle (Junger & Feit 1986) and interchange the source and observation point in the limit procedure described above, that is, the source is now near the edge, and the observation point is at (X, Y, Z). From the reciprocity principle, the value of the field for this interchanged problem is the same as that of the original problem. The diffraction problem with the point source located near the edge is the auxiliary problem, and the solution under the appropriate limit is $\hat{\phi}(x, y, z; l)$. Hence the function $K(l)$ is

$$K(l) = 4 \lim_{R\to\infty}[Re^{-\mathrm{i}k_0R}\hat{\phi}(X, Y, Z; l)]. \tag{2.16}$$

Using equation (2.10), we obtain that

$$K(l) = -\frac{2}{\pi}\hat{D}(\theta_x^{\mathrm{in}}, \theta_y^{\mathrm{in}}; l). \tag{2.17}$$

That is, the edge behaviour of the physical problem is represented in terms of the far field of the auxiliary solution. We are now in a position where we can write the directivity of the physical problem entirely in terms of that found for the auxiliary problem.

We substitute the relation (2.17) into the embedding formula (2.15) to get:

$$H_x[\phi] = -\frac{1}{\pi}\int_\Gamma \hat{D}(\theta_x^{\mathrm{in}}, \theta_y^{\mathrm{in}}; l)\hat{\phi}(x, y, z; l)\sin\Theta(l)\, dl. \tag{2.18}$$

The left- and right-hand sides of this are now evaluated in the far field, the operator H_x acting on the far-field of ϕ yields a coefficient $\mathrm{i}k_0(\cos\theta_x + \cos_x^{\mathrm{in}})D$. The right hand side yields the directivity $\hat{D}$ from $\hat{\phi}$, after some cancellation one obtains the embedding formula

$$D(\theta_x, \theta_y; \theta_x^{\mathrm{in}}, \theta_y^{\mathrm{in}}) = \frac{\mathrm{i}}{\pi k_0\,(\cos\theta_x + \cos\theta_x^{\mathrm{in}})}\int_\Gamma \hat{D}(\theta_x, \theta_y; l)\hat{D}(\theta_x^{\mathrm{in}}, \theta_y^{\mathrm{in}}; l)\sin\Theta(l)dl. \tag{2.19}$$

So far we have worked entirely with H_x. It is interesting to note that another embedding formula emerges by applying the operator H_y, and repeating the arguments above. Then one obtains the embedding formula

$$D(\theta_x, \theta_y; \theta_x^{\mathrm{in}}, \theta_y^{\mathrm{in}}) =$$

$$-\frac{\mathrm{i}}{\pi k_0\left(\cos\theta_y+\cos\theta_y^{\mathrm{in}}\right)}\int_\Gamma \hat{D}(\theta_x,\theta_y;l)\hat{D}(\theta_x^{\mathrm{in}},\theta_y^{\mathrm{in}};l)\cos\Theta(l)dl. \tag{2.20}$$

The arguments remain the same when the boundary conditions on the crack are chosen to be either Neumann or impedance conditions.

(*e*) *Deriving embedding formulae in two dimensions*

We now consider planar cracks in two dimensions, and we choose the (x,z) plane to be that in which the scatterers are located. That is, we could utilize the formulae of the earlier section where the scatterer now extends into the y direction such that it has infinite length, and no variation in y, and we take a cross section of the scatterer. Now ϕ is $\phi(x,z)$.

The derivation of an embedding formula in two dimensions is closely related to the three dimensional case that we have just presented, but it has some special features that are worth highlighting.

First, the auxiliary solutions should be redefined. Instead of a point source located near the edge, a line source should now be taken. To maintain the asymptotic behaviour

$$\hat{\phi}\sim\ r^{-1/2}\sin\frac{\varphi}{2}+O(r^{1/2}),$$

local to the edge, the strength of the line source is $-\pi\epsilon^{-1/2}$, and is located at a distance ϵ from the edge. As in three dimensions, the limit as $\epsilon\to 0$ should be studied.

Second, the directivity of the scattered field is now associated with a far-field cylindrical wave, and the far field is typically

$$\phi^{\mathrm{sc}}(r,\theta)\sim D(\theta,\theta^{\mathrm{in}})\frac{e^{\mathrm{i}[k_0r-\pi/4]}}{(2\pi k_0 r)^{1/2}},\qquad r^2=x^2+z^2 \tag{2.21}$$

An identical far field occurs for the auxiliary function, although the directivity is then distinguished by the hat decoration.

Third, only the operator H_x can now be applied to the scattered field. The embedding formula then expresses the directivity $D(\theta,\theta^{\mathrm{in}})$ as a (discrete) linear combination of several functions $\hat{D}(\theta)$ rather than as an integral over some contour.

We will investigate several examples of embedding formulae for two dimensional problems.

(*f*) *A weight function interpretation of the auxiliary functions, valid in two dimensions*

It is possible to generate a quite general approach to embedding formulae using ideas based upon weight functions. This then translates directly across to elasticity using the existing formalism available in that literature.

We now use the idea of the so-called weight functions Bueckner 1970, Rice 1989, Burridge 1976) that use overly singular solutions with the reciprocal theorem to deduce singular behaviour (stress intensity factors) for arbitrarily loaded cracks as a weighted integral over the crack faces.

The near edge behaviour of the field is vital to our proposed scheme. In elasticity, the physically relevant situation is to take the local stress behaviour at the tip of the crack to be square root singular with respect to the radial distance from the edge, $\sigma_{ij} \sim K r^{-\frac{1}{2}}$, and this behaviour is characterized by a stress intensity factor K and some angular behaviour; this ensures that the energy at the crack tip/ edge is finite. We can incorporate the idea of a source lying very close to the edge, as used in section 2, and avoid explicitly taking some limiting procedure, by directly considering unphysically singular solutions that have the displacements square root singular at the edge, $\hat{u}_i \sim \hat{K} r^{-\frac{1}{2}}$. All these overly singular states and quantities associated with them, are distinguished by the hat decoration. This overly singular avenue is the usual approach taken when applying weight function ideas in fracture mechanics. Analogous ideas can be utilized in acoustics and electromagnetism where the physically relevant solutions have $\phi \sim r^{1/2}$ and the overly singular solutions have $\hat{\phi} \sim r^{-1/2}$.

In acoustics, for a slit with Neumann boundary conditions on $z = 0$ along $x < 0$, the asymptotic behaviour at the edge of the slit is that as $r \to 0$:

$$\phi(r,\theta) \sim K\ r^{\frac{1}{2}} \sin(\theta/2) \quad \text{and} \quad \hat{\phi}(r,\theta) \sim \hat{K}\ r^{-\frac{1}{2}} \sin(\theta/2), \tag{2.22}$$

with r, θ as polar coordinates based at the edge, ($\theta = 0$ is on the fracture plane ahead of the slit), and $K, \hat{K}$ are coefficients characterizing the edge behaviour. In anti-plane elasticity K is the mode III stress intensity factor.

Both the theories of elasticity (see section 3)(c)) and acoustics have Green's formulae that relate two different states, for acoustics this is just

$$\int_S (\phi^\star \phi_j - \phi \phi_j^\star) n_j dS = 0. \tag{2.23}$$

Here n_j is the outward pointing normal to a source-free domain with surface S. We assume the states are source-free, which they are when S is the domain exterior to the cracks. The starred and unstarred fields are independent states in the body. Later we choose one state to be more singular at one of the crack tips/edge than the physically relevant solution, and it also satisfies zero boundary conditions on each crack; we choose the other state to be the scattered physical field. The last equation can be used instead of the reciprocal principle to link the edge behaviour of the field due to plane wave incidence and the far-field behaviour of the auxiliary function. We find (2.23) more convenient in the two dimensional case.

3. Examples of embedding formulae in two dimensions

(*a*) *Scattering by several parallel strips*

We consider N finite cracks/ strips, $z = z_j, a_j^- < x < a_j^+$ for $j = 1...N$, all concentrated in a compact region and let $\mathcal{L}$ denote the union of the crack lines. We set one state to be an overly singular one, overly singular at $z = z_j, x = a_j^+$, with the coefficient of the overly singular behaviour, $\hat{K}(a_j^+)$, set to be unity. The other state is the scattered field due to an incident plane wave.

We shall require the directivity of this overly singular solution, and we denote this by $\hat{D}(\theta; a_j^+)$; the second argument denoting that it is generated by the overly singular behaviour at $z = z_j, x = a_j^+$. We shall also require the directivity generated by overly singular behaviour introduced at $z = z_j, x = a_j^-$.

For clarity, let us treat an acoustic problem with the cracks having the Neumann condition $\frac{\partial \phi}{\partial z} = 0$ on them and incident field $\phi^{\rm in} = \exp[-{\rm i}k_0(x\cos\theta^{\rm in} + z\sin\theta^{\rm in})]$. Let us apply the reciprocal theorem, then

$$\int_{\mathcal{L}} \phi_z^{\rm in}(\mathbf{x})[\hat{\phi}(\mathbf{x}; a_j^+)]ds(\mathbf{x}) = -\int_{\text{crack tip at } a_j^+} \left(\hat{\phi}_n \phi^{\rm sc} - \phi_n^{\rm sc}\hat{\phi}\right) ndS. \tag{3.1}$$

The integral over $\mathcal{L}$ is over all of the cracks, whilst the overly singular solution extracts only the behaviour at $z = z_j, x = a_j^+$. Using the known edge behaviour, one obtains

$$\pi K(\theta^{\rm in}, a_j^+) = \int_{\mathcal{L}} \phi_z^{\rm in}(\mathbf{x})[\hat{\phi}(\mathbf{x}; a)]ds(\mathbf{x}) \tag{3.2}$$

this latter integral is related to the directivity of the overly singular state via

$$\hat{D}(\theta) = \frac{\rm i}{2}\int_{\mathcal{L}} [\hat{\phi}(\mathbf{x}; a)]\phi_z^{\rm in}(\mathbf{x})ds(\mathbf{x}). \tag{3.3}$$

One could view this as the spectrum of $[\hat{\phi}(\mathbf{x}; a_j^+)]$, thus, and using a similar calculation for an overly singular state based at $x = a_j^-$, equation (3.2) becomes

$$\pi K(\theta^{\rm in}, a_j^\pm) = -2{\rm i}\hat{D}(\theta^{\rm in}; a_j^\pm). \tag{3.4}$$

This illustrates an important and powerful property of overly singular states which is that the edge behaviour of the physical problem, characterized by $K(a_j^\pm)$, can be deduced from the far field of the overly singular states.

As in section 2, we now introduce the differential operator $H_x = \partial/\partial x + {\rm i}k_0\cos\theta^{\rm in}$, and apply this to the physically relevant solution. So we consider $\overline{\phi} = H_x[\phi(\mathbf{x})]$, and this generates an overly singular solution. The local behaviour at each crack tip has $\phi \sim Kr^{1/2}$, and applying H_x to this leads to the overly singular local behaviour of $\overline{\phi}$ at each edge; this is identified as $\overline{\phi} \sim \pm K/2\; r^{-1/2}$, with the signs dependent upon which end of the crack we are considering. By considering the local behaviour of the overly singular solutions that we initially introduced, $\hat{\phi}$ (2.22), and recalling that we specifically chose their coefficients $\hat{K} \equiv 1$ we see that local to each crack tip (positioned at $z = z_j, x = a_j^\mp$) $\overline{\phi} = \pm K(\theta^{\rm in}, a_j^\mp)\hat{\phi}(\mathbf{x}; a_j^\mp)/2$. Finally, we invoke uniqueness (Jones 1986) and deduce that

$$\overline{\phi}(\mathbf{x}) = H_x[\phi(\mathbf{x})] = -\frac{1}{2}\sum_{j=1}^{N}\left[K(\theta^{\rm in}, a_j^+)\hat{\phi}(\mathbf{x}; a_j^+) - K(\theta^{\rm in}, a_j^-)\hat{\phi}(\mathbf{x}; a_j^-)\right], \tag{3.5}$$

that is, it is a weighted function of the individual overly singular states. By analysing the behaviour of this function one easily extracts the directivity, $D(\theta, \theta^{\rm in})$, associated with ϕ, as just

$$D(\theta, \theta^{\rm in}) = \sum_{j=1}^{N} \frac{[\hat{D}(\theta^{\rm in}; a_j^+)\hat{D}(\theta; a_j^+) - \hat{D}(\theta^{\rm in}; a_j^-)\hat{D}(\theta; a_j^-)]}{\pi k_0[\cos\theta + \cos\theta^{\rm in}]}. \tag{3.6}$$

This directivity relies upon the assumed far field cylindrical wave behaviour (2.21) being correct, and in some circumstances, for example the semi-infinite crack of section 4 (a) one should exercise care when $\theta = \pi - \theta^{\rm in}$, and the denominator of (3.6) is zero. The resultant infinite directivity is connected with a non-uniformity in the far field pattern and this then requires transition formulae in the form of Fresnel integral corrections (see Noble 1958). For the analogous situation of $\theta = \pi - \theta^{\rm in}$ in the finite length crack example of section 5 the numerator is also zero, and a finite limit emerges and the directivity retains its validity. Thus our directivity has the same advantages and disadvantages as that deduced in the conventional manner, it is completely equivalent, but written in an alternative way.

The scheme for extracting the directivity for N cracks for any $\theta^{\rm in}$ is to solve $2N$ integral equations for each overly singular state (this is independent of $\theta^{\rm in}$) and then manipulate the resulting directivity; this is a considerable saving numerically.

Often one can utilize underlying symmetries, that is, for a single crack along $z = 0$, $|x| < a$, or any system symmetric about the z axis, $\hat{D}(\theta; -a) = \hat{D}(\pi - \theta; a)$ so only N overly singular solutions are required.

The scheme presented above for generating embedding formulae is very versatile and carries across to cracks in elasticity, as well as cracks beneath wavebearing surfaces, and one can easily alter the boundary conditions along the cracks. For instance, a similar calculation for N cracks with the Dirichlet condition $\phi = 0$ as the boundary condition on each crack gives:

$$D(\theta, \theta^{\rm in}) = \sum_{j=1}^{N} \frac{[\hat{D}(\theta^{\rm in}; a_j^-)\hat{D}(\theta; a_j^-) - \hat{D}(\theta^{\rm in}; a_j^+)\hat{D}(\theta; a_j^+)]}{\pi k_0[\cos\theta + \cos\theta^{\rm in}]}, \tag{3.7}$$

where

$$\hat{D}(\theta) = \frac{-\mathrm{i}}{2} \int_{\mathcal{L}} [\hat{\phi}_z(\mathbf{x}; a)] \phi^{\rm in}(\mathbf{x}) ds(\mathbf{x}),$$

and the angular argument of $\phi^{\rm in}(\mathbf{x})$ is taken to be θ.

One can alter the situation to have some cracks with the Dirichlet and others with the Neumann condition. If the cracks have Dirichlet conditions on the upper surface, and Neumann on the lower surface then one has to alter the edge behaviour. For local behaviour about the edge of a crack lying along $x > 0, z = 0$, one has that $\phi \sim K r^{1/4} \sin(\theta/4)$ and then embedding formulae follow again.

(*b*) *Scattering by several parallel cracks above a wavebearing surface*

Let us leave aside most of these generalizations, and consider N Neumann cracks above a wavebearing surface, the simplest prototype has the boundary condition $\phi_z + \alpha\phi = 0$ on $z = 0$; see figure 3.

We take the incident field to now include a contribution from the wavebearing surface, that is,

$$\phi^{\rm in}(x, z) = e^{-\mathrm{i}k_0 x \cos\theta^{\rm in}} \left(e^{-\mathrm{i}k_0 z \sin\theta^{\rm in}} + \mathcal{R} e^{\mathrm{i}k_0 z \sin\theta^{\rm in}} \right), \quad \mathcal{R} = \frac{\mathrm{i}k_0 \sin\theta^{\rm in} - \alpha}{\mathrm{i}k_0 \sin\theta^{\rm in} + \alpha}. \tag{3.8}$$

The application of the reciprocal theorem leads again to the same representation (3.5), although now the scattered field has the additional behaviour that $\phi^{\rm sc}(x, z) \sim A^{\pm} \exp(-\alpha z \pm \mathrm{i}x\sqrt{\alpha^2 + k_0^2})$ as $x \to \pm\infty$, that is, we have surface waves that propagate along the surface. Their amplitudes also follow from an embedding

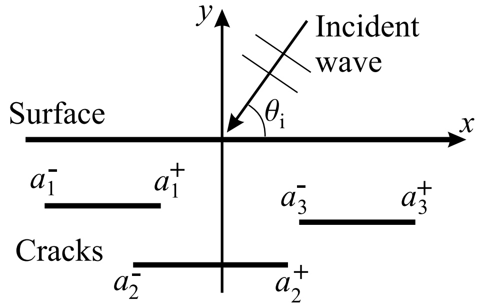

Figure 3. A schematic of a compact array of parallel cracks with incident wave $\theta^{\rm in}$.

formula:

$$A^{\pm}(\theta^{\rm in}) = \sum_{j=1}^{N} \frac{[\hat{D}(\theta^{\rm in}; a_j^+)\hat{A}^{\pm}(a_j^+) - \hat{D}(\theta^{\rm in}; a_j^-)\hat{A}^{\pm}(a_j^-)]}{\pi[\pm\sqrt{\alpha^2+k_0^2} + k_0\cos\theta^{\rm in}]} \tag{3.9}$$

where $\hat{A}^{\pm}(a_j^{\pm})$ are the surface wave amplitudes created by overly singular solutions at $a_j^{\pm}$. The directivities and scattered surface wave amplitudes created by incoming surface waves follow from the formulae we have already deduced where $k_0\cos\theta$ is replaced by $\pm\sqrt{k_0^2+\alpha^2}$, that is, θ is now a complex angle.

(c) *Elastic cracks*

Diffraction caused by elastic waves interacting with cracks is both interesting, and practically important in areas such as non-destructive testing and the evaluation of structures. There are now shear and compressional waves that couple at boundaries, and additionally surface, Rayleigh, or interfacial, Stoneley, waves can occur. Thus we expect to get embedding formulae that incorporate two diffraction coefficients (one for compressional and one for shear waves) and possibly a surface wave amplitude coefficient. For brevity we shall consider parallel, co-planar elastic cracks in an infinite medium, since they are co-planar this means that we can utilize symmetry (or anti-symmetry) about the fracture plane and this can be used to decouple the governing integral equations; the more general case can also be considered, but is somewhat more lengthy. Surface wave behaviour is illustrated by allowing one crack to be semi-infinite, and Rayleigh waves can then propagate along the crack faces.

For isotropic, homogeneous elasticity the governing equations are

$$\sigma_{ij,j} = -\rho\omega^2 u_i \text{ where } \sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \text{ with } \epsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}),$$

time harmonic $e^{-\mathrm{i}\omega t}$ dependence is assumed here and henceforth. The stresses and displacements are σ_{ij} and u_i, and we consider isotropic media (although the methodology carries across to anisotropy); μ and λ are the Lamé constants, and ρ the density. We adopt a Cartesian geometry x, y, z corresponding to $1, 2, 3$. It is often convenient to utilize displacement potentials that follow from $\mathbf{u} = \nabla\chi + \nabla\times\psi\hat{z}$, that is,

$$\nabla^2\chi + k_d^2\chi = 0, \quad \nabla^2\psi + k_s^2\psi = 0$$

where the compressional and shear wavenumbers are given as $k_d^2 = \rho\omega^2/(\lambda+2\mu)$, $k_s^2 = \rho\omega^2/\mu$. Although the potentials satisfy uncoupled equations all realistic problems are posed in terms of, or have boundary conditions involving, the physical stresses or displacements and this couples the shear and compressional components after reflection at interfaces.

The scattered far field for the compressional and shear potentials in elasticity are the same as (2.21), where there are now two directivities $D_d(\theta,\theta^{\rm in})$, $D_s(\theta,\theta^{\rm in})$ for each type of incident wave, that is, we could have incident compressional, shear, or under some circumstances Rayleigh or interfacial waves. There may also be amplitude coefficients related to surface or interfacial waves if these are in the physical problem.

In elasticity the near-edge behaviour is similar to that in acoustics, but with algebraically more complex angular behaviour, for simplicity, let us consider the opening, tensile, mode for a stress-free crack, then on the fracture plane $z=0$ for $x>0$

$$\sigma_{33}(x,0) = \frac{K_1}{(2\pi x)^{\frac{1}{2}}}, \quad \hat{\sigma}_{33}(x,0) = \frac{\hat{K}_1}{(2\pi x^3)^{\frac{1}{2}}} \tag{3.10}$$

and along the crack, $x<0$,

$$u_3(x,0) = -\frac{k_s^2}{\mu(k_d^2-k_s^2)} K_1 \left(\frac{-x}{2\pi}\right)^{\frac{1}{2}}, \quad \hat{u}_3(x,0) = -\frac{k_s^2}{\mu(k_d^2-k_s^2)} \frac{\hat{K}_1}{(-2\pi x)^{\frac{1}{2}}}. \tag{3.11}$$

Note, it is conventional to take the opening and shear crack modes to occur for cracks lying in the (x,y) plane, or choice of axis labels differs and our opening cracks lie in the (x,z) plane. Here K_1 is the mode one, opening mode, stress intensity factor, the full angular behaviours, and the corresponding shear mode formulae, can be found in Atkinson & Craster (1995) and the references therein.

Let us consider the symmetric problem so $\hat{\sigma}_{xz}=0$ along the fracture plane, $\hat{\sigma}_{zz}=0$ on the cracks and $\hat{u}_2$ is unknown on the cracks.

Following the same prescription as for acoustics, and using the reciprocal theorem

$$\int_S (\sigma_{ij}^{\star} u_i - \sigma_{ij} u_i^{\star}) n_j dS = 0,$$

we find, for overly singular behaviour at a_j^+ that

$$\int_{\mathcal{L}} e^{-{\rm i}k_d x\cos\theta} \hat{u}_3(x,0)dx = -\frac{k_s^2 K_1(\theta;a_j^+)}{2\mu(k_d^2-k_s^2)} = \frac{-2{\rm i}k_s^2 \hat{D}_d(\theta;a_j^+)}{2k_d^2\cos^2\theta - k_s^2}, \tag{3.12}$$

$$\int_{\mathcal{L}} e^{-{\rm i}k_s x\cos\theta} \hat{u}_3(x,0)dx = \frac{-{\rm i}\hat{D}_s(\theta;a_j^+)}{\cos\theta\sin\theta}. \tag{3.13}$$

If $\sigma_{33}^{\rm in} = \exp(-{\rm i}k_d\cos\theta^{\rm in} - {\rm i}k_d z\sin\theta^{\rm in})$ then by utilizing the operator $H_x = \partial/\partial x + {\rm i}k_d\cos\theta^{\rm in}$, and uniqueness again, one finds that

$$H_x[\chi(\mathbf{x})] = -\frac{1}{2}\sum_{j=1}^{N}\left[K_1(\theta^{\rm in};a_j^+)\hat{\chi}(\mathbf{x};a_j^+) - K_1(\theta^{\rm in};a_j^-)\hat{\chi}(\mathbf{x};a_j^-)\right]. \tag{3.14}$$

An identical formula follows involving ψ, where ψ replaces χ. Looking at the far-field then gives the embedding formulae

$$D_{p,s}(\theta,\theta^{\rm in}) = \frac{1}{2}\sum_{j=1}^{N}\frac{[K_1(\theta^{\rm in};a_j^-)\hat{D}_{p,s}(\theta;a_j^-) - K_1(\theta^{\rm in};a_j^+)\hat{D}_{p,s}(\theta;a_j^+)]}{{\rm i}[k_{d,s}\cos\theta + k_d\cos\theta^{\rm in}]}, \tag{3.15}$$

with K_1 related to the directivity of the overly singular state via (3.12). If one crack is semi-infinite, say the Nth crack, we ignore terms involving a_N^- and the compression and shear amplitudes of the Rayleigh wave, $\chi(x,0) \sim A_p e^{-ik_r x}$, $\psi(x,0) \sim A_s e^{-ik_r x}$ that propagates along the crack face towards minus infinity follow from

$$A_{p,s}(\theta^{\rm in}) = \frac{1}{2}{\sum_{j=1}^{N}}'\frac{[K_1(\theta^{\rm in};a_j^-)\hat{A}_{p,s}(a_j^-) - K_1(\theta^{\rm in};a_j^+)\hat{A}_{p,s}(a_j^+)]}{{\rm i}[-k_r + k_d\cos\theta^{\rm in}]}. \tag{3.16}$$

Here $\hat{A}(a_j^\pm)$ is the Rayleigh wave amplitude generated by the overly singular state at $x = a_j^\pm$ and k_r is the Rayleigh wavenumber, and the $\sum'$ denotes that we ignore the term involving a_N^-. Clearly, embedding is a general property of diffraction, whether it be in acoustics, elasticity or electromagnetism and it encompasses surface wave propagation.

4. Comparison with exact solutions

(a) *The Sommerfeld problem*

The classical problem of a plane wave incident upon an acoustically hard half plane lying along $-\infty < x < 0$ on $z = 0$, where the incident field is

$$\phi^{\rm in}(x,z) = \exp(-{\rm i}k_0 x\cos\theta^{\rm in} - {\rm i}k_0 z\sin\theta^{\rm in}) \tag{4.1}$$

has been treated by many authors, and is a clear pedagogical example for us to begin with. The full solution details are in, say, Noble (1958), but let us assume that we were in ignorance of the full solution, and that we could only calculate the overly singular solution.

The overly singular solution is determined using half-range Fourier transforms,

$$\hat{\Phi}'_+(\xi) = \int_0^\infty \hat{\phi}_z(x,0)e^{{\rm i}\xi x}dx, \quad \hat{\Phi}_-(\xi) = \int_{-\infty}^0 \hat{\phi}(x,0)e^{{\rm i}\xi x}dx, \tag{4.2}$$

in brief, we use the symmetry of the problem, and consider $z > 0$. The plus (minus) subscripts denote functions analytic in the upper (lower) complex ξ planes, and denote the transforms of quantities that are unknown along $z = 0$ for $x > 0$ ($x < 0$). Fourier transforming the governing equations, and applying the zero boundary conditions $\hat{\phi}_z = 0$ for $x < 0$ and $\hat{\phi} = 0$ for $x > 0$ on $z = 0$, we then get a Wiener–Hopf equation relating the unknown transforms

$$-\gamma_-(\xi)\hat{\Phi}_-(\xi) = \frac{\hat{\Phi}'_+(\xi)}{\gamma_+(\xi)} = -\sqrt{\pi}{\rm i}_+^{-\frac{1}{2}}. \tag{4.3}$$

We use the hat decoration to denote the solutions to the overly singular problem and $i_+^{\frac{1}{2}} = \exp({\rm i}\pi/4)$. Here $\gamma_\pm(\xi) = (\xi \pm k_0)^{\frac{1}{2}}$ (the functions have branch points at $\pm k_0$) and we have used the local behaviour near the edge which is that these solutions are unphysically singular at the edge, $\hat{\phi} \sim r^{-1/2}$, to deduce, via Liouville's theorem, that the right hand side of this functional equation is a constant. The form of this constant has been chosen so that $\hat{\phi}(r,\theta) \sim r^{-1/2} \sin\frac{\theta}{2}$. We also deduce the far field behaviour of the overly singular solution as

$$\hat{\phi}(r,\theta) \sim \hat{D}(\theta)\frac{e^{\mathrm{i}[k_0 r - \pi/4]}}{(2\pi k_0 r)^{\frac{1}{2}}}, \quad \hat{D}(\theta) = k_0 \sin\theta\ \hat{\Phi}_-(-k_0\cos\theta). \tag{4.4}$$

We apply the embedding formula (3.6) for a single crack with $a_j^+ = 0$, and no term involving a_j^-, then

$$D(\theta,\theta^{\mathrm{in}}) = \frac{\hat{D}(\theta)\hat{D}(\theta^{\mathrm{in}})}{(\cos\theta + \cos\theta^{\mathrm{in}})}\frac{1}{\pi k_0} \tag{4.5}$$

which, using (4.4), is the well-known solution (Noble 1958). Also, noting the evident symmetry of the problem, we can apply the Dirichlet embedding formula (3.7) with $a_j^- = 0$ and no term involving a_j^+ and this again recovers (4.5). Thus the directivity for the physical problem, which depends upon two angular variables the angle of incidence and that of the observer, is the product of the directivities of the overly singular problem, each a function of a single variable.

A direct generalization of the above can be used to verify the surface amplitude embedding formula (3.9): let us consider a semi-infinite Neumann strip lying parallel to, and a distance d, above a wavebearing surface, that is, $\phi_z = 0$ on $z = d$, $x < 0$ and $\phi_z + \alpha\phi = 0$ on $z = 0$ $-\infty < x < \infty$. It is convenient to introduce a function, $L(\xi)$,

$$L(\xi) = \frac{\gamma - \alpha}{(\gamma-\alpha) - (\gamma+\alpha)e^{-2\gamma d}} \tag{4.6}$$

which has the property that $L \to 1$ as $|\xi| \to \infty$ and one can split this function into a product of functions, $L = L_+L_-$, analytic in the upper (lower) halves of the complex ξ plane such that $L_+(-\xi) = L_-(\xi)$. The overly singular problem is translated via Fourier transforms to the functional equation

$$-\frac{\gamma_-(\xi)}{2L_-(\xi)}\hat{\Phi}_-(\xi) = \frac{L_+(\xi)}{\gamma_+(\xi)}\hat{\Phi}'_+(\xi) = -\frac{\sqrt{\pi}}{\mathrm{i}_+^{\frac{1}{2}}} \tag{4.7}$$

where now $\hat{\Phi}_-$ is the transform of the unknown jump in $\hat{\phi}$ across $z = 0, x < 0$. This is easily unwrapped to give the directivity and surface wave amplitude (in $x > 0$) as

$$\hat{D}(\theta) = \frac{k_0\sin\theta(\pi\mathrm{i})_+^{\frac{1}{2}}}{L_-(k_0\cos\theta)\gamma_+(k_0\cos\theta)}e^{-\mathrm{i}k_0 d\sin\theta}, \quad \hat{A} = \frac{(\pi/\mathrm{i})_+^{\frac{1}{2}}e^{\alpha d}L_+(\sqrt{k_0^2+\alpha^2})}{\gamma_+(\sqrt{k_0^2+\alpha^2})L'(-\sqrt{k_0^2+\alpha^2})}, \tag{4.8}$$

the prime denoting differentiation with respect to ξ, applying (3.9) gives a surface wave amplitude (in $x > 0$) that is the same as that obtained from treating a subsurface semi-infinite strip with incoming wave (3.8).

(*b*) *Scattering by a half-plane crack in an elastic medium*

We consider waves incident upon a stress free semi-infinite crack lying on $z = 0$, $x < 0$ and we follow our recipe above. For a crack in an unbounded material we can split the problem into symmetric and antisymmetric pieces (Achenbach *et al* 1982) and, for brevity we shall consider the symmetric case:

$$\sigma^{\rm sc}_{xz} = 0 \quad \text{on } x = 0, \quad u_3 = 0 \quad \text{on } x > 0, \tag{4.9}$$

$$\sigma^{\rm sc}_{zz} = -\exp[-{\rm i}k_d x\cos\theta^{\rm in} - {\rm i}k_d z\sin\theta^{\rm in}] \quad \text{on } x < 0. \tag{4.10}$$

Let us deduce the directivities and Rayleigh wave amplitude on the crack faces using only our knowledge of the overly singular state. We define $\hat{T}_+$ as the unknown tensile stress along the fracture plane, and $\hat{U}_-$ as the unknown normal crack displacement, then the following functional equation is deduced:

$$\frac{\gamma_{d+}(\xi)\hat{T}_+(\xi)}{(\xi + k_r)L_+(\xi)} = \frac{2\mu(k_d^2 - k_s^2)(\xi - k_r)L_-(\xi)\hat{U}_-(\xi)}{\gamma_{d-}(\xi)k_s^2} = -{\rm i}_+^{-\frac{1}{2}}\sqrt{2} \tag{4.11}$$

where $\hat{T}_+$ and $\hat{U}_-$ are the half range Fourier transforms of the unknown stress σ_{zz} on the fracture plane ahead of the crack, and the unknown opening displacement on the crack itself, defined analogously to (4.2). The displacement is unphysically singular at the crack tip and the local behaviour there is given by (3.11) with $\hat{K}_1 = 1$.

For equation (4.11) we require the following function:

$$L(\xi) = \frac{(2\xi^2 - k_s^2)^2 - 4\xi^2(\xi^2 - k_s^2)^{\frac{1}{2}}(\xi^2 - k_d^2)^{\frac{1}{2}}}{2(k_d^2 - k_s^2)(\xi^2 - k_r^2)}. \tag{4.12}$$

This function has no zeros in the cut plane and can be split such that $L_\pm(\xi) \to 1$ as $|\xi| \to \infty$, and $L_-(-\xi) = L_+(\xi)$ (see for instance Achenbach *et al* 1982). The wavenumbers k_r are the Rayleigh wavenumbers that are the zeros of the numerator of $L(\xi)$.

The embedding formulae (3.15-3.16) give

$$D_d(\theta, \theta^{\rm in}) = \frac{-K_1(\theta^{\rm in}; 0)\hat{D}_d(\theta)}{2{\rm i}k_d(\cos\theta + \cos\theta^{\rm in})} = \frac{\mu(k_d^2 - k_s^2)(2k_d^2\cos^2\theta - k_s^2)}{2k_s^4 k_d(\cos\theta + \cos\theta^{\rm in})}\hat{U}_-(-k_d\cos\theta)\hat{U}_-(-k_d\cos\theta^{\rm in}),$$

$$D_s(\theta, \theta^{\rm in}) = \frac{-K_1(\theta; 0)\hat{D}_s(\theta)}{2{\rm i}(k_s\cos\theta + k_d\cos\theta^{\rm in})} = \frac{\mu(k_d^2 - k_s^2)\sin\theta\cos\theta}{k_s^2[k_s\cos\theta + k_d\cos\theta^{\rm in}]}\hat{U}_-(-k_s\cos\theta)\hat{U}_-(-k_d\cos\theta^{\rm in})$$

and

$${\rm i}(-k_r + k_d\cos\theta^{\rm in})A_{p,s}(\theta^{\rm in}) = -K_1(\theta^{\rm in}; 0)\hat{A}_{p,s}/2.$$

After some manipulation these can be shown to be identical to the directivities that are deduced by using Wiener-Hopf directly on the physical problem; using the overly singular state is a viable approach for deducing all the features of the scattering problem.

5. Numerical simulations

It is a straightforward matter to solve the standard integral equations in acoustics, elasticity and electromagnetism numerically; typically one utilizes a Chebyshev expansion of the unknown. The Dirichlet and Neumann cases from acoustics are prototypical examples as the same expansion functions are used in many other examples allbeit in more complicated scenarios, say, inclined subsurface cracks in elasticity Van der Hijden & Neerhoff (1984); Craster (1998). Alternatively, one could transform the integral equation to a second-kind one (Porter & Chu 1986), indeed there are several numerical methods one could adopt: boundary integral methods, finite elements, finite differences; the precise numerical method is irrelevant to the embedding formulae themselves. However, it is important to demonstrate that the overly singular states do not introduce new numerical difficulties or instabilities. Here we treat a Dirichlet example from acoustics, that is, a single finite length crack with $\phi = 0$ on $|x| < 1$ on $z = 0$. The standard integral equation for an incoming plane wave $\phi^{\text{in}} = \exp[-\text{i}k_0(x\cos\theta^{\text{in}} + z\sin\theta^{\text{in}})]$ is

$$\exp(-\text{i}k_0 x'\cos\theta^{\text{in}}) = \int_{-1}^{1} \frac{\text{i}}{4\pi}\int_C \left[\frac{\partial\phi^{\text{sc}}(x,0)}{\partial z}\right] \frac{e^{\text{i}k(x-x')}}{\sqrt{k_0^2-k^2}}dk\ dx \qquad (5.1)$$

where C is the real axis, indented below (above) the branch points on the positive (negative) real axis, the scattered field is ϕ^{sc}. The unknown $[\phi_z^{\text{sc}}(x,0)]$ is then expanded in some suitable basis function, see for instance de Hoop (1955) and many subsequent authors. We set

$$[\phi_z^{\text{sc}}(x,0)] = 2\sum_{n=0}^{\infty} a_n \frac{T_n(x)}{(1-x^2)^{\frac{1}{2}}}, \quad \text{and} \quad T_n(x) = \cos(n\cos^{-1}x), \qquad (5.2)$$

where the coefficients a_n are unknown and, notably,

$$\int_{-1}^{1} \frac{T_n(x)}{(1-x^2)^{\frac{1}{2}}} e^{\text{i}kx} = \pi e^{\text{i}\pi n/2} J_n(k).$$

Thus we can transfer the numerical setting from the physical to the spectral, k, domain. We multiply (5.1) by $T_m(x')/(1-{x'}^2)^{\frac{1}{2}}$ and integrate over the crack with respect to x' to obtain a linear system of equations $b_m = \sum_{n=0}^{\infty} K_{mn}a_n$. This system is truncated at some N, typically $O(2k_0)$ and one can check convergence by increasing N. The b_m and K_{mn} are

$$b_m = e^{-\text{i}\pi n/2} J_n(k_0\cos\theta^{\text{in}}), \quad K_{mn} = \frac{i}{2}\int_C e^{\text{i}\pi(n-m)/2}\frac{J_n(k)J_m(k)}{\sqrt{k_0^2-k^2}}dk; \qquad (5.3)$$

this latter integral can be simpified to lie along the positive real axis, and the slow convergence of the resulting integral can be accelerated by extracting known integrals. One extracts the directivity, $D(\theta,\theta^{\text{in}})$, as

$$D(\theta,\theta^{\text{in}}) = -i\sum_{n=0}^{N} a_n \pi e^{-\text{i}\pi n/2} J_n(k_0\cos\theta^{\text{in}}).$$

Let us now consider the overly singular state. The overly singular state introduces a source at $x = 1$, and the integral equation (5.1) then requires some adjustments; it is natural to utilize generalized functions to deal with the singularity at $x = 1$. We define $x_+^n = x^n H(x), x_-^n = (-x)^n H(-x)$ where $H(x)$ is the Heaviside function, and then we use the known edge behaviour at $x = 1$ to deduce the integral equation

$$(x'-1)_+^{-1/2} = -\int_{-1}^{1}\left[\frac{\partial\hat{\phi}(x,0)}{\partial z}\right]\int_C \frac{\mathrm{i}}{4\pi\sqrt{k_0^2-k^2}}e^{\mathrm{i}k(x-x')}dkdx. \qquad (5.4)$$

We pose an expansion of the form

$$\frac{1}{2}\left[\frac{\partial\hat{\phi}(x,0)}{\partial z}\right] = \frac{1}{\sqrt{2}}(1+x)^{-\frac{1}{2}}(1-x)_+^{-3/2} + \sum_{n=0}^{\infty} a_n \frac{T_n(x)}{(1-x^2)^{\frac{1}{2}}}, \qquad (5.5)$$

that is, we explicitly introduce the imposed overly singular edge behaviour in the basis expansion. After manipulating the integrals, utilizing the identities

$$\int_{-1}^{1} e^{\mathrm{i}kx}(1+x)^{-\frac{1}{2}}(x-1)_-^{-3/2}dx = \pi k(J_1(k) - \mathrm{i}J_0(k))$$

and

$$\int_{-1}^{1}(x-1)_+^{-\frac{1}{2}}T_m(x)(1-x)_-^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} = \frac{\pi}{2\sqrt{2}}, \qquad (5.6)$$

we then find that for the overly singular solution b_m in (5.3) is replaced by

$$\hat{b}_m = -\frac{1}{2\sqrt{2}} - \frac{\mathrm{i}}{\sqrt{2}}e^{-\mathrm{i}m\pi/2}\int_C k(J_1(k) - \mathrm{i}J_0(k))\frac{J_m(k)}{(k_0^2-k^2)^{\frac{1}{2}}}dk. \qquad (5.7)$$

Once again, this integral can be manipulated to lie along the positive real axis, and known integrals can be subtracted from it to improve numerical convergence. We then solve a linear system of equations, as above, and extract the directivity $\hat{D}(\theta)$.

Numerically evaluating the directivities for the overly singular state and utilizing them in the embedding formula, and comparing with those directivities evaluated directly, yields identical results (see figure 4). The advantage is that the embedding formula is very much more rapid to evaluate across many θ^{in}.

6. High frequency asymptotics

One is not limited to dealing with exact, or numerical, solutions to the overly singular states, it is perfectly viable to adopt an asymptotic approach for, say, high frequencies and utilize this within the embedding framework. So although the embedding formulae themselves are valid for arbitrary ratios of wavelength to the size of the scatterer, it is also worthwhile constructing the short wave/high frequency approximation and applying it to the embedding formulae. For high frequencies, an explicit approximation for the auxiliary function $\hat{\phi}$ is easy to find and this enables us to write down a complete approximate solution for the diffraction problem.

The embedding formula for the directivity is in the same form as directivities that arise from application of the geometric theory of diffraction (Keller 1962;

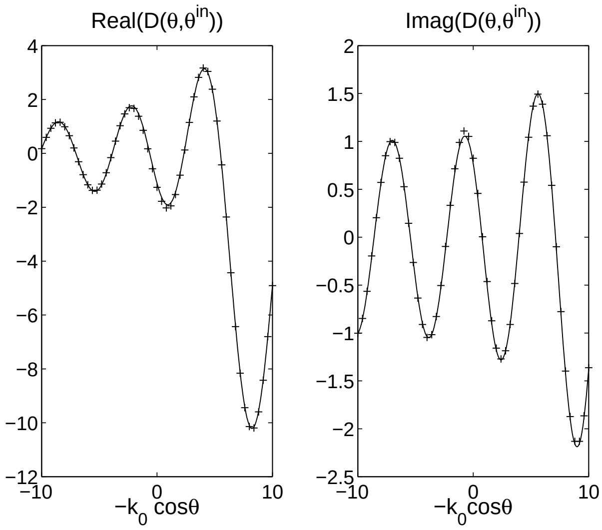

Figure 4. Typical directivities obtained using the direct (solid lines) and embedded overly singular state (+) approach. These are for $k_0 = 10$ and $\theta^{\rm in} = \pi/6$.

Achenbach *et al* 1982). The embedding approach therefore provides another mathematical route to these asymptotic solutions, and provides justification for their efficiency and accuracy even at mid to low frequencies when they might be supposed to be poor.

As an illustration, let us briefly consider the Dirichlet acoustic problem of section 5, we can utilize the semi-infinite overly singular state, analogous to that of section 4(a), to deduce

$$\hat{D}(\theta) \sim (2\pi k_0)^{\frac{1}{2}} \cos\left(\frac{\theta}{2}\right) \ \exp[{\rm i}(\pi/4 - k_0\cos\theta)]. \tag{6.1}$$

This is the directivity due to overly singular behaviour at $z = 0, x = 1$. As one naturally expects from our knowledge of the non-singular physical applications of GTD (Keller 1962) this turns out to be a very good asymptotic approximation to $\hat{D}(\theta)$. In figure 5 we show a numerical comparison for $k_0 = 4$, that is, for quite low frequencies. For higher frequencies the asymptotic and numerical solutions are virtually indistinguishable. This also provides a good verification of our numerical method for extracting the overly singular states. Incidentally, this provides an asymptotic representation for the "stress intensity factor", K of the physical problem via $K(\theta^{\rm in}, 1) = -2{\rm i}\hat{D}(\theta)/\pi$, and that is a useful and not often appreciated property of overly singular states. Inserting this asymptotic directivity, $\hat{D}$, in the embedding formulae yields the well-known result (Keller 1962)

$$D(\theta, \theta^{\rm in}) \sim \frac{-2{\rm i}\left[\cos\left(\frac{\theta+\theta^{\rm in}}{2}\right)\cos(k_0[\cos\theta + \cos\theta^{\rm in}]) - {\rm i}\cos\left(\frac{\theta-\theta^{\rm in}}{2}\right)\sin(k_0[\cos\theta+\cos\theta^{\rm in}])\right]}{\cos\theta + \cos\theta^{\rm in}},$$

and this asymptotic result is virtually indistinguishable from the numerically generated curves in our earlier figure 4. That is partly because the field generated by

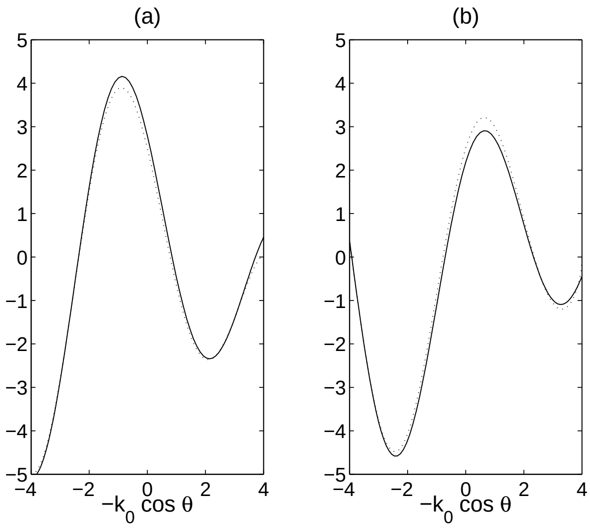

Figure 5. We show in (a) the real, and in (b) the imaginary, parts of $\hat{D}(\theta)$ generated as the numerical solution of the integral equation of section 5 using the directivity deduced using (5.7), solid line, and from (6.1), as dotted line; this is for $k_0 = 4$.

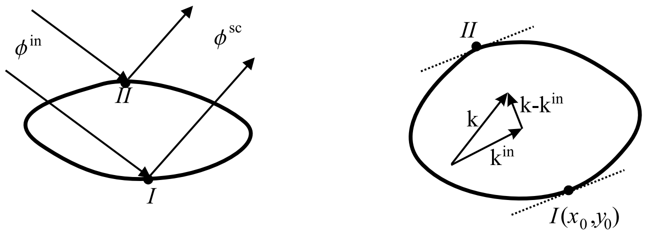

Figure 6. The geometry for the planar case.

the overly singular state satisfies the condition $\phi_z = 0$ on $z = 0$ for $x < -1$ to a high order. The point of this is that one can utilize the overly singular state to deduce valuable details of the physical problem, and an asymptotic representation of the overly singular state can yield very good usable results.

Consider now the general planar three dimensional example from section 2. The calculation of the directivity function $\hat{D}(\theta_x, \theta_y; l)$ is a very complicated problem. Using the approach of section 2, if the wavelength is much smaller than the characteristic size of the scatterer then the behaviour local to the edge dominates, and the more remote parts of the crack do not play an important role in diffraction. It is then natural to approximate $\hat{D}$ by the corresponding function for a half-plane crack. Using the exact solution of the half-plane problem with point source incidence one finds that,

$$\hat{D}(\theta_x, \theta_y; l) \approx -\sqrt{-\pi \mathrm{i}} \left(\sqrt{k_0^2 - k_\tau^2} + k_\eta \right)^{1/2} e^{\mathrm{i}(k_x x_0 + k_y y_0)} \tag{6.2}$$

where (x_0, y_0) are the coordinates of the point of the edge, k_η and k_τ are the projections of the wavenumber k on the directions normal to Γ and tangential to it, respectively. These values can be calculated using the relations

$$k_\eta = -k_0 \cos\theta_x \sin\Theta + k_0 \cos\theta_y \cos\Theta, \quad k_\tau = k_0 \cos\theta_x \cos\Theta + k_0 \cos\theta_y \sin\Theta. \quad (6.3)$$

We then substitute the function (6.2) into the embedding formula (2.19) and consider the exponential factor. The integrand oscillates rapidly everywhere except at stationary points of Γ, that is, where the vector dl is orthogonal to the difference of the vectors $\mathbf{k} = (k_0 \cos\theta_x, k_0 \cos\theta_y)$ and $\mathbf{k}^{\mathrm{in}} = -(k_0 \cos\theta_x^{\mathrm{in}}, k_0 \cos\theta_y^{\mathrm{in}})$. These stationary points provide the main terms in the asymptotic behaviour of the field.

Let us consider the case $\mathbf{k} \neq \mathbf{k}^{\mathrm{in}}$. There are two stationary points, I and II, at which Γ is orthogonal to $\mathbf{k} - \mathbf{k}^{\mathrm{in}}$ (see Fig. 6). At each point we (first, say, for the point I) use the local coordinates η and τ, and calculate the components of the vectors (k_τ, k_η) and $(k_\tau^{\mathrm{in}}, k_\eta^{\mathrm{in}})$ using the transformation formulae (6.3). Note that $k_\tau = k_\tau^{\mathrm{in}}$.

Using the method of stationary phase we obtain

$$D_I \approx D_I^{\mathrm{e}} \times D_I^{\mathrm{a}} \times D_I^{\mathrm{c}}, \quad (6.4)$$

where D_I^{e}, D_I^{a} and D_I^{c} are the exponential, angular and curvature factors, respectively:

$$\begin{aligned}
D_I^{\mathrm{e}} &= \exp\{-\mathrm{i}k_0[x_0(\cos\theta_x + \cos\theta_x^{\mathrm{in}}) + y_0(\cos\theta_y + \cos\theta_y^{\mathrm{in}})]\},\\
D_I^{\mathrm{a}} &= \frac{(\sqrt{k_0^2 - k_\tau^2} - k^{\mathrm{in}}{}_\eta)^{1/2}(\sqrt{k_0^2 - k_\tau^2} + k_\eta)^{1/2}}{k_\eta - k_\eta^{\mathrm{in}}},\\
D_I^{\mathrm{c}} &= \left(\frac{\pi \mathrm{i}}{(k_\eta - k_\eta^{\mathrm{in}})d\Theta/dl}\right)^{1/2}.
\end{aligned}$$

Analogously, the term corresponding to the stationary point II (and all other stationary points, if there are any others) should be estimated, and the sum over all of them should be taken.

The expression (6.4) has the structure of the classical ray asymptotics of the Geometrical Theory of Diffraction (Keller 1962).

7. Concluding remarks

Overly singular states are clearly a useful device for extracting directivities using embedding, and allows for a physical interpretation in terms of a line or point source incidence. We have demonstrated that embedding is related to high frequency asymptotic techniques, and that it is numerically feasible to utilize the overly singular states within conventional numerical schemes. The approach is also useful in combination with Wiener-Hopf techniques and embedding is clearly applicable to elasticity and surface waves. Thus embedding should become a method of choice when solving integral equations in diffraction theory.

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