Wave modes in periodic systems of thin tubes

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The problem of wave propagation in a periodic system of thin tubes is studied. The waves are described by the dispersion diagrams, i.e. by the dependencies of the propagation constants on the temporal frequency. The main question under consideration is how many modes can propagate in a particular system. The paper presents a method of finding the dispersion diagrams, and an inequality for the number of modes, linking the number of modes with the topological properties of the system.

1 Introduction

Consider an infinite periodic set of tubes filled with compressible lossless linear gas (or liquid). The tubes are assumed to be one-dimensional, i.e. only propagation of the piston mode is taken into account. The tubes are connected with each other by small nodes. We assume that for each node the sum of incoming flows is equal to zero. There can be dead ends (i.e. closed nodes), or open nodes. The flow from the close node is equal to zero, and the pressure at an open node is equal to zero. An example of the system is shown in Fig. 1.

Here "periodic" means single-periodic, i.e. the system is infinite along only one dimension (from left to right in the figures). A stationary (time harmonic) problem is studied. The time dependence has form $e^{-i\omega t}$ and is omitted everywhere.

The smallest period of the system will be called a *cell* of the system. The cuts separating a cell will be called the *terminals* of the cell. The terminals are divided into input and output terminals (we assume that the positive direction of mode propagation along the cell is from the input terminals to the output terminals). The choice of the cell shape is not unique.

We assume that each terminal is connected with only one tube within the cell, i.e. the cut points are chosen somewhere in middles of tubes before splitting the system into the cells, rather than cutting across the nodes connecting three or more tubes.

Note that for many cells even the number of input/output terminals is not uniquely defined. For example, in Fig. 1 there is a possibility to split the system into cells having two or three input terminals. However, the number of output terminals should always be equal to the number of input terminals.

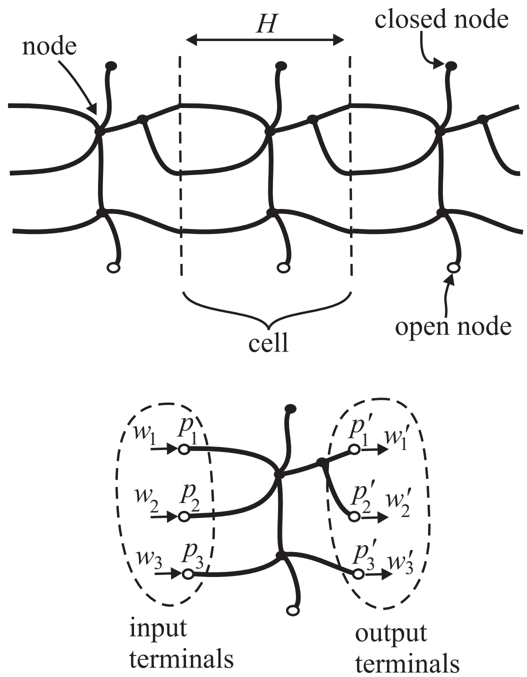

Figure 1: Periodic system of tubes (top) and a single cell (bottom)

As a practical application we have in mind water supply or ventilation systems in tall buildings. A cell in such a system is naturally the set of ducts belonging to a single floor.

Let the input terminals be numbered, and let the output terminals be numbered by the same numbers in a matching order. We associate with the input terminals the pressures p_m, $m = 1, \dots, M$ and the gas flows w_m directed *into* the cell. Accordingly, we associate with the output terminals the pressures p'_m and the flows w'_m directed *from* the cell (see Fig. 1). A wave mode is a solution of the governing equations in the tubes and boundary conditions in the nodes obeying the relations

$$p'_m = \lambda p_m, \quad w'_m = \lambda w_m, \tag{1}$$

where

$$\lambda = \lambda(\omega) = \exp\{i\beta(\omega)\} \tag{2}$$

is common for all terminals. The value $\beta(\omega)$ will be called a propagation constant.

As usually, there can be propagating modes corresponding to real β and evanescent modes corresponding to β having a non-zero complex part. A propagating mode can propagate either in the positive or in the negative direction. The direction can be established by studying the group velocity

$$v_{\rm gr} = H(d\beta/d\omega)^{-1}, \tag{3}$$

where H is the length of the cell. As we shall see below, for each mode having propagation constant β there exists a mode having the propagation constant equal to $-\beta$, i.e. a mode propagating or decaying similarly, but in the opposite direction.

For complicated cells for a fixed time frequency there can exist different types of modes. The main task of the current work is to estimate the number of modes and to compute the dispersion diagrams, i.e. the dependencies $\beta_j(\omega)$ for different modes labelled by the index j.

If there exists a matrix T connecting the pressures and flows at the output with the pressures and flows at the input, i.e.

$$(p'_1, \dots, p'_M, w'_1, \dots, w'_M)^t = T \cdot (p_1, \dots, p_M, w_1, \dots, w_M)^t \tag{4}$$

then the values λ_j would be simply the eigenvalues of $T(\omega)$. From this consideration it is clear that the number of modes cannot be more than $2M$, and the number of modes running in the positive direction cannot be bigger than the number of input terminals. The problem is that in many cases it is impossible to construct such a matrix T. For example, for the cell shown in Fig. 1 such a matrix does not exist. A natural explanation is that instead of a cell with 3 terminals one could select a splitting for which each cell has only two terminals (for such a splitting a matrix T does exist). So one can suppose that the number of modes, propagating in the positive direction is equal to the minimum number of terminals available for some splitting. However this is not true, and below we discuss it in details. We develop another technique applicable to any cell and any splitting.

2 Mathematical formalism

Consider a single tube of the length L (see Fig. 2). Introduce a coordinate l along the tube. The cross-section S of the tube may depend on l. Denote the pressures at the ends of the tube by p_1 and p_2, and the flows from corresponding ends by w_1 and w_2. Construct a matrix C connecting the flows with the pressures (i.e. a DtN mapping for a tube):

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = C \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{5}$$

In the general case this matrix can be obtained by solving the Webster's equation

$$\frac{d^2 p}{dl^2} + \frac{1}{S}\frac{dS}{dl}\frac{dp}{dl} + k^2 p = 0, \quad w(l) = \frac{S(l)}{i\omega\rho_0}\frac{dp}{dx}, \tag{6}$$

where $k = \omega/c_0$, c_0 and ρ_0 are the speed of sound and the density of the medium. The flow in the second relation is taken with respect to the positive l direction. There are two linearly independent solutions of the Webster's equation, so one can find the integration constants from the boundary values (p_1, p_2), and then reconstruct the boundary values (w_1, w_2). Here we write down the form of matrix C for the case of a constant cross-section:

$$C = -\frac{iS}{\rho_0 c_0 \sin(kL)} \begin{pmatrix} \cos(kL) & -1 \\ -1 & \cos(kL) \end{pmatrix}. \tag{7}$$

Also it is easy to find such a matrix for the cross-section depending on l linearly or quadratically.

Figure 2: A single tube

Note that the matrix is symmetrical, i.e.

$$C^1_2 = C^2_1. \tag{8}$$

This relation is true for arbitrary shape of the tube, i.e. for an arbitrary function $S(l)$. The proof follows immediately from the Green's theorem. This fact is important for what follows.

Let us construct an equation for finding the parameter λ. Consider a waveguide mode in a cell described above. Denote the pressures at the nodes of the cell by $p_1, \dots, p_N$. The first M values, i.e. $p_1, \dots, p_M$ are related to the input terminals, and the values $p_{M+1}, \dots, p_N$ are the pressures at the internal nodes of the cell. The pressures at the output terminals are not taken as independent unknowns since they are equal to $\lambda p_1, \dots, \lambda p_M$. The pressures at the open nodes are also not taken into account, since they are known to be zero. Thus, for the cell shown in Fig. 1 $N = 7$.

We assume that no tube connects directly an input and an output terminal. If this happens, one can place an additional node somewhere on the tube.

To determine λ and $p_1, \dots, p_N$ consider the flows going to each node (including the input terminals). The sum of these flows should be zero, and this constitutes N linear equations, which can be written in the matrix form as follows:

$$\begin{pmatrix} W_1 \\ \vdots \\ W_N \end{pmatrix} \equiv G \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} = 0, \qquad (9)$$

where W_n is the sum of all flows into the node n.

The matrix G can be obtained from (7) as follows:

$$G = \sum_\nu G_\nu. \qquad (10)$$

The index ν runs over all tubes. The matrices G_ν are constructed by the following rules. Let C_ν be the matrix (5) found for the tube with index ν. Let this tube connect the nodes n_1 and n_2 which are either internal nodes of the cell or input terminals. Then

$$(G_\nu)^{m_1}_{m_2} = \delta_{m_1,n_1}\delta_{m_2,n_1}(C_\nu)^1_1 + \delta_{m_1,n_1}\delta_{m_2,n_2}(C_\nu)^1_2 +$$
$$\delta_{m_1,n_2}\delta_{m_2,n_1}(C_\nu)^2_1 + \delta_{m_1,n_2}\delta_{m_2,n_2}(C_\nu)^2_2. \qquad (11)$$

Here and below δ is the Kronecker's delta. This formula means that the only non-zero elements of G_ν stand on the crossings of rows and the columns with indices n_1, n_2 and these elements are equal to corresponding elements of the matrix C_ν.

If the tube ν connects the internal node n with an open node, and the open node corresponds to the second node in Fig. 2, then

$$(G_\nu)^{m_1}_{m_2} = \delta_{m_1,n}\delta_{m_2,n}(C_\nu)^1_1. \qquad (12)$$

If the tube connects the output terminal n_1 with the internal node n_2 then the structure of the wave mode should be taken into account. This structure results in the non-diagonal terms' dependence on λ:

$$(G_\nu)^{m_1}_{m_2} = \delta_{m_1,n_1}\delta_{m_2,n_1}(C_\nu)^1_1 + \delta_{m_1,n_1}\delta_{m_2,n_2}\frac{(C_\nu)^1_2}{\lambda} +$$
$$\delta_{m_1,n_2}\delta_{m_2,n_1}\lambda(C_\nu)^2_1 + \delta_{m_1,n_2}\delta_{m_2,n_2}(C_\nu)^2_2 \qquad (13)$$

This dependence reflects the fact that the tube connects nodes belonging to different cells in the chain.

Thus, for the case of tubes having constant cross-sections, the matrix G is defined explicitly. This matrix depends on λ and ω. If for some λ there exits a waveguide mode, then due to (9),

$$\det[G(\lambda, \omega)] = 0. \qquad (14)$$

Being considered as a function of λ, the determinant (14) is a polynomial of λ and λ^{-1}. By construction and by taking into account (8),

$$G(\lambda, \omega) = Y_1(\omega) + \lambda Y_2(\omega) + \lambda^{-1} Y_2^t(\omega), \qquad (15)$$

where Y_1 is a symmetrical matrix. This means that $G(\lambda) = G^t(\lambda^{-1})$, and

$$\det[G(\lambda, \omega)] = g_\omega(\lambda + \lambda^{-1}), \qquad (16)$$

where $g_\omega(z)$ is a polynomial. Each root of this polynomial, if not equal to ± 2, corresponds to a pair of different $\lambda_{1,2}$, and thus to a pair of propagation constants $\beta_{1,2}$, such that $\beta_2 = -\beta_1$. If all roots of g_ω are simple and not equal to ± 2, and the degree of the polynomial in equal to η, then there exist η different wave modes propagating in each direction. In any case, the numbed of modes propagating in the positive (or negative) direction is not bigger than η. The main aim of the rest of the paper is to estimate η.

3 Some examples

Here we study two simple examples, namely a cell with an open node, and a cell with a close node (see Fig. 3). For these cells we construct a matrix G by the algorithm described above and find the dispersion diagram.

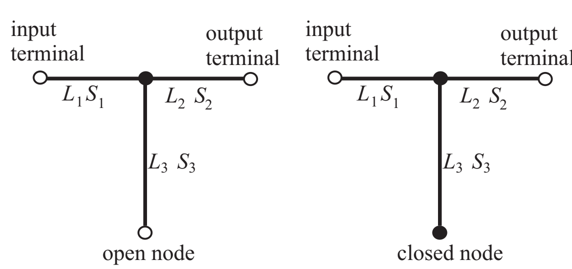

Figure 3: Two simple examples of cells

Consider a cell shown in Fig. 3 (left). According to the procedure outlined above, the G-matrix has form of

$$i\rho_0 c_0 G = S_1 \begin{pmatrix} \cot(kL_1) & -\csc(kL_1) \\ -\csc(kL_1) & \cot(kL_1) \end{pmatrix} + S_2 \begin{pmatrix} \cot(kL_2) & -\lambda^{-1}\csc(kL_2) \\ -\lambda\csc(kL_2) & \cot(kL_2) \end{pmatrix} + S_3 \begin{pmatrix} 0 & 0 \\ 0 & \cot(kL_3) \end{pmatrix}. \tag{17}$$

One can see that the polynomial g_ω defined by (16) is linear for this case. This means that generally there are two modes, one in each direction.

We have computed the propagation constants $\beta(\omega)$ with the following set of dimensionless parameters: $\rho_0 = c_0 = 1$, $S_1 = S_2 = S_3 = 1$, $L_1 = L_2 = 1$, $L_3 = 10$. The real and imaginary parts of β are shown in Fig. 4. Note that the real part takes values from $-\pi$ to π.

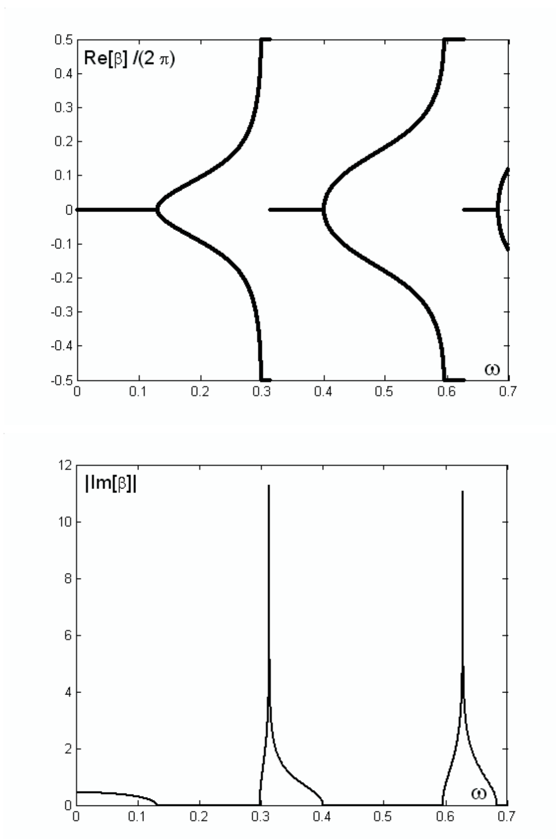

Figure 4: Dispersion diagram for a cell with an open node

The ω axis is split into the segments of transparency (where $\mathrm{Im}[\beta] = 0$) and opacity. The diagram starts with an opacity region. In the transparent regions there are two opposite values of $\mathrm{Re}[\beta]$. The positive $\mathrm{Re}[\beta]$ corresponds to the mode travelling in the positive direction, and the negative one corresponds to the mode travelling in the negative direction (note that positive values of β correspond to positive $d\beta/d\omega$). In the opacity regions there are also two modes, decaying in the positive and negative direction.

The decay parameter tends to infinity at the resonance frequencies of the tube with an open end.

Consider similarly the cell shown in Fig. 3, right. Again, performing the computations according to the method described above,

$$i\rho_0 c_0 G = S_1 \begin{pmatrix} \cot(kL_1) & -\csc(kL_1) & 0 \\ -\csc(kL_1) & \cot(kL_1) & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_2 \begin{pmatrix} \cot(kL_2) & -\lambda^{-1}\csc(kL_2) & 0 \\ -\lambda\csc(kL_2) & \cot(kL_2) & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cot(kL_3) & -\csc(kL_3) \\ 0 & -\csc(kL_3) & \cot(kL_3) \end{pmatrix} \tag{18}$$

The propagation constants computed similarly to the previous case are shown in Fig. 5. The computations were conducted for the same parameters.

The properties of the diagram are mostly the same as discussed above. A major difference is that the first (i.e. low-frequency) zone is transparent. This means that in a closed system there is a possibility of propagation of low-frequency low-dispersive modes, whose properties are very close to usual longitudinal waves, but the velocity is lower than the velocity of sound in the medium.

4 Topological estimation of the number of modes

Let us estimate the degree of the polynomial g_ω, i.e. find the maximum number of modes travelling or decaying in the positive direction. In the examples considered in the previous section this question was quite trivial: since there was only one input terminal per a cell, the maximum number of modes travelling or decaying in the positive direction was one. As we can see from the dispersion diagrams, for almost all temporal frequencies this estimation was exact. However, for bigger amount of input / output terminals the problem becomes non-trivial.

Let us formulate the main result of the section and the paper. Define a *closed loop* in a cell as follows. Take a single cell of the system. Attach the

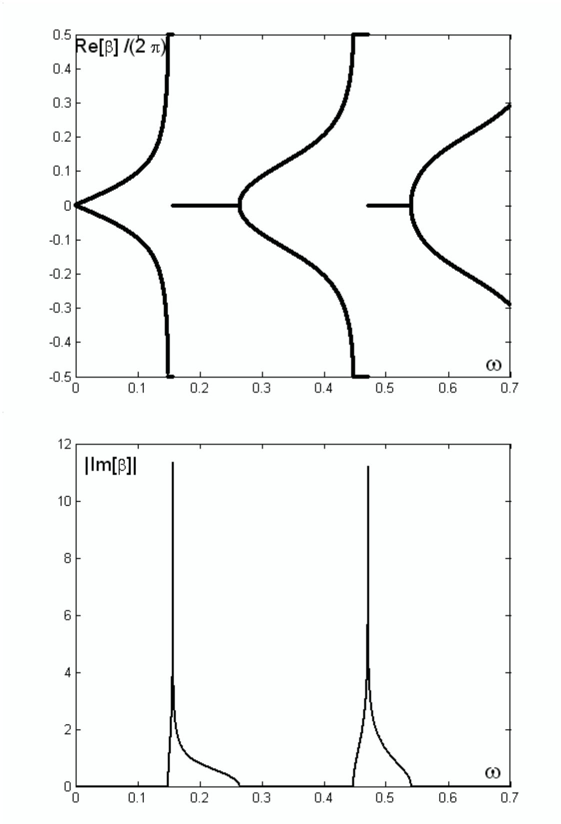

Figure 5: Dispersion diagram for a cell with a closed node

output terminals to corresponding input terminals. The result will be called a *closed cell*. A closed loop is a closed oriented path along the closed cell, passing each node no more than one time. This path should go from node to node along the tubes. The path is not necessarily connected, i.e. it can consist of several closed oriented sub-loops. We allow as a possibility a path consisting of a single node, and a path consisting of two nodes connected by a tube (however both of these possibilities are not important for what follows). The main feature of the closed loop is that a double pass through a node is not allowed.

Introduce the order of the closed loop as follows. At each point where the loop crosses a terminal, assign $+1$ if the loop goes from the output to the input terminal, and -1 if the loop goes in the opposite direction. The order of the loop is the sum of all assigned values. In Fig. 6 we represent several cells and loops in them. Note that in (c) only the points marked by circles are nodes. Simple intersections in the figure correspond to tubes passing one above another without having common points.

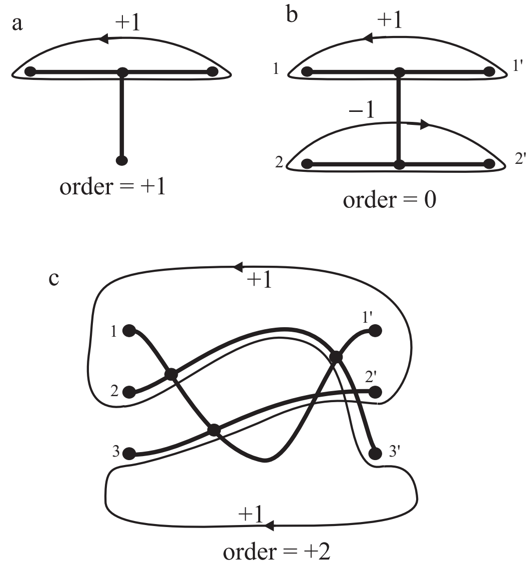

Figure 6: Examples of closed loops on cells

The main result is as follows:

Theorem 1 *The degree of g_ω is not bigger than the maximum possible order of closed loop for a given cell.*

Before proving this statement let us demonstrate that it is useful and non-trivial. Consider the cell shown in Fig. 7. It is not easy to guess the number of modes in such a cell, and the intuitive estimation is 3, since any cross-section of the system crosses at least three tubes. However, the maximum order of loop is 2, and the analysis based on the G matrix confirms this estimation of the number of modes.

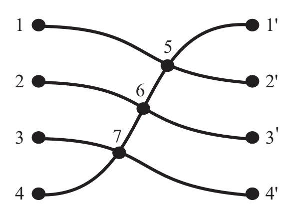

Figure 7: An example of a cell

Let us prove the theorem. First, make a preparatory step. If an inner node is connected both with some input terminal and some output terminal, insert an additional node between the inner node and the input terminal. Obviously, putting a node on an existing tube changes only mathematical expressions, but not the physics and not the answer. For example, for the cell in Fig. 7 it is necessary to insert nodes between 1 and 5, 2 and 6, 3 and 7, 4 and 7. As the result, dimension of matrix G becomes bigger, but each non-zero non-diagonal element corresponds exactly to one tube.

Consider the standard representation of $\det[G]$:

$$\det[G] = \sum_d \varepsilon(d) G^1_{d_1} G^1_{d_2} G^1_{d_3} \dots G^1_{d_N}, \tag{19}$$

where N is the number of nodes (inner nodes plus input terminals), d is a permutation of the numbers $1 \dots N$, $\varepsilon = \pm 1$ depending on the parity of the permutation. Obviously, if λ^η is the maximum power of λ contained in $\det[G]$, then there exists such a permutation d that the product

$$G^1_{d_1} G^1_{d_2} G^1_{d_3} \dots G^1_{d_N} \tag{20}$$

is non-zero and contains λ^η. Note that each factor in (20) can be either a constant with respect to λ, or be proportional to $\lambda^{\pm 1}$. The statement of the theorem follows from the fact:

Each non-zero product having form of (20) corresponds to a closed loop on the cell. The power η is equal to the order of the loop.

To prove the last fact note that $G^m_{d_m} \neq 0$ means that there exists a tube connecting the nodes with the numbers m and d_m. Split the permutation d into a set of non-intersecting cycles. Consider a cycle

$$m_1 \to m_2 \to m_3 \to \dots \to m_h \to m_1. \tag{21}$$

There should exist tubes connecting m_1 with m_2, m_2 with m_3, etc up to m_h with m_1. So, we can define a loop composed of sub-loops connecting the nodes in the order (21). All conditions of the definition of a closed loop will be fulfilled. Each sub-loop is defined on a closed cell, since the input and output terminals are labelled by the same indices.

If a tube connects two inner node or an inner node with the input terminal, then corresponding $G^{m_n}_{m_{n+1}}$ is a constant with respect to λ. If the tube goes from an inner node m to an output terminal n, then corresponding G^m_n is proportional to λ. Note that the same pass adds $+1$ to the order of the loop. If a tube goes from an output terminal n to an inner node m, then corresponding factor G^m_n is proportional to λ^{-1}, and of course this pass adds -1 to the order of the loop.

Thus, the proof of the theorem is completed. Although the theorem states an inequality, a less humble (and more adequate) formulation would be "almost always equal to" instead of "no bigger than".

5 Conclusion

On the base of Dirihlet-to-Neumann approach the number of modes propagation through one-periodical 1D pipe system is linked to the degree of the polynomial g_ω. A topological estimation of the degree of the polynomial g_ω is obtained.

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