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WEINSTEIN'S DIFFRACTION PROBLEM: EMBEDDING FORMULA AND SPECTRAL EQUATION IN PARABOLIC APPROXIMATION*

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Abstract. A short-wave problem of reflection and radiation by an open end of a two-dimensional planar waveguide is studied. The incident mode is assumed to have frequency close to the cut-off. The problem is studied in the parabolic approximation. A recently developed approach based on the embedding formula and the "spectral" equation for the directivity of an edge Green's function is applied to the problem.

Key words. waveguide modes, Fabri–Perrot resonator, parabolic approximation, Wiener–Hopf method

AMS subject classifications. 74J20, 78A50

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1. Introduction.

A classical problem of diffraction theory is revisited, namely the stationary problem of a travelling mode reflection by an open end of a planar waveguide. The width of the waveguide is assumed to be *large* comparatively to the wavelength, and the partial waves of the incident mode are assumed to be propagating almost *normally* to the walls (i.e., the frequency is close to the cut-off frequency of the incident mode).

The short-wave reflection from an outlet of a waveguide seems to be one of the most difficult solved problems in the diffraction theory. Two features responsible for the difficulties can be explained in terms of the ray theory. First, the process is highly reverberant; i.e., plenty of successive diffractions by the edges of the waveguide should be taken into account. Second, for each diffraction act the edge of the scatterer lies on the boundary of the shadow zone of the field generated by the previous diffraction act. Each diffraction order possesses a more complicated structure of the penumbral field. Keller's ray approximation fails in this case, and significantly more sophisticated techniques should be used (e.g., a uniform theory of diffraction).

The problem is described by two dimensionless parameters. The first is the product ka, where k is the wavenumber, and a is the width of the waveguide. The second is the "incidence" angle θ_{in} measured between the direction of propagation of the partial wave in the waveguide and the normal to the waveguide wall. The short-wave consideration studies the case $ka \gg 1$. Here we also impose the limitation of the skew-incidence, i.e., $\theta_{\text{in}} \ll 1$.

A very important parameter of the problem is the combination $\beta_{\text{in}} = \sqrt{ka}\,\theta_{\text{in}}$. Since it is a product of a large and a small number, its magnitude is not specified. A practically important case corresponds to $\beta_{\text{in}} \ll 1$. As is known [1], the reflection coefficient in this case is close to -1. (This should not be confused with a quite opposite case of the long-wave reflection of a piston mode by an open end of a waveguide, for which the coefficient is also close to -1.) The result finds its applications in the

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theory of Fabri–Perrot resonators with planar mirrors. The fact that the reflection coefficient is close to -1 is responsible for a high Q-factor of such a resonator. A small correction for the reflection coefficient enables one to estimate the diffraction losses in the resonator.

Three approaches are known to be applied to this problem. The Wiener–Hopf method [1, 2, 3] was used to obtain an exact solution of the problem. The ray method and its modification, uniform theory of diffraction, was used to analyze the diffracted fields appearing as the result of multiple diffractions [4, 5]. Later, the methods based on the uniform theory of diffraction were used to study more complicated diffraction problems [6, 7, 8]. The Fox–Li integral equation [9] can be used to study numerically the process mainly for the resonator case. It is necessary to note that an exact summation of the diffraction series, performed, e.g., in [6, 7], uses the Wiener–Hopf method as well. Unfortunately, the Wiener–Hopf method can be applied to a rather restricted set of problems. If the problem admits an integral equation formulation, the kernel should have a difference form. The aim of this paper is to develop an alternative approach to the Weinstein problem, namely the one that is not using the Wiener–Hopf formalism. Later, we plan to apply this approach to a wide set of more complicated waveguide problems, namely to reflection by an end of a waveguide with nonparallel walls, a waveguide with a partially closed end, and some others.

In the present paper a novel technique is applied to the Weinstein problem. The method comprises derivation of the embedding formula (see, e.g., [10, 11, 12, 13, 14]) and solving the spectral equation (see, e.g., [15]).

The term "embedding formula" was introduced by Williams in [10]. Initially it was used as a sophisticated technique applicable to integral equations of a certain class. Later, two different methods (see, say, [12] and [14]) made this method less complicated. Now the method is remarkably simple and its results are of considerable importance for diffraction theory. The class of problems to which the method is applicable has also significantly broadened. The spectral equation was first introduced for a strip problem [10] and for a finite diffraction grating problem [16]. Unlike for the Helmholtz strip or grating problem, for the current problem we get the spectral equations that can be easily solved in a closed form.

The structure of the paper is as follows. In section 2 the initial problem is formulated.

In section 3 the method of reflections is applied, and the waveguide problem is reformulated as a problem of wave propagation on a branched surface. The interior of the waveguide is expanded onto a half-plane, and the waves reflected by the walls of the waveguide are eliminated. As the price paid for this, an infinite number of branch points emerge, corresponding to the edges of the waveguide and their reflections.

In section 4 a parabolic equation is formulated instead of the Helmholtz equation. This approximation simplifies the problem considerably, since the details of edge diffraction are coarsened. Namely, Keller's edge diffraction is omitted, and only Fresnel's diffraction is left.

In section 5 the edge Green's function is introduced for the branched surface. This function is the field on the surface that is generated by a pair of point sources located near one of the branch points (i.e., it is a solution of corresponding inhomogeneous parabolic equation obeying the radiation condition). Directivities $S^{\mathrm{i}}(\theta)$ and $S^{n}(\theta)$ of the edge Green's function are also introduced.

In section 6 the embedding formula is derived. For this, a differential operator H is applied to the field u. The operator has the following properties: it preserves the parabolic equation on the branched surface, and it nullifies the incident wave.

However, the operator leads to the appearance of the singularities at the branch points. These singularities can be interpreted as the sources for the field $H[u]$. Due to uniqueness, $H[u]$ can be written as a linear combination of the edge Green's functions with the sources located at the branch points. A reciprocity principle is applied to find the source amplitudes for $H[u]$. As the result, we get an embedding formula expressing the reflection coefficient and the radiation directivity in terms of the directivities S^{i} and S^n related to the edge Green's function.

In section 7 we derive spectral equations for S^{i} and S^n. For this, an operator K, which is an analogue of differentiation with respect to the angular coordinate, is applied to the edge Green's function v. The argument is quite close to the derivation of the embedding formula: this operator leads to the appearance of new singularities, which can be treated as sources of the field $K[v]$.

In section 8 the spectral equations are solved in terms of the series over the values of the Fresnel integral.

In section 9 the solution is analyzed and compared with the Weinstein's one.

2. Problem formulation. Consider a two-dimensional acoustic stationary problem. The time dependence has the form $e^{-i\omega t}$ and is omitted everywhere. The wavelength of a plane wave is equal to $k = \omega/c$ (here c is the phase velocity of waves). Study a planar waveguide formed by two half-lines $x = 0$, $y > 0$ and $x = a$, $y > 0$ (see Figure 2.1). The Neumann boundary conditions are fulfilled on the faces of the walls; i.e., the normal derivative of the field on the walls is equal to zero.

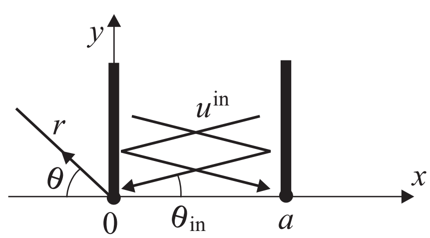

FIG. 2.1. *Geometry of the waveguide.*

We are solving a stationary "acoustical" diffraction problem; i.e., we assume that the Helmholtz equation is valid in the medium:

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + k^2\right)\tilde{u} = 0, \tag{2.1}$$

where the field $\tilde{u}(x, y)$ stands for the solution of the "exact" equation (as opposed to the parabolic equation introduced below). All values related to the exact formulation have tilde decoration.

For correct diffraction problem formulation we have to pose edge and radiation conditions as well. However, since we are going to continue the study in the parabolic approximation, the details (being quite standard) are not important now.

We are studying diffraction of a waveguide mode travelling in the negative direction along the y-axis. This problem involves finding the amplitudes of the waveguide modes travelling backwards and the directivity of the wave radiated from the end of the waveguide into the open space.

Define the incidence angle θ_{in} as follows. The incident waveguide mode is assumed to be composed of two partial plane waves of unit amplitude. The incidence angle is measured between the direction of propagation of one of the partial waves and the x-axis (see Figure 2.1). Formally, the incident wave is as follows:

$$\tilde{u}_{\rm in} = 2\cos(kx\cos\theta_{\rm in})\exp\{-iky\sin\theta_{\rm in}\}. \tag{2.2}$$

Factor 2 indicates that each of the two partial waves has unit amplitude. The angle $\theta_{\rm in}$ belongs to the set $\{\theta_n\}$ describing the waveguide modes. This set is defined by the equation

$$\exp\{2ika\cos\theta_n\} = 1. \tag{2.3}$$

We assume that the angle of incidence $\theta_{\rm in}$ is small.

3. Reformulation of the diffraction problem using a branched surface. Apply the reflection principle to the problem. Cut the physical plane, on which $\tilde{u}$ is defined, along the walls of the waveguide. Obviously, the field is discontinuous on the cuts. Reflect the physical plane with respect to one of the waveguide walls and attach the shores of the cuts of the physical plane to those of the reflected copy. As a result, the field becomes smoothly continued through the cuts, and a branch point emerge instead of the scatterer. However, the resulting surface again contains two cuts. Attach copies of the physical plane to each of them, and repeat this process many times. As a result, obtain the surface with branch points of second order located at the points $x = ma$, $y = 0$, $m \in \mathbb{Z}$. A scheme of the surface obtained as the result of this procedure is shown in Figure 3.1.

The next step is to transform the representation of the branched surface. Take the same surface and make the cuts going from the branch points along the lines $x = ma$, $y < 0$. The surface will be decomposed into a single plane having an infinite number of cuts and a series of planes having a single cut each (see Figure 3.2).

Our consideration will be held on the surface decomposed as shown in Figure 3.2. Denote the field on the upper sheet (with an infinite number of cuts) by $\tilde{u}^{\rm i}$, and the fields on the lower sheets by $\tilde{u}^n$, where n is the number of the sheet. The sheet having number n is assumed to be connected with the upper sheet by the branch point located at $x = an$, $y = 0$.

The upper sheet (in the figure) corresponds to the space inside the waveguide ($y > 0$) and inside its continuation ($y < 0$). The lower sheets correspond to the space outside the waveguide and its continuation, i.e., to $x < 0$ and $x > a$.

The formulation on the branched surface is equivalent to the initial formulation with the waveguide, provided that the incident field chosen for the branched surface is obtained from the initial field by corresponding reflections. This means that we should put two incident plane waves on the upper sheet. Here, however, we prefer to study a single plane wave as the incident wave. Namely, let the incident wave be a plane wave coming along the upper sheet from positive infinity:

$$\tilde{u}^{\rm i}_{\rm in} = \exp\{ik(x\cos\theta_{\rm in} - y\sin\theta_{\rm in})\}, \tag{3.1}$$

where $\theta_{\rm in}$ is small positive. There are no incident components on the lower sheets. Formally, a complete solution of the initial problem can be obtained by adding a mirror reflection with respect to the y-axis.

We are looking for the reflection coefficient $\tilde{R}(\theta_n;\theta_{\rm in})$ and the diffraction coefficient $\tilde{D}(\theta;\theta_{\rm in})$. The first value is defined as follows. The field on the upper sheet for $y > 0$ can be represented as a series:

$$\tilde{u}^{\rm i} = \tilde{u}^{\rm i}_{\rm in} + \sum_n \tilde{R}_n(\theta_n;\theta_{\rm in})\exp\{ik(x\cos\theta_n + y\sin\theta_n)\}, \tag{3.2}$$

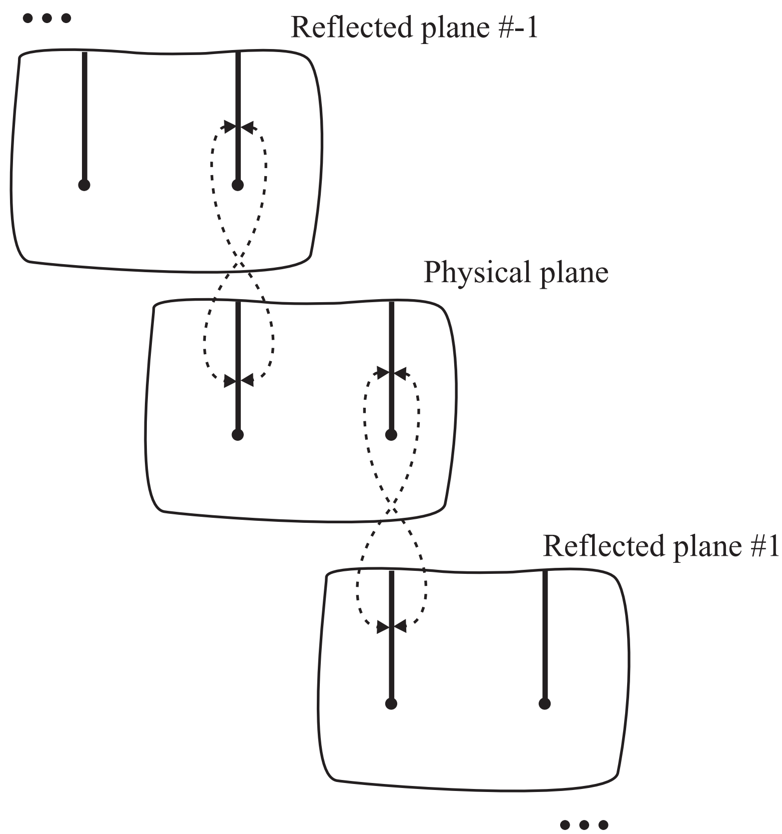

FIG. 3.1. *Branched surface.*

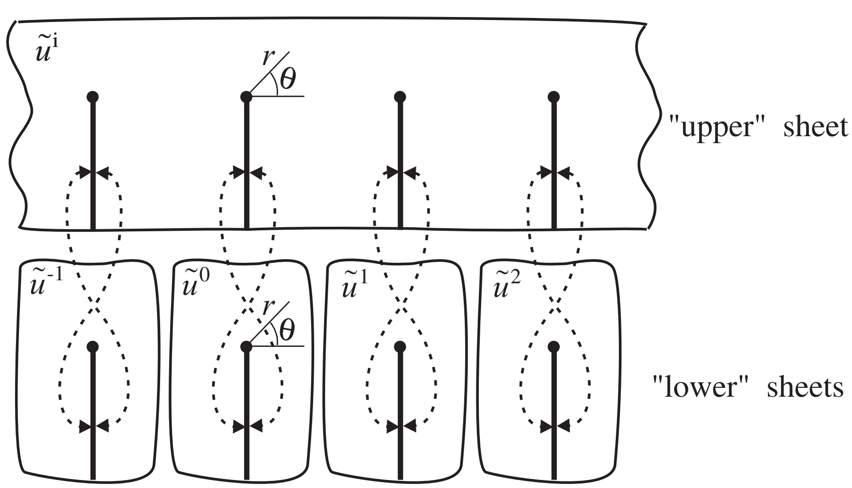

FIG. 3.2. *An alternative representation of the branched surface.*

where θ_n are not necessarily real. This relation can be treated as the definition of the coefficients $\tilde{R}(\theta_n; \theta_{\mathrm{in}})$. The coefficients $\tilde{R}_{m,n} = \tilde{R}(\theta_n; \theta_m)$ describe cross-scattering of the modes by the open end of the waveguide.

Consider the field on the lower sheet having index 0. The far field of $\tilde{u}^0$ can be represented as a modulated field of a point source:

$$\tilde{u}^0(\theta, r) = \sqrt{\frac{k}{2\pi r}} \exp\left\{ikr - \frac{i\pi}{4}\right\} \tilde{D}(\theta;\theta_{\rm in}) + O(e^{ikr} r^{-3/2}), \tag{3.3}$$

where r is the distance between the origin and the observation point. This relation is the definition for $\tilde{D}$. According to the position of the cut, angle θ in (3.3) takes values from the segment $(-\pi/2, 3\pi/2)$. The directivity $\tilde{D}^{\rm t}$ of the field radiated by the open waveguide in the physical formulation can be obtained from $\tilde{D}$ by adding a mirror reflection:

$$\tilde{D}^{\rm t}(\theta, \theta_{\rm in}) = \tilde{D}(\theta, \theta_{\rm in}) + \tilde{D}(\pi - \theta, \theta_{\rm in}). \tag{3.4}$$

The directivity $\tilde{D}^{\rm t}$ is studied only for $-\pi/2 < \theta < \pi/2$. We assume that backward scattering on the branched surface is negligibly small comparatively to the scattering in the forward direction, so $\tilde{D}^{\rm t} \approx \tilde{D}$.

Below we are finding approximations to the values $\tilde{R}$ and $\tilde{D}$.

4. Parabolic equation on the branched surface. We assume that the incident angle $\theta_{\rm in}$ is small. Moreover, we are interested in the reflection and diffraction coefficients for small scattering angles θ. In other words, the x-axis plays the role of "optical axis" of the system, and we are studying only paraxial processes. Note that we do not choose the axis of the waveguide, i.e., the y-axis, as the optical axis of the system. The cause is that the most interesting case is the scattering of a mode close to the cut-off state. Partial waves for such a mode travel almost normally to the walls of the waveguide.

In this paraxial case the problem can be significantly simplified by using the parabolic approximation. The idea of the parabolic equation is to look for the solution $\tilde{u}$ of (2.1) in the form

$$\tilde{u} = e^{ikx} u(x, y), \tag{4.1}$$

where dependence of u on x is slow comparatively to that of the exponential term. The expression (4.1) is substituted into (2.1), and the term containing the second derivative of u on x is omitted. As a result, we get the parabolic equation

$$\left(\frac{\partial^2}{\partial y^2} + 2ki\frac{\partial}{\partial x}\right) u = 0. \tag{4.2}$$

A detailed review of the parabolic approximation can be found in [17].

Let us make here a rough estimation of the accuracy of our assumption. The main error of the parabolic approximation is caused by broadening of the angular spectrum as a result of diffraction by the edges. The amplitude of wave diffracted to large angles is proportional to the area of a screen quite close to the wedge edge, namely about a wavelength from the edge, i.e., $\sim k^{-1}$. At the same time, the area responsible for the penumbral phenomena, which is of great importance in this case and which is correctly described by the parabolic equation, is about the size of the first Fresnel zone at the distance a. The size of this area is $\sim \sqrt{a/k}$. Thus, a ratio describing the accuracy of the parabolic approximation is $(ka)^{-1/2}$. This parameter is small by the problem formulation.

The incident wave has the following form:

$$u_{\mathrm{in}} = \exp\left\{-\frac{ik\theta_{\mathrm{in}}^2}{2}x - ik\theta_{\mathrm{in}}y\right\}. \tag{4.3}$$

The expression (4.3) can be obtained by using series approximations of the sine and cosine functions in (3.1).

Corresponding approximation of (2.3) is given by

$$\theta_n = \sqrt{2}\left(1 - \frac{\pi n}{ka}\right)^{1/2}, \quad n = 0, 1, 2 \ldots. \tag{4.4}$$

Since θ_{in} is small, mode indices n of the incident/reflected waves should be close to ka/π. Note that θ_n are not necessarily real. Positive imaginary θ_n correspond to inhomogeneous waves decaying in the y-direction.

A field on the branched surface can be described by boundary conditions on the cuts, i.e., on the lines $x = an$, $y < 0$:

$$u^{\mathrm{i}}(an+0, y) = u^n(an-0, y)\,\Xi(-y) + u^{\mathrm{i}}(an-0, y)\,\Xi(y), \tag{4.5}$$

$$u^n(an+0, y) = u^{\mathrm{i}}(an-0, y)\,\Xi(-y) + u^n(an-0, y)\,\Xi(y), \tag{4.6}$$

where $\Xi(y)$ is the Heaviside function

$$\Xi(y) = \begin{cases} 1, & y > 0, \\ 0, & y < 0. \end{cases}$$

Note that for the parabolic equation the waves can travel only from left to right, so the boundary conditions are remarkably simple.

The reflection coefficient $\tilde{R}$ and diffraction coefficient $\tilde{D}$ are approximated by the values R and D obtained from the parabolic problem. The expressions (3.2) and (3.3) are substituted by

$$u^{\mathrm{i}} = u^{\mathrm{i}}_{\mathrm{in}} + \sum_n R(\theta_n; \theta_{\mathrm{in}}) \exp\left\{-\frac{ik\theta_n^2}{2}x + ik\theta_n y\right\} \tag{4.7}$$

and

$$u^0(x, y) = g(x, y)(D(\theta; \theta_{\mathrm{in}}) + O(1/x)), \qquad \theta = y/x, \tag{4.8}$$

where $g(x, y)$ is the Green's function of an entire plane:

$$g(x, y) = \sqrt{\frac{k}{2\pi x}} \exp\left\{\frac{iky^2}{2x} - \frac{i\pi}{4}\right\}. \tag{4.9}$$

The angle θ in the parabolic approximation is the tangent of that in the exact formulation. Since small angles are considered, there is no difference between the tangent and the angle itself.

5. Edge Green's function on the branched surface. Consider the branched surface introduced above and parabolic equation (4.2) on it. Define a field $v(x, y)$ on this surface generated by a special "dipole" source located near the origin. The source is constructed as follows. There are two point sources having strength equal to

$+1$ and -1. The first one is located at the point $(+0,0)$ on the upper sheet, and the second source is located at the point $(+0,0)$ on the lower sheet with index 0. There are no other sources on the surface.

The components v^{i} and v^0 of v obey the following inhomogeneous equations:

$$\left(\frac{1}{2ik}\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial x}\right)v^{\mathrm{i}}=\delta(x-0)\delta(y), \tag{5.1}$$

$$\left(\frac{1}{2ik}\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial x}\right)v^{0}=-\delta(x-0)\delta(y). \tag{5.2}$$

Here δ is the Dirac delta-function. Note that the field v^{i} is rather sophisticated. It exists on the upper sheet and has nonzero continuations to all lower sheets with indices greater than 0. Conversely, the field v^0 is trivial, and it exists only on the lower sheet with index 0.

Function $v(x,y)$ composed of the components v^{i} and v^n is the *edge Green's function* of the problem. The reflection coefficient R and the directivity D will be expressed in terms of this function.

It is easy to write an explicit expression for the edge Green's function. Obviously, $v(x,y)\equiv 0$ for all $x<0$. For $0<x<a$

$$v^{\mathrm{i}}(x,y)=g(x,y),$$

where g is the Green's function of a plane introduced in (4.9). Moreover, for all $x>0$

$$v^0(x,y)=-g(x,y).$$

Consider the upper sheet, i.e., the component v^{i}. Let

$$x=an+\Delta x,\quad n>0,\quad 0<\Delta x\le a.$$

Then the explicit formula is as follows:

$$\begin{aligned} &v^{\mathrm{i}}(x,y) \\ &=\int_0^\infty\dots\int_0^\infty g(a,y_1)\,g(a,y_2-y_1)\dots g(a,y_n-y_{n-1})\,g(\Delta x,y-y_n)\,dy_1\dots dy_n. \end{aligned}\tag{5.3}$$

All integrations are taken along the positive half-axis.

Now consider the field on one of the lower sheets, i.e., the component v^n. Note that $v^n(x,y)\equiv 0$ for $n<0$ or $x<an$, so consider $n>0$ and $x>an$. Let

$$x=an+\Delta x,\quad n>0,\quad \Delta x>0.$$

Then

$$\begin{aligned} &v^{\mathrm{i}}(x,y) \\ &=\int_{-\infty}^0\int_0^\infty\dots\int_0^\infty g(a,y_1)\,g(a,y_2-y_1)\dots g(a,y_n-y_{n-1})\,g(\Delta x,y-y_n)\,dy_1\dots dy_n; \end{aligned}\tag{5.4}$$

i.e., all the integrals over $y_1\dots y_{n-1}$ are taken along the half-line $(0,\infty)$, and the integral over y_n is taken along $(-\infty,0)$.

Introduce the directivities $S^{\mathrm{i}}(\theta)$, $S^n(\theta)$ of the edge Green's function. The value S^{i} is related to the upper sheet, and the values S^n are related to the lower sheets. Obviously, the far field of v should have the form of a modulated cylindrical wave with the directivity as a modulation coefficient.

For the upper sheet, large positive x, and positive θ let

$$v^{\mathrm{i}}(x, \theta x) = g(x, \theta x) S^{\mathrm{i}}(\theta) + O(x^{-3/2}). \tag{5.5}$$

For the lower sheet number n and for large positive $(x - an)$ let

$$v^n(x, \theta(x - an)) = g(x - an, \theta(x - an)) S^n(\theta) + O((x - an)^{-3/2}). \tag{5.6}$$

In (5.5) θ stands for y/x, and in (5.6) $\theta = y/(x - an)$. Note that the center of the polar system for v^n is chosen at the point $(an, 0)$.

Since the set of the scatterers is infinite, we can expect that the asymptotic expressions (5.5) will be uniform only for $\theta > \epsilon$ for any small positive ϵ. For our needs, however, it will be enough to choose ϵ such that $\epsilon \ll \theta_{\mathrm{in}}$.

6. Embedding formula. Consider the problem with a plane wave incidence. Apply the *embedding operator* to the total field u. The embedding operator has the form

$$H[u] = \left(\frac{\partial}{\partial y} + ik\theta_{\mathrm{in}} \right) u. \tag{6.1}$$

Consider the function $w(x, y) = H[u]$. This function has the following properties.

— It obeys the parabolic equation everywhere except the branch points $(an, 0)$. This property follows from the fact that the parabolic equation operator

$$L = \frac{1}{2ik} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x}$$

commutes with H. Thus, we can treat w as a wave field.

— The field w contains no wave components coming from infinity. This is because the only component of u that comes from infinity, namely $u^{\mathrm{i}}_{\mathrm{in}}$, is nullified by the operator H.

Consider the behavior of w near the branch point $(0, 0)$. Let

$$u^{\mathrm{i}}(-0, 0) = C, \tag{6.2}$$

where C is an unknown value depending on θ_{in}. Note that $u^0(-0, y) \equiv 0$. By applying the operator H to the boundary conditions (4.5) and (4.6), we obtain

$$w^{\mathrm{i}}(+0, y) = C\delta(y) + w^{\mathrm{i}}(-0, y)\, \Xi(y), \tag{6.3}$$

$$w^0(+0, y) = -C\delta(y) + w^{\mathrm{i}}(-0, y)\, \Xi(-y). \tag{6.4}$$

Substituting (6.3) and (6.4) into the parabolic equation we conclude that the delta-functions can be treated as sources of the field; i.e., in the vicinity of the point $(0, 0)$ the components of w obey the following equations:

$$L[w^{\mathrm{i}}] = C\delta(x - 0)\delta(y), \qquad L[w^0] = -C\delta(x - 0)\delta(y).$$

Using the Floquet property we can derive similar relations for vicinities of all other branch points. Namely, in the vicinity of the point $(an, 0)$ the following equations are valid:

$$L[w^{\mathrm{i}}] = C \exp\{-iank\theta_{\mathrm{in}}^2/2\}\delta(x - an - 0)\delta(y), \tag{6.5}$$

$$L[w^n] = -C \exp\{-iank\theta_{\mathrm{in}}^2/2\}\delta(x - an - 0)\delta(y). \tag{6.6}$$

The sources at the branch points have the same dipole structure as that of the edge Green's functions introduced above.

Thus, w is a solution of the parabolic equation with known sources having no components coming from infinity (i.e., w obeys a radiation condition). *By applying the uniqueness theorem*, we conclude that the field is a linear combination of the fields generated by point sources, i.e., a linear combination of the edge Green's functions:

$$w^{\mathrm{i}}(x, y) = C \sum_{m=-\infty}^{\infty} \exp\{-ikam\theta_{\mathrm{in}}^2/2\} v^{\mathrm{i}}(x - am, y), \tag{6.7}$$

$$w^n(x, y) = C \sum_{m=-\infty}^{\infty} \exp\{-ikam\theta_{\mathrm{in}}^2/2\} v^{n-m}(x - am, y). \tag{6.8}$$

Relations (6.7) and (6.8) contain an unknown value C implicitly depending on θ_{in}. The next step will be to express C in terms of v. Reciprocity is used for that.

A reciprocity theorem is valid for the parabolic equation, particularly, when it is set on a branched surface. Let $G(x, y; x', y')$ be the Green's function of the surface. The notation implies that source is located at the point (x', y'), and the observation point is (x, y). Both points belong to the upper sheet. The reciprocity relation enables one to interchange the source and the observation point. Note that the direction of propagation should be reversed also. In our case the most compact form of the reciprocity relation is as follows:

$$G(x, y; x', y') = G(-x', y'; -x, y). \tag{6.9}$$

This relation can be easily checked by an explicit formula.

The reciprocity relation enables one to express $C(\theta_{\mathrm{in}})$ in terms of the directivity $S^{\mathrm{i}}(\theta)$. Namely, consider the point source of unit strength located at the point $(-x', \theta_{\mathrm{in}} x')$ on the upper sheet (x' is large and positive). This source produces a field asymptotically close to the incident plane wave multiplied by $g(x', \theta_{\mathrm{in}} x')$. Thus, the field of this source taken not far from the origin is close to the total field produced by an incident plane wave:

$$G(x, y; -x', \theta_{\mathrm{in}} x') \approx g(x', \theta_{\mathrm{in}} x') u^{\mathrm{i}}(x, y). \tag{6.10}$$

Coefficient C can be calculated by finding the field at the edge $(0, 0)$ and taking the limit $x' \to \infty$:

$$C = \lim_{x' \to \infty} G(0, 0; -x', \theta_{\mathrm{in}} x')(g(x', \theta_{\mathrm{in}} x'))^{-1}. \tag{6.11}$$

Apply the reciprocity theorem and take into account that by definition $v^{\mathrm{i}}(x, y) = G(x, y; 0, 0)$:

$$C = \lim_{x' \to \infty} v^{\mathrm{i}}(x', \theta_{\mathrm{in}} x')(g(X, \theta_{\mathrm{in}} X))^{-1}. \tag{6.12}$$

Finally, using (5.5), we obtain

$$C = S^{\mathrm{i}}(\theta_{\mathrm{in}}). \tag{6.13}$$

Thus, the embedding formulae for the fields have the form

$$H[u^{\mathrm{i}}](x,y) = S^{\mathrm{i}}(\theta_{\mathrm{in}}) \sum_{m=-\infty}^{\infty} \exp\{-ikam\theta_{\mathrm{in}}^2/2\} v^{\mathrm{i}}(x-am,y), \tag{6.14}$$

$$H[u^n](x,y) = S^{\mathrm{i}}(\theta_{\mathrm{in}}) \sum_{m=-\infty}^{\infty} \exp\{-ikam\theta_{\mathrm{in}}^2/2\} v^{n-m}(x-am,y). \tag{6.15}$$

Express the reflection coefficient $R(\theta;\theta_{\mathrm{in}})$ using the embedding formula (6.14). Let the angle of incidence θ_{in} and every angle of scattering θ_n belong to the set $\{\theta_m\}$ determined by (4.4).

According to the Floquet theory, the field w^{i} can be represented for $y > 0$ as a series:

$$w^{\mathrm{i}}(x,y) = \sum_n W_n \exp\{-ikx\,\theta_n^2/2 + iky\,\theta_n\}. \tag{6.16}$$

By performing the Fourier transform, we obtain an expression for the coefficients:

$$W_n = \frac{1}{a} \int_{x_0}^{x_0+a} w^{\mathrm{i}}(x,y) \exp\left\{ \frac{ikx\,\theta_n^2}{2} - iky\,\theta_n \right\} dx \tag{6.17}$$

for arbitrary x_0 and arbitrary $y > 0$.

Using (6.14) the last expression can be rewritten as

$$W_n = \frac{S^{\mathrm{i}}(\theta_{\mathrm{in}})}{a} \int_{-\infty}^{\infty} v^{\mathrm{i}}(x,y) \exp\left\{ \frac{ikx\,\theta_n^2}{2} - iky\,\theta_n \right\} dx. \tag{6.18}$$

A standard argument based on studying the values of the field far from the origin and applying the stationary phase method connects the directivity S^{i} of v^{i} and the Fourier transform of v^{i}. In our case the link is as follows:

$$\int_{-\infty}^{\infty} v^{\mathrm{i}}(x,y) \exp\{ikx\,\theta_n^2/2 - iky\,\theta_n\} dx = \theta S^{\mathrm{i}}(\theta). \tag{6.19}$$

By comparing the series (6.16) and (4.7) and by studying the action of the operator H on a plane wave, we find that

$$W_n = ik(\theta_n + \theta_{\mathrm{in}}) R(\theta_n;\theta_{\mathrm{in}}). \tag{6.20}$$

Using (6.19) and (6.20), we obtain from (6.18)

$$R(\theta_n;\theta_{\mathrm{in}}) = \frac{S^{\mathrm{i}}(\theta_n) S^{\mathrm{i}}(\theta_{\mathrm{in}})}{ika\,(\theta_n + \theta_{\mathrm{in}})\theta_n}. \tag{6.21}$$

This result agrees with the exact representation obtained by the Wiener–Hopf method. Namely, (6.21) can be compared with equation (3.11.20) of [2]. Function $S^{\mathrm{i}}(\theta)$ corresponds to $(i\gamma + k)G_+(i\gamma)$ in the notation of [2].

This representation of the reflection coefficient has, in fact, a reciprocal form (although this may not be clear from the first glance). A detailed study shows that the reciprocity relation for the reflection coefficient has the form

$$\theta_n R(\theta_n; \theta_{\mathrm{in}}) = \theta_{\mathrm{in}} R(\theta_{\mathrm{in}}; \theta_n). \tag{6.22}$$

Studying the far-field asymptotics of (6.15) we get the expression for D:

$$D(\theta; \theta_{\mathrm{in}}) = \frac{S^{\mathrm{i}}(\theta_{\mathrm{in}})}{ik(\theta + \theta_{\mathrm{in}})} \sum_{n=0}^{\infty} \exp\left\{ \frac{ikan\theta_{\mathrm{in}}^2}{2} \right\} S^n(\theta). \tag{6.23}$$

Here θ is arbitrary, i.e., not necessarily belonging to the set $\{\theta_n\}$.

Formulae (6.21) and (6.23) are the main results of this section. In what follows we are concentrating our efforts on finding the directivities $S^{\mathrm{i}}(\theta)$ and $S^n(\theta)$ of the edge Green's function v.

7. Derivation of the spectral equations. Apply operator

$$K = x\frac{\partial}{\partial y} - iky \tag{7.1}$$

to the edge Green's function v. The function $K[v]$ has the following properties.

— It obeys parabolic equation (4.2), everywhere except the branch points, since the operators K and L commute. Operator K is listed as one of the symmetry operators for the parabolic equation in the monograph [18].

— It obeys the radiation condition; i.e., $K[v]$ does not contain components coming from infinity.

— Operator K nullifies the Green's function of a free plane (4.9); therefore the identities $K[v^{\mathrm{i}}] \equiv 0$ and $K[v^0] \equiv 0$ are valid for $x < a$.

We are going to apply the uniqueness argument developed in the previous section. According to this argument, $K[v]$ is a wave field (i.e., a solution of the parabolic equation) on the branched surface introduced above. Since there are no incident components, the field is generated by a set of sources located at the branch points. The next task is to reveal these sources.

The sources at the point $(0, 0)$ are nullified by the operator, as follows from the last property. The behavior of $K[v]$ at the branch points $(an, 0)$, $n > 0$, can be studied by the method proposed above. The result is as follows:

$$L[K[v^{\mathrm{i}}]] = anC_n\delta(x - an - 0)\delta(y), \tag{7.2}$$

$$L[K[v^n]] = -anC_n\delta(x - an - 0)\delta(y), \tag{7.3}$$

where we introduce the following notation:

$$C_n \equiv v^{\mathrm{i}}(an - 0, 0), \quad n = 1, 2, \dots; \tag{7.4}$$

i.e., C_n are the values of the edge Green's function at the branch points. The coefficients C_n play an important role in the study. They will be calculated later.

According to the uniqueness argument, the field $K[v]$ can be written as the sum of edge Green's functions taken with the amplitudes defined by (7.2) and (7.3):

$$K[v^{\mathrm{i}}](x, y) = a\sum_{m=1}^{\infty} mC_m v^{\mathrm{i}}(x - am, y), \tag{7.5}$$

$$K[v^n](x, y) = a\sum_{m=1}^{\infty} mC_m v^{n-m}(x - am, y). \tag{7.6}$$

Study the far-field asymptotics of (7.5) and (7.6). A direct check shows that the operator K acts on S^{i} as differentiation with respect to the angle θ:

$$S^{\mathrm{i}} \xrightarrow{K} \frac{dS^{\mathrm{i}}}{d\theta}. \tag{7.7}$$

Note that the far field of the function $v^{\mathrm{i}}(x-am,y)$ can be written as follows:

$$v^{\mathrm{i}}(x-am,y) = g(x,y)\exp\{ikam\theta^2/2\}S^{\mathrm{i}}(\theta) + O(x^{-3/2}), \qquad \theta = y/x; \tag{7.8}$$

i.e., a shift along the coordinate x leads to multiplication of the directivity by an exponential factor. Using (7.7) and (7.8), one can derive the following relation from (7.5):

$$\frac{dS^{\mathrm{i}}(\theta)}{d\theta} = a\, S^{\mathrm{i}}(\theta)\sum_{n=1}^{\infty} nC_n \exp\left\{\frac{ikna\theta^2}{2}\right\}. \tag{7.9}$$

This is the *spectral equation* for S^{i}. It is an ordinary differential equation.

Derivation of the spectral equation for S^n is a bit more sophisticated. First, study the far-field asymptotics of (7.6). Taking into account the shift of the origin, the result is as follows:

$$\frac{d}{d\theta}\left[\exp\left\{\frac{ikan\theta^2}{2}\right\}S^n(\theta)\right] = a\exp\left\{\frac{ikan\theta^2}{2}\right\}\sum_{m=1}^{\infty} mC_m S^{n-m}(\theta). \tag{7.10}$$

This relation taken for different n is a set of ordinary differential equations with respect to an infinite number of unknown functions. Our aim is to split the equations and solve them. For that, we make the next step, namely introduce a new variable

$$\hat{S}^n(\theta) = \exp\{ikan\theta^2/2\}S^n(\theta)$$

and rewrite (7.10) as

$$\frac{d\hat{S}^n(\theta)}{d\theta} = a\sum_{m=1}^{\infty} mC_m \exp\left\{\frac{ikam\theta^2}{2}\right\}\hat{S}^{n-m}(\theta). \tag{7.11}$$

Note that (7.11) has a convolution structure with respect to the indices. Thus, it is now possible to apply the Fourier method. Introduce the Fourier transforms

$$\bar{C}(p;\theta) = a\sum_{m=1}^{\infty} mC_m \exp\{ikam\theta^2/2 + ipm\},$$

$$\bar{S}(p;\theta) = \sum_{m=0}^{\infty}\exp\{ipm\}\hat{S}^m = \sum_{m=0}^{\infty}\exp\{ikam\theta^2/2 + ipm\}S^m.$$

According to the general properties of the Fourier transformation, (7.11) can be rewritten as

$$\frac{d\bar{S}(p;\theta)}{d\theta} = \bar{C}(p;\theta)\bar{S}(p;\theta), \tag{7.12}$$

i.e., as an ordinary differential equation for each fixed p.

An inverse Fourier transformation can be used to find the values S^m. However, according to embedding formula (6.23), we need to find the combination

$$\sum_{n=0}^{\infty} \exp\left\{\frac{ikan\theta_{\text{in}}^2}{2}\right\} S^n(\theta) = \bar{S}(p_*;\theta) \quad \text{for} \qquad p_* = \frac{ka}{2}(\theta_{\text{in}}^2 - \theta^2). \tag{7.13}$$

The main results of this section are equations (7.9) and (7.12). They should be solved (the second one for p defined by (7.13)), and the solutions should be substituted into the embedding formulae (6.21) and (6.23).

There is an ambiguity in the definition of (7.12) since p_* depends on θ according to (7.13). A correct procedure is as follows. Equation (7.12) is solved for each fixed real parameter $0 < p < 2\pi$. Thus, $\bar{S}(p;\theta)$ is found as a function of two arguments. Then for each θ an appropriate value of p_* is found and substituted into (7.13).

8. Solution of the spectral equations. To solve (7.9) and (7.12) one should provide physically motivated boundary conditions. Since both equations are of first order, each of them requires a single boundary condition.

Study the asymptotics of the unknown functions S^{i} and $\bar{S}(p;\theta)$ as $\theta \to \pm\infty$. Take into account the terms having order $O(1)$. Note that only direct rays going from one of the sources to infinity produce terms of order $O(1)$. Any diffracted ray gives a term of order $O(1/|\theta|)$ or smaller.

For positive θ there is a single direct ray going from the origin along the upper sheet and a single ray going along the lower sheet with index 0. Thus, we get boundary conditions

$$S^{\mathrm{i}}(+\infty) = 1, \tag{8.1}$$

$$\bar{S}(p, +\infty) = -1. \tag{8.2}$$

The first one will be used for solving (7.9) and the second one for (7.12).

A very important additional boundary condition can be obtained for $\bar{S}(p;\theta)$ as $\theta \to -\infty$. Note that there are *two* direct rays: one which belongs to directivity S^0 and another to S^1 (see Figure 8.1). As a result, for large negative θ the asymptotics are as follows:

$$S^0(\theta) = -1 + O(1/|\theta|), \tag{8.3}$$

$$S^1(\theta) = \exp\{-ika\theta^2/2\} + O(1/|\theta|). \tag{8.4}$$

All other S^n do not have terms of order $O(1)$.

According to the definition of $\bar{S}(p;\theta)$,

$$\bar{S}(p;\theta) = -(1 - e^{ip}) + O(1/|\theta|),$$

and, finally,

$$\bar{S}(p;-\infty) = -(1 - e^{ip}), \tag{8.5}$$

which is an additional boundary condition for (7.12). Below we use this condition to find the coefficients C_n.

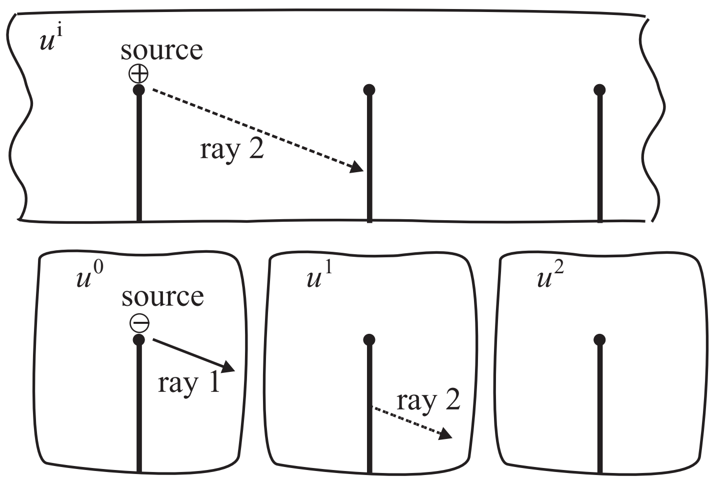

FIG. 8.1. *Direct rays for $\theta < 0$ on the lower sheets.*

The solutions of spectral equations (7.9) and (7.12) with boundary conditions (8.1) and (8.2) are as follows:

$$S^{\mathrm{i}}(\theta) = \exp\left\{ -\sqrt{\frac{\pi i a}{2k}} \sum_{m=1}^{\infty} \sqrt{m} C_m \mathrm{erfc}\left(\sqrt{-\frac{ikma}{2}}\,\theta \right) \right\}, \tag{8.6}$$

$$\bar{S}(p;\theta) = -\exp\left\{ -\sqrt{\frac{\pi i a}{2k}} \sum_{m=1}^{\infty} \sqrt{m} C_m e^{ipm} \mathrm{erfc}\left(\sqrt{-\frac{ikma}{2}}\,\theta \right) \right\}, \tag{8.7}$$

where $\mathrm{erfc}(z)$ is the complementary error function

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-\tau^2} d\tau. \tag{8.8}$$

Let us find the coefficients C_n. For this, calculate $\bar{S}(p, -\infty)$ from (8.7) and compare it with (8.5). Note that $\mathrm{erfc}(-\infty) = 2$. Thus,

$$-\exp\left\{ -\sqrt{\frac{2\pi i a}{k}} \sum_{m=1}^{\infty} \sqrt{m} C_m e^{ipm} \right\} = -(1 - e^{ip}). \tag{8.9}$$

Taking the logarithm of (8.9) and using the relation

$$\log(1 - e^{ip}) = -\sum_{m=1}^{\infty} \frac{e^{ipm}}{m}$$

we obtain that

$$C_n = \frac{1}{n} \sqrt{\frac{k}{2\pi i a n}} = \frac{g(an, 0)}{n}. \tag{8.10}$$

This result coincides with the expression obtained by Anis and Lloyd [19] and later by Boersma [5, 20].

These values can be substituted into (8.6) and (8.7) to get the solutions of the spectral equation in a closed form. To simplify the final expressions, introduce the normalized angles

$$\beta = \sqrt{ka}\,\theta, \qquad \beta_{\rm in} = \sqrt{ka}\,\theta_{\rm in}. \tag{8.11}$$

By substituting (8.6) and (8.7) into (6.21) and (6.23), we obtain

$$R(\theta;\theta_{\rm in}) = \frac{\exp\{s^{\rm i}(\beta) + s^{\rm i}(\beta_{\rm in})\}}{i\beta(\beta+\beta_{\rm in})}, \tag{8.12}$$

$$s^{\rm i}(\beta) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{\operatorname{erfc}(\sqrt{-in/2}\,\beta)}{n} \tag{8.13}$$

and

$$D(\theta;\theta_{\rm in}) = -\sqrt{\frac{a}{k}}\frac{\exp\{s^{\rm i}(\beta) + s(\beta,\beta_{\rm in})\}}{i(\beta+\beta_{\rm in})}, \tag{8.14}$$

$$s(\beta;\beta_{\rm in}) = -\frac{1}{2}\sum_{n=1}^{\infty} \exp\left\{\frac{in(\beta_{\rm in}^2-\beta^2)}{2}\right\}\frac{\operatorname{erfc}(\sqrt{-in/2}\,\beta)}{n}. \tag{8.15}$$

Note that if both θ and $\theta_{\rm in}$ belong to the set $\{\theta_n\}$ of the waveguide modes, then

$$s(\beta;\beta_{\rm in}) = s^{\rm i}(\beta).$$

9. Solution analysis. The function s introduced above is Weinstein's special function U:

$$s(\alpha,\beta) = U(\alpha,\beta^2/(4\pi)) \tag{9.1}$$

(after a correction of an obvious sign typo in the formula (B.22) of [1]). Thus, our results can be compared with those of Weinstein. The results are the same.

Numerical computation of the function $s^{\rm i}(\beta)$ can be performed using a direct summation of (8.13). The amount of terms should be taken sufficiently large. The results are shown in Figure 9.1.

One can note the singular points on the graphs corresponding to

$$\beta_m = \sqrt{4\pi m}, \qquad m = 1,2,\dots. \tag{9.2}$$

For these points the series (8.13) converges, but a similar series for the derivative of the directivity diverges. That is why there are "sharp edges" on the graph.

Study the most important case, namely $\beta \ll 1$. An elementary analysis gives the following asymptotics:

$$s^{\rm i}(\beta) = \log(\sqrt{-2i}\beta) + \zeta(1/2)\sqrt{-i/(2\pi)}\beta + O(\beta^3), \tag{9.3}$$

where ζ is the Riemann zeta-function.

Let be $\theta = \theta_{\rm in}$, and $\theta_{\rm in} \ll 1$. Substitute (9.3) into (6.21) to obtain the relation

$$R(\theta_{\rm in};\theta_{\rm in}) \approx -\exp\left\{-0.824(1-i)\sqrt{ka}\,\theta_{\rm in}\right\}, \tag{9.4}$$

which coincides with the known expression [1, 2]. Here we used the relation

$$\frac{\zeta(1/2)}{\sqrt{\pi}} \approx -0.824.$$

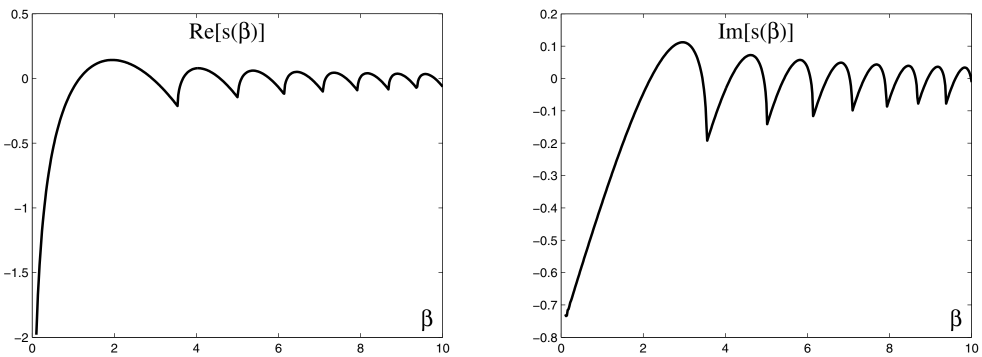

FIG. 9.1. *Numerical computation of* $s^{\mathrm{i}}(b)$.

10. Conclusion. The problem of diffraction of a waveguide mode is studied in the parabolic approximation. The axis of the parabolic equation is chosen perpendicular to the waveguide direction; i.e., we are focused on the modes with the propagation constant being small compared to the wavenumber in the medium. The problems of reflection of a wave mode and radiation from the waveguide into the open space are studied.

The waveguide problem is transformed into a propagation problem on a branched surface. The reflection principle is used for this.

The problem is treated by the method earlier developed for the Helmholtz equation. Two main steps are performed. First, embedding formulae (6.21) and (6.23) are derived for the problem. As a result, the reflection coefficient and the directivity become represented as combinations of the directivities related to the edge Green's function of the problem. Thus, the problem becomes reduced to finding the directivities of the edge Green's function.

Second, spectral equations (7.9) and (7.12) are derived for the directivities of the edge Green's function. The spectral equations are homogeneous linear ordinary differential equations with rather complicated coefficients. They contain unknown parameters C_n, which are the values of the edge Green's function at the edge points. Fortunately, in our case there exists a compact representation (8.10) for C_n.

The solution of the spectral equations is given by the formulae (8.6) and (8.7). The solution can be expressed in terms of Weinstein's special function.

The main aim of the paper is to develop a framework of solving a wide range of waveguide problems avoiding Wiener-Hopf method. It is shown that such a technique can be developed using the embedding formula and the spectral equation.

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