An approximate theory for waves in a slender elastic wedge immersed in liquid

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The problem is considered of wave propagation along the tip of a slender elastic wedge immersed in liquid. Interaction between wedge mode and acoustic waves is studied. Namely, attenuation of wedge wave occurs if the velocity of wedge mode is greater than the velocity of acoustic waves and wedge mode velocity changes in the opposite case A functional-differential equation is derived for the problem. This equation is solved asymptotically by using Wiener-Hopf method in the case of very light liquid loading.

Keywords: Wiener-Hopf technique, wedge waves, thin plate flexural vibrations

Introduction

The theory of diffraction and wave propagation in sectorial-shaped region is a challenging branch of mathematical physics. In the early 50-s, a powerful technique was developed to solve 2D stationary diffraction problems for liquid (or gas) angles subject to arbitrary boundary conditions (see Maljuzhinetz (1955)). The main idea was to reduce the problem to a set of functional equations. These equations were defined on a complex plane and linked the values of an unknown function at two different points. There were also some important restrictions on the behaviour of the unknown functions on the complex plane. Some methods for solving such functional equations have been developed. The approach enabled one to avoid solving boundary integral equations. A number of papers appeared developing the method of functional equations in order to be able to to apply it to a great variety of problems (for example, see Osipov & Norris (1997) and a large bibliography therein).

The main restriction of the method of functional equations is that the problem can involve only one wave velocity. Such important problems as diffraction of electromagnetic waves on a dielectric wedge and diffraction of acoustic waves in a solid wedge involve wave modes with at least two different velocities, and they cannot yet be solved explicitly using functional equations, since although functional equations can be derived for each of these problems, the equations cannot be solved using known methods.

There is an important aspect that attracts a great deal of attention for acoustic diffraction in a solid wedge: such a wedge can act as a waveguide for the so-called wedge waves (Lagasse (1973)). The bulk of information about wedge waves has been obtained using numerical calculations, (see e.g., Moss et al. (1973), Hladky-Hennion et al. (1997)), and no analytical theory is available for waves in a wedge with arbitrary apex angle. However, analytical theories have been developed successfully for two important cases possessing small parameters: symmetrical wave modes in a wedge with the apex angle close to π and antisymmetric (flexural) modes in a slender wedge. In the present paper we shall study the latter case.

The wedge modes have some important properties. Their energy is localized in the vicinity of the edge, and their velocities can be essentially lower than the velocities of bulk elastic waves in the wedge material. These properties make wedge modes attractive for such applications as nondestructive testing of certain engineering structures (with edges).

The theory of wedge waves in a slender wedge is well-developed. The wedge can be treated as a thin plate of variable thickness, and the dynamic equation of thin plates can be either directly solved (as in McKenna et al. (1974)) or a geometrical acoustics approach can be applied using the dispersion curves of flexural modes (as in Krylov (1989) or Mozhaev (1989)). Both methods lead to similar results and are in good agreement with the analytical derivation based on the expansion of an elastic field in a slender wedge as power series in apex angle θ (see Parker (1992)). Namely, it is known that there is a number of antisymmetric modes roughly estimated as $\sim \pi/\theta$, where θ is the apex angle of a wedge. The lowest mode has velocity of order $\sim c_k\theta$, where c_k is the velocity of a shear wave in the solid. Higher modes have a complicated structure and their velocities are approximately proportional to the number of a mode.

In this paper we study wave propagation in a slender elastic wedge surrounded by liquid or gas. One can expect such effects as attenuation of wedge waves (in the supersonic case) and modification of their velocities.

In what follows we pursue the following aims:

— We study wedge wave propagation in immersed wedges. This is an interesting engineering problem of structural acoustics (Hladky-Hennion et al. (1997), Krylov (1994), Krylov (1998)). The results can be used in noise control and in nondestructive testing applications. There are also some ideas to use such waves for modern underwater propulsion techniques (Krylov (1994)).

— We derive a new type of functional equation. This equation is similar to the classical Wiener-Hopf type, but includes a differential operator. As far as we know, there is no known method for solving such equations. It can be expected that such equations emerge in a variety of diffraction problems. Therefore any attempt to solve this equation would be a valuable contribution to the theory of Wiener-Hopf equations.

— We attempt to use functional equations as a starting point for all calculations in this paper. Functional equations proved to be an effective tool in the previous research of one of the authors (Shanin (1996, 1997)). In the present paper, functional equations replace boundary integral equations and enable us to avoid using special functions.

1. Formulation of the problem

Consider the space (x, y, z) filled with a homogenous liquid and a thin elastic plate of variable thickness (a slender wedge) occupying the area $x > 0$, $-\infty < z < \infty$. The geometry of the problem is shown in Fig. 1. The local thickness of the wedge is presumed to be negligible everywhere (the apex angle of the wedge θ is very small), so we can say that the y-coordinate of the points of the wedge is everywhere equal to zero.

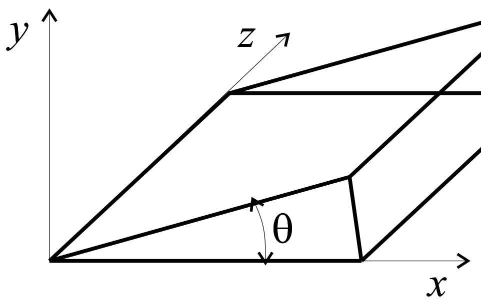

Figure 1. Geometry of the wedge

We are going to use the plate theory equation to decribe the motion of the wedge and its interaction with the fluid. Of course, it is a simplification of the original problem (wave propagation in a solid wedge with fluid loading), but only this assumption enables us to derive some analytical results. We must note, that our approach is not bad, because wedge waves are known to have exponential decay with the distance from the edge, and the waves in liquid have either exponential or algebraic decay, so at the distances from the edge where the local thickness of the wedge is considerable (comparable with wavelength) both elastic and acoustic values are small.

We shall consider the wedge of finite thickness only in the Appendix in order to derive the equations of the limiting case $\theta \to 0$.

Suppose that all variables (displacements of the wedge and pressure in the liquid) have the dependence on the z coordinate and the time of the type $\mathrm{e}^{-\mathrm{i}(\beta z+\omega t)}$. Note that the wave travels in the negative direction of the z-axis, when $\beta > 0$, $\omega > 0$.

We assume in this paper that β is fixed and ω is the parameter to be determined. Instead of the attenuation of the wave we study the damping in time of a spatially sinusoidal wave, which corresponds to ω with a negative imaginary part. Note that in practice it is usually an imaginary part of β which is sought, with ω fixed. For small attenuation, we can use the relationship $Im[\beta] = -Im[\omega]/c$ to find the estimation of spatial attenuation of the wedge wave (c is the velocity of undisturbed wedge wave).

Below we seek only the solutions antisymmetric in y. The reason for this is following. Only antisymmetric modes exist in a slender wedge in vacuum. Symmetric mode appears only for θ close to π. Besides, antisymmetric modes in slender wedge are much slower than bulk modes in the solid, so for some θ the velocities of wedge mode and acoustic wave coincide and the interaction between them can be very effective.

There are two subsystems under consideration: fluid one and elastic one. For each subsystem we must formulate the equation of motion, boundary and radiation conditions and, besides, we must desribe the interaction between the subsystems.

(*a*) *Fluid subsystem*

Helmholtz' equation is satisfied in the liquid

$$\Delta p + k_0^2 p = 0, \tag{1.1}$$

where p is the amplitude of the acoustic pressure (factor $\mathrm{e}^{\mathrm{i}\omega t}$ is assumed), and $k_0 = \omega/c_0$ is the wavenumber of sound in the liquid, ω is (complex) circular frequency, and c_0 is the velocity of sound.

As the pressure takes the form $p(x, y, z) = p(x, y)\mathrm{e}^{-\mathrm{i}\beta z}$, we have the following equation for the function $p(x, y)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p - \lambda^2 p = 0, \tag{1.2}$$

where

$$\lambda^2 = \beta^2 - k_0^2. \tag{1.3}$$

There are two different cases: $Re[\beta] > Re[k_0]$ (subsonic case) and $Re[\beta] < Re[k_0]$ (supersonic case). In the supersonic case, the wave travelling along the edge is faster than sound in the liquid, so that excitation of non-decaying waves is possible. This case is more difficult for analysis because formally one should obtain the solution for p that shows exponential increase at $y \to \infty$. The subsonic case is more simple, since the solution must decay exponentially with y. We use a simple model for the supersonic case by supposing that the liquid possesses some absorption (i.e., by considering an imaginary part of c_0). This is enough to exclude the exponential growth of the solution. We formulate the radiation condition for the liquid in the following way: there must be no wave components coming from infinity or having exponential growth at infinity. This condition will be satisfied by choosing an appropriate branch of square root in the Fourier integral (see the next section).

Edge conditions must be imposed, describing the behaviour of p in the vicinity of the edge. Acoustic energy must be integrable in this area, and energy flow from the origin must be equal to zero. Taking into account that an antisymmetric solution is sought, we conclude that

$$p(r) = o(1), \qquad \nabla p = o(1/r), \tag{1.4}$$

where r is the distance from the edge.

(*b*) *Elastic subsystem*

For the decription of antisymmeric motion of the wedge we will use the theory of flexural vibrations of plates with variable thickness. Namely, the thickness of the plate (wedge) grows linarly with the x-coordinate (as in McKenna et al. (1974)).

Let the y-component of the dislpacement of the wedge be denoted by

$$w(x, z, t) = w(x)\mathrm{e}^{-\mathrm{i}(\beta z + \omega t)}.$$

The equation of flexural vibrations of the plate is

$$L[w(x)] = p(x, +0) - p(x, -0), \tag{1.5}$$

where the operator $L[w]$ is equal to (see Timoshenko & Woinovsky-Krieger (1959))

$$L[w] = \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{zz}}{\partial z^2} - 2\frac{\partial^2 M_{xz}}{\partial x \partial z} + \omega^2 m(x) w, \tag{1.6}$$

where

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial z^2}\right), \qquad M_{zz} = -D\left(\frac{\partial^2 w}{\partial z^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),$$

$$M_{xz} = D(1-\sigma)\frac{\partial^2 w}{\partial x \partial z},$$

$$D = D_0 x^3 \theta^3, \qquad D_0 = \frac{E}{12(1-\sigma^2)},$$

$$m(x) = \rho_s \theta x,$$

E is Young's modulus of the material, σ is Poisson's ratio, ρ_s is the mass density of the solid and θ is the apex angle of the wedge.

The values M_{xx} and M_{zz} are the bending moments per unit length of sections of the wedge perpendicular to the x and z axes, respectively, M_{xz} is the twisting moment per unit length of a section of the wedge perpendicular to the x axis, and $m(x)$ is the mass per unit area.

Since the problem is chosen to be antisymmetrical, we can consider the fluid motion only in the region $y > 0$ and rewrite (1.7) in the form

$$L[w(x)] = 2p(x, 0). \tag{1.7}$$

We seek for the solutions $w(x)$ decaying at infinity. Beside that, the tip of the wedge is unconstrained, so w must satisfy the boundary conditions at $x = 0$ (Timoshenko & Woinovsky-Krieger (1959)):

$$M_{xx} = 0, \qquad \frac{\partial M_{xx}}{\partial x} - 2\frac{\partial M_{xz}}{\partial z} = 0. \tag{1.8}$$

(c) *Interaction between elastic and fluid subsystems*

Consider the liquid near the plate. It follows from the equations of fluid acoustics that the acceleration of liquid is proportional to the gradient of pressure. It is obvious that the y-component of the displacement is continuous at the plate, so we conclude that

$$w(x) = \left.\frac{1}{\rho_l \omega^2}\frac{\partial p}{\partial y}\right|_{y=0}, \tag{1.9}$$

where ρ_l is the mass density of the liquid.

Substituting (1.9) into (1.7), we obtain a higher-order boundary condition for the liquid. A detailed study shows that the following asymptotic series satisfying Meixner's conditions for the fluid and boundary conditions for the solid can be substituted for p near the edge

$$p \sim \sum_{n=1}^{\infty} a_n r^n \sin n\varphi + \tag{1.10}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(b_{n,m}\frac{\partial^m r^\nu \sin\nu\varphi}{\partial\nu^m}+c_{n,m}\frac{\partial^m r^{\nu+1}\sin[(\nu+1)\varphi]}{\partial\nu^m}\right)\Bigg|_{\nu=2n+1/2},$$

where φ is the angle from the negative direction of the x-axis. Note that the second series contains logarithmic terms. Such behaviour is peculiar to the solutions of impedance boundary problems. Anzats (1.10) enables to make an estimation of $\hat{w}$ near the edge:

$$w(x) = w(0) + xw'(0) + o(x). \tag{1.11}$$

2. Derivation of the functional equation

Function $p(x, y)$ in the region $y > 0$ can be represented in the form of Fourier integral

$$p(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{p}(k)\mathrm{e}^{-\mathrm{i}kx-\sqrt{k^2+\lambda^2}y}\mathrm{d}k \tag{2.1}$$

The appropriate choice of the branch of the square root is necessary for satisfying the radiation condition in the liquid. Namely, the real part of the root must be positive. Note, that a special choice of the contour of integration can help to overcome some difficulties in the the supersonic case with real c_0, in which the exponential growth of p can appear. However, here we do not consider this case. Also, we expect the exponential decay of w and p as $|x| \to \infty$, so the Fourier integrals here and below are convergent.

Taking into account, that $p(x, 0) = 0$ for $x < 0$ since the problem is antisymmetric, we find that

$$\hat{p}(k) = \int_{0}^{\infty} p(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x. \tag{2.2}$$

Normal derivative of the field can be obtained by differentiation of (2.1):

$$\frac{\partial p(x,y)}{\partial y} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{k^2+\lambda^2}\hat{p}(k)\mathrm{e}^{-\mathrm{i}kx-\sqrt{k^2+\lambda^2}y}\mathrm{d}k. \tag{2.3}$$

Note, that $\partial p/\partial y$ at the plate can be found from (1.9), so the inverse Fourier transform of (2.3) for $y = 0$ can be written in the form

$$-\sqrt{k^2+\lambda^2}\hat{p}(k) = \hat{s}(k) + \rho_l\omega^2\hat{w}(k), \tag{2.4}$$

where

$$\hat{s}(k) = \int_{-\infty}^{0}\frac{\partial p}{\partial y}e^{\mathrm{i}kx}\mathrm{d}x, \tag{2.5}$$

$$\hat{w}(k) = \int_0^\infty w(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x, \tag{2.6}$$

Let $\hat{L}$ be the Fourier image of the operator $L/2$, i.e. an operator, such that

$$\hat{L}[\hat{w}(k)] = \hat{p}(k). \tag{2.7}$$

The factor 1/2 is chosen for convenience. The Fourier transform yields

$$\hat{L}[\hat{w}(k)] = -\frac{\mathrm{i}D_0\theta^3}{2}\left[(k^4 + 2\beta^2 k^2 + \beta^4)\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}k^3} + 6k(k^2+\beta^2)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}k^2} + \right. \tag{2.8}$$
$$\left. 6(k^2 + \sigma\beta^2 + (1-\sigma)k_T^2/\theta^2)\frac{\mathrm{d}\hat{w}}{\mathrm{d}k}\right],$$

where $k_T^2 = \omega^2/c_T^2$ is the wavenumber of shear waves in the solid, and $c_T^2 = E/[2(1-\sigma)\rho_s]$.

We show in the Appendix how equation (2.8) can be obtained directly from the governing equations for the solid wedge.

Note that the operator L is defined only for $x > 0$. It might seem that the Fourier transform of (1.7) should include terms corresponding to the values of functions at $x = 0$ (as in formulas concerning Laplace's transform). However, boundary conditions (1.8) and edge conditions, following from (1.11) precisely make these terms zero. The behaviour of unknown functions at infinity does not cause any problems, because under the assumptions made above, all functions have exponential decay both in the supersonic and the subsonic case.

Equation (2.4) can be rewritten in the form

$$\hat{s}(k) + \rho_l\omega^2\hat{w}(k) + \sqrt{k^2+\lambda^2}\,\hat{L}[\hat{w}(k)] = 0. \tag{2.9}$$

Unknown functions $\hat{w}$ and $\hat{s}$ have some important properties following from (2.5), (2.6). The function $\hat{w}(k)$ must be analytic in the upper half of the complex k-plane and have no singularities on the real positive half-axis. The function $\hat{s}(k)$ must be analytic in the lower half of the complex k-plane and have no singularities on the real negative half-axis.

Besides, $\hat{w}(k)$ and $\hat{s}(k)$ have the behavior at infinity

$$\hat{w}(k) = w(0)\frac{\mathrm{i}}{k} - w'(0)\frac{1}{k^2} + o(\frac{1}{k^2}), \qquad \hat{s}(k) = o(1). \tag{2.10}$$

Note that equation (2.9) is similar to the classical homogeneous functional Wiener-Hopf equation (see Noble (1958)). However, it contains a differentional operator, so we shall call (2.9) a *functional-differentional equation* (FDE).

3. Application of the perturbation technique to the FDE in the case of a very light fluid

Unfortunately, no theory is known for equations like (2.9) with analytic restrictions on unknown functions. Here we shall consider only a particular case containing a small parameter.

Let the liquid surrounding the wedge be very light. By using the perturbation technique, we shall construct the solution of equations (2.9), (2.8) in the form of a series in a small dimensionless parameter.

We start from the solution for the none-immersed wedge (zero-order approximation):

$$\hat{L}[\hat{w}_0(k)] = 0. \tag{3.1}$$

Note that equation (3.1) is a second order equation for $\hat{w}'$ and can be solved using standard methods. Equation (3.1) has two singular points in the finite part of the complex k-plane: $k = \mathrm{i}\beta$ and $k = -\mathrm{i}\beta$. The behaviour of the solution at $k = \mathrm{i}\beta$ is of great importance because the solution must be analytic in the upper half plane.

The general solution of (3.1) is

$$\hat{w}_0(k) = C_1 \left(\frac{k - \mathrm{i}\beta}{k + \mathrm{i}\beta}\right)^{\gamma} + C_2 \left(\frac{k - \mathrm{i}\beta}{k + \mathrm{i}\beta}\right)^{-\gamma} + C_3, \tag{3.2}$$

where

$$\gamma = \frac{1}{2}\sqrt{4 + 6(1-\sigma)(\frac{k_T^2}{\theta^2\beta^2} - 1)}.$$

The solution (3.2) must have no singularities in the upper half plane, particularly at the point $k = \mathrm{i}\beta$. So, we must insist that γ is an integer and $C_2 = 0$. The constant C_3 is chosen such that $\hat{w}_0(k) \to 0$ as $k \to \infty$.

Finally, we rewrite the zero-order approximation of the solution in the form

$$\hat{w}_0 = \left(\frac{k - \mathrm{i}\beta}{k + \mathrm{i}\beta}\right)^{N} - 1, \qquad N = 1, 2, 3\ldots \tag{3.3}$$

where

$$\beta = \frac{k_T}{\theta}\sqrt{\frac{3(1-\sigma)}{(2N^2 + 1 - 3\sigma)}}. \tag{3.4}$$

The last equation determines the eigenvalues of the slender wedge as a waveguide. It coincides with the condition from McKenna et al. (1974). It can be shown that the eigenwaves corresponding to (3.2) can be expressed by Laguerre polynomials in the same way as in McKenna et al. (1974).

Let us introduce a small parameter of the problem

$$\varepsilon = \frac{\rho_l \omega_0^2}{D_0 \theta^3 \beta^2}. \tag{3.5}$$

It is easy to show that

$$\varepsilon \sim \frac{\rho_l N^2}{\rho_s \theta}. \tag{3.6}$$

This parameter is small for a very light liquid (or gas) loading.

Introducing the notations

$$\hat{s}^*(k) = \frac{\hat{s}(k)}{D_0\theta^3\beta^2},$$

$$\hat{L}^*[\hat{w}(k)] = \frac{\hat{L}[\hat{w}(k)]}{D_0\theta^3\beta},$$

we rewrite (2.9) in the form

$$\hat{s}^*(k) + \sqrt{k^2+\lambda^2}\hat{L}^*[\hat{w}(k)] = -\varepsilon\hat{w}(k). \tag{3.7}$$

Traditional perturbation techniques can be applied to such equation (Nayfeh, 1981). We suppose that the parameter β is fixed and look for a correction to ω_0, i.e., to the initial value of ω. Since the problem is regularly perturbed, the variables can be expanded as series in ε:

$$\begin{aligned} \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots, \\ \hat{w}(k) &= \hat{w}_0(k) + \varepsilon\hat{w}_1(k) + \varepsilon^2\hat{w}_2(k) + \ldots, \\ \hat{s}^*(k) &= \varepsilon\hat{s}_1^*(k) + \varepsilon^2\hat{s}_2^*(k) + \ldots \end{aligned}$$

Each term of the expansions for $\hat{w}$ and $\hat{s}^*$ must obey the same analytic and edge restrictions, as formulated for $\hat{w}$ and $\hat{s}^*$ themselves.

The operator $\hat{L}^*$ depends on $k_T^2(\omega)$, and λ is the function of ω. Therefore, $\hat{L}^*$ and λ also must be expanded as a power series in ε:

$$\begin{aligned} \hat{L}^*[\hat{w}(k)] &= \hat{L}_0^*[\hat{w}(k)] + \varepsilon\hat{L}_1^*[\hat{w}(k)] + \varepsilon^2\hat{L}_2^*[\hat{w}(k)] + \ldots \\ \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 \ldots \end{aligned}$$

We shall find first-order term of the expansion for ω, i.e. ω_1. This term provides the information about the energy losses due to wave radiation into the liquid and the changes of edge mode velocity. For this purpose we substitute the expansions for the unknown functions into (3.7) and obtain the equation at $O(\varepsilon)$:

$$\hat{s}_1^*(k) + \sqrt{k^2+\lambda_0^2}\hat{L}_0^*[\hat{w}_1(k)] = -\hat{w}_0(k) - \sqrt{k^2+\lambda_0^2}\hat{L}_1^*[\hat{w}_0(k)], \tag{3.8}$$

where

$$\hat{L}_1^* = -\omega_1\frac{6\mathrm{i}\omega_0(1-\sigma)}{\beta^2\theta^2c_T^2}\frac{\mathrm{d}}{\mathrm{d}k} \tag{3.9}$$

Note that $\hat{w}_1(k)$ is analytic in the upper half-plane of k. Therefore, the function

$$\hat{f}(k) \equiv \hat{L}_0^*[\hat{w}_1(k)]$$

is also analytic in the same region. According to (2.10), we conclude that $\hat{L}_0[\hat{w}_1(k)] \sim O(1)$ at infinity.

We seek the functions $\hat{s}_1^*(k)$ and $\hat{f}(k)$ by using the standard procedure of Wiener-Hopf method. First decompose into factors the coefficient on the left-hand side of the equation:

$$\frac{\hat{s}_1^*(k)}{\sqrt{k-\mathrm{i}\lambda_0}} + \sqrt{k+i\lambda_0}\hat{f}(k) = -\frac{\hat{w}_0(k)}{\sqrt{k-\mathrm{i}\lambda_0}} - \sqrt{k+\mathrm{i}\lambda_0}\hat{L}_1^*[\hat{w}_0(k)]. \tag{3.10}$$

In agreement with the restrictions imposed on $\hat{s}(k)$ and $\hat{w}(k)$, the factors are chosen so that the term containing $\hat{s}_1^*$ is analytic in the lower half-plane while the term involving $\hat{f}(k)$ is analytic in the upper one.

Next the r.-h.s. of (3.10) is similarly split as the sum of two terms, one of which is regular in the upper half plane and another one is regular in the lower half-plane.

Note that the function $\sqrt{k+\mathrm{i}\lambda_0}\hat{L}_1^*[\hat{w}_0(k)]$ is regular in the upper half plane and $\hat{w}_0(k)/\sqrt{k-\mathrm{i}\lambda_0}$ has only a pole of order N at $k=-\mathrm{i}\beta$ (i.e. in the lower half-plane). The required decomposition can be obtained by subtracting a rational fraction from the last expression and adding it to the first.

Let be

$$\frac{\hat{w}_0(k)}{\sqrt{k-\mathrm{i}\lambda_0}} = -\frac{1}{\sqrt{k-\mathrm{i}\lambda_0}} + (k+\mathrm{i}\beta)^{-N} g(k) \tag{3.11}$$

where

$$g(k) = \frac{(k-\mathrm{i}\beta)^N}{\sqrt{k-\mathrm{i}\lambda_0}} = \sum_{n=0}^{N-1} \frac{1}{n!} g^{(n)} (k+\mathrm{i}\beta)^n + R_N(k), \tag{3.12}$$

so that $(k+\mathrm{i}\beta)^{-N} R_N(k)$ is analytic in the lower half-plane.

Taking into account the conditions (2.10), we conclude that

$$\frac{\hat{s}_1^*(k)}{\sqrt{k-\mathrm{i}\lambda_0}} = -\frac{\hat{w}_0(k)}{\sqrt{k-\mathrm{i}\lambda_0}} + \sum_{n=0}^{N-1} \frac{g^{(n)}(k+\mathrm{i}\beta)^{n-N}}{n!}, \tag{3.13}$$

$$\sqrt{k+i\lambda_0}\hat{f}(k) = \omega_1 \frac{6\mathrm{i}(1-\sigma)\omega_0\sqrt{k+\mathrm{i}\lambda_0}}{\beta^2\theta^2 c_T^2} \frac{d\hat{w}_0(k)}{dk} - \sum_{n=0}^{N-1} \frac{g^{(n)}(k+\mathrm{i}\beta)^{n-N}}{n!}. \tag{3.14}$$

After performing some manipulations we get

$$\hat{L}_0^*[\hat{w}_1(k)] = -\omega_1 \frac{12(1-\sigma)N\omega_0}{\beta\theta^2 c_T^2} \frac{(k-\mathrm{i}\beta)^{N-1}}{(k+\mathrm{i}\beta)^{N+1}} - \frac{1}{\sqrt{k+\mathrm{i}\lambda_0}} \sum_{n=0}^{N-1} \frac{g^{(n)}(k+\mathrm{i}\beta)^{n-N}}{n!}. \tag{3.15}$$

For example, for $N=1$

$$g^{(0)} = -\frac{2\mathrm{i}\beta}{\sqrt{-\mathrm{i}(\beta+\lambda_0)}}, \tag{3.16}$$

$$\hat{L}_0^*[\hat{w}_1(k)] = -\omega_1 \frac{12(1-\sigma)\omega_0}{\beta\theta^2 c_T^2} \frac{1}{(k+\mathrm{i}\beta)^2} + \frac{2\mathrm{i}\beta}{\sqrt{(\lambda_0-\mathrm{i}k)(\beta+\lambda_0)}(k+\mathrm{i}\beta)}. \tag{3.17}$$

Equation (3.15) is an inhomogenous linear ordinary differential equation with respect to the function $\hat{w}_1$. This equation has a singular point in the upper half-plane (at $k=\mathrm{i}\beta$). The solution of such equation generally has a singularity at the singular point, so we must find a condition, which guarantees the existence of a solution regular in the upper half-plane. This condition will enable to find ω_1.

4. Solvability condition for the first-order term equation

Rewrite (3.15) in the form

$$\hat{L}_0^*[\xi(k)] = \hat{f}(k), \tag{4.1}$$

where $\hat{f}(k)$ is the function obtained in the previous section (namely, the r.-h.s. of (3.15)), and ξ stands for $\hat{w}_1$. The function $\hat{f}$ is by construction analytic in the upper k-plane, particularly at the point $k = \mathrm{i}\beta$. We need a condition providing the analiticity of ξ at this point.

Let ξ be analytic at $k = \mathrm{i}\beta$. Introduce the notation $\tau = k - \mathrm{i}\beta$. Represent $\hat{f}(k)$ and $\xi(k)$ near the point $\tau = 0$ as power series in τ:

$$\hat{f}(k) = \sum_{n=0}^{\infty} \hat{f}_n \tau^n, \tag{4.2}$$

$$\xi(k) = \sum_{n=0}^{\infty} \xi_n \tau^n; \tag{4.3}$$

Equation (4.1) can be rewritten in the form of a difference equation:

$$K[\xi_n] \equiv P_0(n)\xi_n + P_1(n)\xi_{n-1} + P_2(n)\xi_{n-2} = 2\mathrm{i}\beta \hat{f}_{n-1} \tag{4.4}$$

where $n = 1, 2, 3 \ldots$ and the coefficients are

$$\begin{aligned} P_0(n) &= -4\beta^2 n(n^2 - N^2) \\ P_1(n) &= 2\mathrm{i}\beta n(n-1)(2n-1) \\ P_2(n) &= n(n-1)(n-2), \end{aligned}$$

N is the number of wedge mode (see (3.3)).

The term ξ_0 can be chosen arbitrary, all others must be found recursevely by using (4.4). Note, that $P_0 = 0$ for $n = N$. This means that generally the nth term cannot be calculated. However, if $\hat{f}_n$ has some specific property, namely, if

$$2\mathrm{i}\beta \hat{f}_{N-1} - P_1(N)\xi_{N-1} - P_2(N)\xi_{N-2} = 0, \tag{4.5}$$

then the term ξ_N can be chosen arbitrary and all other terms can be calculated using (4.4).

The restriction (4.5) is not simple to check, because ξ_{N-1} and ξ_{N-2} depend on all $\hat{f}_n$ for $n = 0 \ldots N-2$. Below we formulate a simplier relation involving all $\hat{f}_n$ for $n = 0 \ldots N-1$ in explicit form.

The solution of difference equation (4.4) can be represented in the form

$$\xi_n = 2\mathrm{i}\beta \sum_{m=0}^{\infty} \hat{f}_m G_{m,n} + \hat{h}_n, \tag{4.6}$$

where $\hat{h}_n$ is a solution of the homogenous equation and G is Green's function with the property

$$K[G_{m,n}] = \begin{cases} 1 & \text{for } n = m+1 \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

Let be $m < N$ and $n < N$. It is easy to show directly, that in this case $G_{m,n} = 0$ for $n \le m$. On the other hand, $G_{m,n}$ for $n > m+1$ is a solution of homogeneous difference equation $K[G_{m,n}] = 0$ with respect to n.

Difference equation does not provide a convenient representation of $G_{m,n}$. Fortunately, operator K can be factorized:

$$K = -nK^+K^- = -nK^-K^+, \tag{4.8}$$

where

$$\begin{aligned} K^-[\xi_n] &\equiv 2\beta(n-N)\xi_n - \mathrm{i}(n-1)\xi_{n-1}, & (4.9)\\ K^+[\xi_n] &\equiv 2\beta(n+N)\xi_n - \mathrm{i}(n-1)\xi_{n-1}. & (4.10)\end{aligned}$$

It is obvious that the sequences ξ_n^+ and ξ_n^-, for which $K^\pm[\xi_n^\pm] = 0$ are the solutions of homogeneous equation $K[\xi_n] = 0$. Such sequences can be found explicitly from (4.9), (4.10) in terms of factorials. For example, if $n > m$, $\xi_m^\pm = 1$ and both $m+1$ and n are either greater or smaller than N, then

$$\begin{aligned} \xi_n^+ &= \frac{(n-1)(n-2)\dots m}{(n+N)(n-1+N)\dots(m+1+N)}\left(\frac{\mathrm{i}}{2\beta}\right)^{n-m}\\ \xi_n^- &= \frac{(n-1)(n-2)\dots m}{(n-N)(n-1-N)\dots(m+1-N)}\left(\frac{\mathrm{i}}{2\beta}\right)^{n-m}\end{aligned}$$

We seek $G_{n,m}$ for $m < n$ in the form of a linear combination of ξ_n^+ and ξ_n^-. One can check directly, that Green's function can be written in the form

$$G_{m-1,n} = \frac{1}{8\beta^2 n^2 (2\mathrm{i}\beta)^{(n-m)}}\left[\frac{(m+1)(m+2)\dots(n-1)}{(N-n)(N-n+1)\dots(N-m)} - \right.$$

$$\left.\frac{(m+1)(m+2)\dots(n-1)}{(-N-n)(-N-n+1)\dots(-N-m)}\right] \tag{4.11}$$

for $m < n < N$.

Now we can calculate ξ_{N-1}, ξ_{N-2} and apply (4.5). Elementary, but tedious calculations show that no singularity appears for $n = N$ if and only if

$$\sum_{n=1}^{N} \frac{N!}{n!(N-n)!} f_{n-1} (2\mathrm{i}\beta)^{n-1} = 0. \tag{4.12}$$

Equation (4.12) is the condition of solvability for the inhomogenous equation (4.1).

To conclude this section it would be useful to describe the connection of the results obtained in the Fourier domain with those in x domain.

It is known (see McKenna et al. (1974)) that the expressions for eigenfunctions of a slender wedge contain Laguerre polynomials:

$$w_n(x) = \mathrm{e}^{-\beta x} L_n^1(2\beta x). \tag{4.13}$$

Note that the first term in (3.2) can be reduced to (4.13) by means of the inverse Fourier transform and the known integral representation of Laguerre polynomials. The value of n in (4.13) corresponds to $\gamma - 1$ in (3.2).

Moreover, let $\hat{f}(k)$ be the Fourier transform of $f(x)$. Then condition (4.12) is equivalent to the relation

$$\int_0^\infty f(x)\mathrm{e}^{-\beta x} L_n^1(2\beta x)\, dx = 0, \tag{4.14}$$

which follows from the usual solvability conditions for ODEs.

5. Final results

The condition of solvability (4.12) applied to equation (3.15) provides the first-order correction to the circular frequency in the form

$$\omega_1 = \frac{c_T^2 \beta^3 \theta^2}{3(1-\sigma)\omega_0 N} \sum_{m=0}^{N-1} \frac{N!(2\mathrm{i}\beta)^m}{m!(m+1)!(N-m-1)!} \times$$

$$\times \frac{\mathrm{d}^m}{\mathrm{d}k^m} \left(\frac{1}{\sqrt{k + \mathrm{i}\lambda_0}} \sum_{n=0}^{N-1} \frac{g^{(n)}}{n!(k+\mathrm{i}\beta)^{(N-n)}} \right)\Bigg|_{k=\mathrm{i}\beta} \tag{5.1}$$

The last formula seems rather complicated. Consider the case $N = 1$, so giving

$$\omega_1 = -\frac{\beta^3 \theta^2 c_T^2}{3(1-\sigma)\omega_0} \frac{1}{\beta + \lambda}, \tag{5.2}$$

where λ is the square root of r.-h.s. of (1.3) with the branch chosen to be positive real or negative imaginary.

Note that in the subsonic case this formula describes change in the velocity of a wedge mode. In the supersonic case, one can calculate the imaginary part of ω_1:

$$Im[\omega_1] = \frac{\beta^3 \theta^2 c_T^2}{3(1-\sigma)\omega_0} \frac{\lambda}{\beta^2 - \lambda^2}, \tag{5.3}$$

which describes the attenuation of a wedge mode, in addition to the change in velocity given by $Re[\omega_1]$.

Using equations (1.3), (3.5), (3.8) and (5.2), calculations have been carried out of the ratio $C = c_{wat}/c_{vac} = 1 + \varepsilon\omega_1/\omega_0$ between wedge wave velocities in a brass wedge in water and in the same wedge in vacuum. The following parameters have been used: $\rho_s = 8600\,\mathrm{kg/m^3}$, $\rho_l = 1000\,\mathrm{kg/m^3}$, $c_0 = 1478\,\mathrm{m/s}$, $c_l = 4350\,\mathrm{m/s}$, $c_t = 2127\,\mathrm{m/s}$, and $\sigma = 0.343$. The results are shown in Fig.2 as a function of wedge angle θ (curve — C_1). For comparison, the corresponding ratio following from the geometrical acoustics theory (Krylov (1998)) is displayed in the same picture (curve — C_2). One can see that the agreement between the present theory and the geometrical acoustics one is reasonably good. The discrepancy at smaller angles θ can be explained by the fact that, according to equation (3.9), the parameter ε is no longer small in this region and the present theory becomes inaccurate.

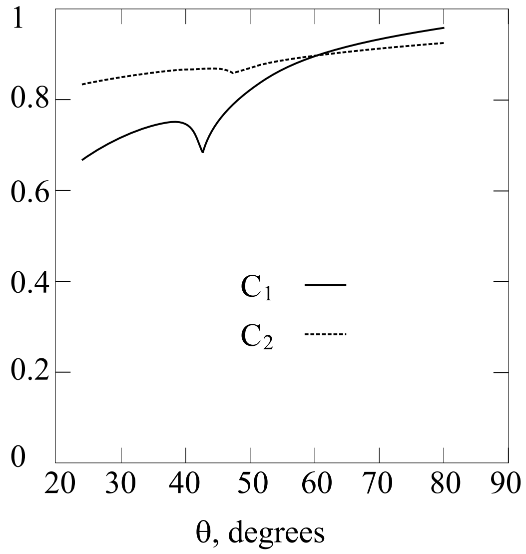

Figure 2. Calculated values of $C = c_{wat}/c_{vac}$ as functions of wedge angle θ for a brass wedge in water: curve C_1 — present theory, curve C_2 — geometrical acoustics theory (Krylov (1998)).

6. Conclusions

It has been demonstrated that the problem of flexural wave propagation along the tip of an immersed wedge can be formulated in terms of functional-differential equation (FDE) which is similar to the classical Wiener-Hopf equation. The first order perturbation solution of this equation has been derived for the case of light liquid loading. The comparison of the perturbation solution applied for a brass wedge in water with the corresponding geometrical acoustics calculations shows that the agreement is reasonably good for relatively large values of wedge apex angle. The increased discrepancy at smaller wedge angles can be explained by the fact that liquid loading in this case is no longer light.

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Appendix A.

Here we derive a homogenous equation for a thin elastic wedge in vacuum starting from the functional equation.

The contents of this section is slightly aside from the rest of the paper, so that some notations below are different from those above.

(*a*) *Formulation of the problem*

Let us describe stationary oscillations of the wedge using the displacement vector $u_i(x, y, z, t) = u_i(x, y)\mathrm{e}^{-\mathrm{i}(\beta z+\omega t)}$, where the index i runs over the numbers $1, 2, 3$ corresponding to coordinates x, y, z respectively (see Fig. 1).

The equation of motion of the solid is

$$\tau_{ij,j} + \rho_s \omega^2 u_i = 0, \tag{A 1}$$

where τ_{ij} is the stress tensor. For an isotropic solid

$$\tau_{ij} = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}\delta_{i,j}u_{k,k} + \frac{E}{2(1+\sigma)}(u_{i,j} + u_{j,i}). \tag{A 2}$$

Here we use the notation implying summation over the repeated indexes, $\delta_{i,j}$ is the Kronecker's delta. Symbol ”$, i$” denotes differentiation.

Boundary conditions corresponding to the free boundary are

$$\tau_{ij}n_j = 0, \tag{A 3}$$

where n_j is the vector of interior normal to the boundary.

We seek an antisymmetric solution with respect to the plane $x \cos(\theta/2) - y \sin(\theta/2) = 0$, that decreases in x direction. We are interested in waveguide modes decreasing exponentially, but the results obtained can be generalized to some other cases.

(*b*) *Derivation of functional equations*

The procedure described below is similar to that performed by Shanin (1997) for a wedge with the apex angle close to π.

Let us apply the second Green's formula to the problem:

$$\int [u_i\tau'_{ij}n_j - u'_i\tau_{ij}n_j]\mathrm{d}s = 0, \tag{A 4}$$

where u'_i is an arbitrary solution of (A 1), τ'_{ij} is corresponding stress tensor. Integration is performed over the surface including a large fragment of the wedge.

Let u'_i be a plane wave travelling from the edge, with β the z-component of its wave vector. The integration in (A 4) can be performed along lines that are projections of the sides onto the plane $z = 0$.

There are three bulk modes in a solid. Each of them can replace u'_i:

$$A_i^1 = -\frac{\mathrm{i}k_T^2}{\rho_s\omega^2}\left\{\begin{array}{c} k \\ \sqrt{k_L^2 - k^2 - \beta^2} \\ \beta \end{array}\right\}\mathrm{e}^{\mathrm{i}(kx+y\sqrt{k_L^2-k^2-\beta^2}+\beta z)}, \tag{A 5}$$

$$A_i^2 = -\frac{\mathrm{i}k_T^2}{\rho_s\omega^2}\left\{\begin{array}{c} -\sqrt{k_T^2 - k^2 - \beta^2} \\ k \\ 0 \end{array}\right\}\mathrm{e}^{\mathrm{i}(kx+y\sqrt{k_T^2-k^2-\beta^2}+\beta z)}, \tag{A 6}$$

$$A_i^3 = -\frac{\mathrm{i}k_T^2}{\rho_s\omega^2}\left\{\begin{array}{c}\beta k\\ \beta\sqrt{k_T^2-k^2-\beta^2}\\ \beta^2-k_T^2\end{array}\right\}\mathrm{e}^{\mathrm{i}(kx+y\sqrt{k_T^2-k^2-\beta^2}+\beta z)}, \qquad (\mathrm{A}\,7)$$

where

$$k_L^2 = \omega^2\rho_s\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)},$$
$$k_T^2 = \omega^2\rho_s\frac{2(1+\sigma)}{E}.$$

To provide the convergency of the integral in (A 4), we chose $k = k' + \mathrm{i}k''$, where k' is positive and k'' is small and positive. having positive imaginary part. The branches of square roots are chosen to have positive imaginary parts.

Let $u_i(x)$ be the values of u_i on the line $y = 0$. The values on the other side of the wedge can be calculated since the solutions are known to be antisymmetric. Note that on the sides of the wedge, the terms A_i^j are exponentials. Introduce the notation

$$\hat{u}_i(k) = \int_0^\infty u_i(x)\mathrm{e}^{\mathrm{i}kx}dx, \qquad (\mathrm{A}\,8)$$

$$\begin{aligned}\bar{k} &= k\cos\theta + \sqrt{k_L^2-k^2-\beta^2}\sin\theta,\\ \tilde{k} &= k\cos\theta + \sqrt{k_T^2-k^2-\beta^2}\sin\theta.\end{aligned} \qquad (\mathrm{A}\,9)$$

Note that k is the projection of wavenumber of u_i' to the side of the angle coinciding with x - axis; $\bar{k}$ and $\tilde{k}$ are projections of a wavevector to the other side (see Fig. 3).

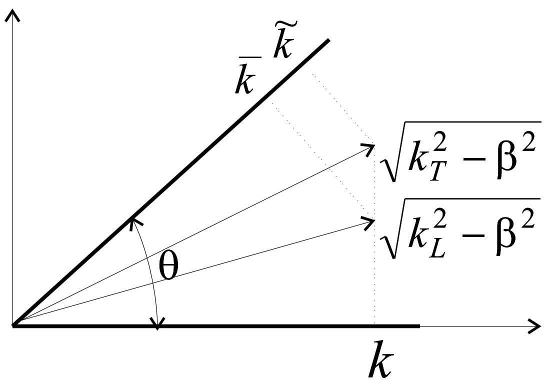

Figure 3. To the definition of $\bar{k}$ and $\tilde{k}$

After calculating τ'_{ij} and performing the integration in (A 4) we obtain for A^1, A^2 and A^3 respectively:

$$l_{1i}(k)\hat{u}_i(k) + l_{1i}(\bar{k})\hat{u}_i(\bar{k}) - 2l_{12}(\bar{k})\hat{u}_2(\bar{k}) = 0, \qquad (\mathrm{A}\,10)$$
$$l_{2i}(k)\hat{u}_i(k) + l_{2i}(\tilde{k})\hat{u}_i(\tilde{k}) - 2l_{22}(\tilde{k})\hat{u}_2(\tilde{k}) = 0, \qquad (\mathrm{A}\,11)$$
$$l_{3i}(k)\hat{u}_i(k) + l_{3i}(\tilde{k})\hat{u}_i(\tilde{k}) - 2l_{32}(\tilde{k})\hat{u}_2(\tilde{k}) = 0, \qquad (\mathrm{A}\,12)$$

where

$$l_{ij}(k) = \begin{pmatrix} 2k\sqrt{k_L^2 - k^2 - \beta^2}; & k_T^2 - 2(k^2 + \beta^2); & 2\beta\sqrt{k_L^2 - k^2 - \beta^2} \\ -k_T^2 + 2k^2 + \beta^2; & 2k\sqrt{k_T^2 - k^2 - \beta^2}; & \beta k \\ 2\beta\sqrt{k_T^2 - k^2 - \beta^2}; & 2\beta(k_T^2 - k^2 - \beta^2); & (2\beta^2 - k_T^2)\sqrt{k_T^2 - k^2 - \beta^2} \end{pmatrix}$$

Equations (A 10)–(A 12) are the functional equations for antisymmetrical oscillations of an arbitrary elastic wedge. They are exact. These equations must be completed by the restrictions concerning the behaviour of functions $\hat{u}_i(k)$ in the complex plane. As follows from (A 8), the function $\hat{u}_i(k)$ must be analytic in the upper half plane and must decrease at infinity no slower than $1/k$.

(*c*) *The limit of thin plates*

We study the solution of equations (A 10)–(A 12) supposing that

$$\frac{k_T}{\beta} = \epsilon, \qquad \frac{k_L}{\beta} = \alpha\epsilon, \tag{A 13}$$

where $\alpha = k_L/k_T = 1 - [2(1-\sigma)]^{-1}$ and

$$\epsilon \sim \theta^2. \tag{A 14}$$

After introducing the notations

$$\xi = \frac{k}{\beta}, \qquad \bar{\xi} = \frac{\bar{k}}{\beta}, \qquad \tilde{\xi} = \frac{\tilde{k}}{\beta} \tag{A 15}$$

equations (A 10)–(A 12) can be rewritten in the form:

$$\begin{gathered} 2\xi\sqrt{\alpha\epsilon - \xi^2 - 1}\hat{u}_1(\xi) + 2\bar{\xi}\sqrt{\alpha\epsilon - \xi^2 - 1}\hat{u}_1\bar{\xi} + \\ [\epsilon - 2(\xi^2+1)]\hat{u}_2(\xi) - [\epsilon - 2(\bar{\xi}^2+1)]\hat{u}_2(\bar{\xi}) + \\ 2\sqrt{\alpha\epsilon - \xi^2 - 1}\hat{u}_3(\xi) + 2\sqrt{\alpha\epsilon - \bar{\xi}^2 - 1}\hat{u}_3(\bar{\xi}) = 0, \end{gathered} \tag{A 16}$$

$$\begin{gathered} (-\epsilon + 2\xi^2 + 1)\hat{u}_1(\xi) + (-\epsilon + 2\tilde{\xi}^2 + 1)\hat{u}_1(\tilde{\xi}) + \\ 2\xi\sqrt{\epsilon - \xi^2 - 1}\hat{u}_2(\xi) - 2\tilde{\xi}\sqrt{\epsilon - \tilde{\xi}^2 - 1}\hat{u}_2(\tilde{\xi}) + \\ \xi\hat{u}_3(\xi) + \tilde{\xi}\hat{u}_3(\tilde{\xi}) = 0, \end{gathered} \tag{A 17}$$

$$\begin{gathered} 2\xi\sqrt{\epsilon - \xi^2 - 1}\hat{u}_1(\xi) + 2\tilde{\xi}\sqrt{\epsilon - \tilde{\xi}^2 - 1}\hat{u}_1(\tilde{\xi}) + \\ 2(\epsilon - \xi^2 - 1)\hat{u}_2(\xi) - 2(\epsilon - \tilde{\xi}^2 - 1)\hat{u}_2(\tilde{\xi}) + \\ (2-\epsilon)\sqrt{\epsilon - \xi^2 - 1}\hat{u}_3(\xi) + (2-\epsilon)\sqrt{\epsilon - \tilde{\xi}^2 - 1}\hat{u}_3(\tilde{\xi}) = 0. \end{gathered} \tag{A 18}$$

Suppose that the solution can be written in the form of the following asymptotic series:

$$\hat{u}_1(\xi) = \theta \hat{u}_1^1(\xi) + \theta^3 \hat{u}_1^3(\xi) + O[\theta^5] \quad \text{(A 19)}$$
$$\hat{u}_2(\xi) = \hat{w}(\xi) + \theta^2 \hat{u}_2^2(\xi) + O[\theta^4] \quad \text{(A 20)}$$
$$\hat{u}_3(\xi) = \theta \hat{u}_3^1(\xi) + \theta^3 \hat{u}_3^3(\xi) + O[\theta^5] \quad \text{(A 21)}$$

We expand equations (A 16)–(A 18) as power series in θ using (A 14). Note that equations (A 16) and (A 18) coincide at $O(1)$ and $O(\epsilon)$. Therefore, equation (A 18) must be replaced by the difference of (A 16) and (A 18) divided by ϵ.

At the first order, the system is equivalent to the conditions

$$\hat{u}_1^1 = -\left(\hat{w}(\xi) + \frac{\xi}{2}\frac{\mathrm{d}\hat{w}(\xi)}{\mathrm{d}\xi}\right), \quad \text{(A 22)}$$
$$\hat{u}_3^1 = -\frac{1}{2}\frac{\mathrm{d}\hat{w}(\xi)}{\mathrm{d}\xi}. \quad \text{(A 23)}$$

At the second order, the equations are satisfied identically.

The third order yields:

$$\begin{gathered} 24(\xi\hat{u}_1^3 + \hat{u}_3^3 + \xi_2^2) + 12(1+\xi^2)\frac{\mathrm{d}\hat{u}_2^2}{\mathrm{d}\xi} + 8\xi\hat{w} + \\ (7 - 6\frac{\epsilon}{\theta^2} + 19\xi^2)\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi} + 9\xi(1+\xi^2)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}\xi^2} + \\ (1 + 2\xi^2 + \xi^4)\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}\xi^3} = 0, \end{gathered} \quad \text{(A 24)}$$

$$\begin{gathered} 12[(1+2\xi^2)\hat{u}_1^3 + \xi\hat{u}_3^3 + (1+2\xi^2)\hat{u}_2^2 + \xi(1+\xi^2)\frac{\mathrm{d}\hat{u}_2^2}{\mathrm{d}\xi}] + \\ 4(1+2\xi^2)\hat{w} + \xi(16 - 6\frac{\epsilon}{\theta^2} + 19\xi^2)\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi} + \\ 3(1 + 4\xi^2 + 3\xi^4)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}\xi^2} + \xi(1 + 2\xi^2 + \xi^4)\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}\xi^3} = 0 \end{gathered} \quad \text{(A 25)}$$

$$\begin{gathered} 12[2\xi(\alpha-1)\hat{u}_1^3 + (-4 + 2\alpha - 2\xi^2)\hat{u}_3^3 + 2\xi(\alpha-1)\hat{u}_2^2 + \\ (\alpha-2)(1+\xi^2)\frac{\mathrm{d}\hat{u}_2^2}{\mathrm{d}\xi}] + 8\xi(\alpha-1)\hat{w} + (-20 + 19\alpha + \\ \frac{6\epsilon(1-\alpha)}{\theta^2} - 32\xi^2 + 31\alpha\xi^2)\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi} + (-24\xi(1+\xi^2) + \\ 21\alpha\xi(1+\xi^2))\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}\xi^2} + (3\alpha - 4)(1 + 2\xi^2 + \xi^4)\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}\xi^3} = 0. \end{gathered} \quad \text{(A 26)}$$

The values $\hat{u}_1^3$, $\hat{u}_3^3$, $\hat{u}_2^2$ and $\mathrm{d}u_2^2/\mathrm{d}\xi$ can be excluded from the equations (A 24)–(A 26). The result is the equation for $\hat{w}$:

$$(1+2\xi^2+\xi^4)\frac{\mathrm{d}^3\hat{w}}{\mathrm{d}\xi^3}+6\xi(1+\xi^2)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}\xi^2}+ \tag{A 27}$$

$$6[(1+\xi^2)+(1-\sigma)(\frac{k_T^2}{\beta^2\theta^2}-1)]\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi}=0.$$

Thus, we have obtained the equation $\hat{L}[\hat{w}]=0$ with the operator $\hat{L}$ in the form (2.8).

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