Diffraction series on a sphere and conical asymptotics

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The problem of finding field components for a conical diffraction problem is studied. All components except the spherical wave diffracted by cone tip are under consideration. As a starting point, the integral formula (7) derived by Babich et.al. is used. A geometrical optics approximation of the spherical Green's function is constructed in the form of diffraction series. There is a finite set of terms of the diffraction series on sphere, to each of which the conical field components correspond. Formula (7) is simplified, giving a convenient field representation (26). In many cases further simplification can be performed, giving formula (30) directly converting the terms of the diffraction series on sphere into the field component in the 3D space.

1 Introduction

The problem of scattering of a plane harmonic wave by a conical obstacle with ideal boundary conditions (Dirichlet or Neumann) is studied. As an example of such a cone here we take a flat cone with the angle Φ at the vertex. However, most of the consideration performed here can be easily applied to any cone of polygonal cross-section. Some ideas can also be useful for the cones of arbitrary cross-sections.

Usually, the main subject of investigation for the conical obstacles is the *diffraction coefficient*, which is the amplitude of scattering into the spherical wave. The diffraction coefficient depends on two directions: the direction of incidence and the direction of scattering. Both directions can be treated as points on a unit sphere. The problem of computing the diffraction coefficient has been studied by Smyshlyaev and co-authors [4]. The result is the so-called Smyshlyaev's formula expressing the diffraction coefficient as a contour integral of the Green's function on a sphere with scatterers.

The spherical wave is a very interesting part of the diffracted wave, but there are many other types of waves emerging as as a result of diffraction of a plane wave by a conical obstacle. Namely, if a polygonal cone is considered, one can mention the geometrically reflected plane waves, cylindrical waves scattered by the edges and multiply diffracted waves (i.e. the waves scattered by one edge and then re-scattered by another edge). Each of these components can be computed by the ray technique, however there is a lot of penumbral zones corresponding to the geometrical boundaries of visibility of each wave component, and finding of the asymptotics in the penumbral zones is a sophisticated problem. Here we are interested in all wave components except the spherical wave.

The case studied in details in the literature is the "conical" penumbral zone where the wave front of the cylindrical scattered wave touches the wave reflected from the surface of a smooth cone. This field is known to have structure described by the function of parabolic cylinder or generalized fresnel integral [1, 2, 3].

Our aim here is to find a convenient formula to describe the wave asymptotics in the numerous penumbral zones. For this we follow Smyshlyaev's approach, and express the field in terms of the Green's function on the sphere. For the spherical problem we construct an asymptotics in the form of diffraction series, i.e. we consider the sequence of successive diffractions acts, and each diffraction term on the sphere corresponds to some wave contribution for the initial conical problem. For each term we restrict the consideration to only the leading asymptotics.

2 Problem formulation and representation of the field

The problem is formulated as follows. Let the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

be valid in the 3D space (x, y, z), from which a conical obstacle Ω is removed. The tip of the cone coincides with the origin of the coordinate system The time dependence of all values has form $e^{-2\pi f t}$, and it is omitted henceforth. The direction of incidence (the direction *to* the source) is denoted by ω_0, the direction of scattering is denoted by ω. Both ω and ω_0 can be treated as points on a unit sphere. The incident wave has form

$$u^{\rm in} = \exp\{-ikr\cos(\theta(\omega, \omega_0))\}, \tag{2}$$

where $\theta(\omega,\omega_0)$ is the angular distance between the points ω and ω_0. Here $r=\sqrt{x^2+y^2+z^2}$ is the radial variable. The total field is a sum of the incident field and the scattered field:

$$u = u^{\rm in} + u^{\rm sc}. \tag{3}$$

The total field obeys Dirichlet or Neumann boundary conditions on the boundary $\partial\Omega$. Also, the field obeys vertex conditions

$$u = o(r^{-3/2}). \tag{4}$$

Also the scattered field should obey the radiation condition, and the total field should obey edge conditions on the edges of the cone Ω. In our simplest case Ω is the flat cone (see Fig. 1).

$$\Omega = \{(x,y,z)\,|\, z=0,\, x>0,\, 0<y<x\tan\Phi\}. \tag{5}$$

We assume here that $\Phi < \pi/2$. Φ is the "opening angle" of the flat cone. This cone has two edges, for which the edge condition has form

$$u = O(\rho^{1/2}), \tag{6}$$

where ρ is the distance between the observation point and the edge.

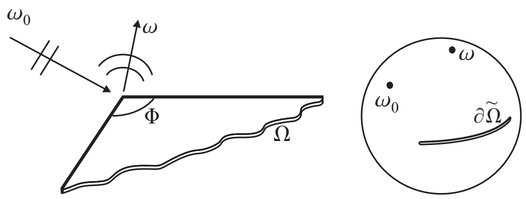

Figure 1: Diffraction by a flat cone

The total field obeying the conditions of the problem can be found by using the formula taken from [4]:

$$u(r,\omega;\omega_0) = 2e^{3\pi i/4}\sqrt{\frac{2\pi}{kr}}\times$$

$$\int_\gamma J_\nu(kr)e^{-i\pi\nu/2}g(\omega,\omega_0,\nu)\nu d\nu \tag{7}$$

The observation point is described by the coordinates (r,ω), J_ν is the Bessel's function. Contour γ is shown in Fig. 2. Function $g(\omega,\omega_0,\nu)$ is the Green's function on the sphere for the Laplace–Beltrami equation

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) g = \delta(\omega-\omega_0). \tag{8}$$

where $\tilde{\Delta}$ is the Laplace–Beltrami differential operator on sphere acting on variable ω. Function g obeys Dirichlet or Neumann boundary condition (depending on the initial problem) on the contour $\partial\tilde{\Omega}$, which is the boundary of the cross-section of the conical body Ω by the unit sphere. Equation (8) is valid outside the cross-section $\tilde{\Omega}$.

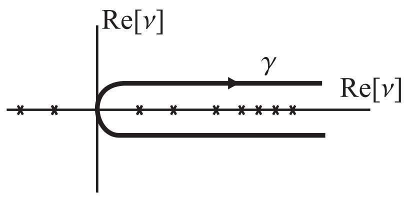

Figure 2: Contour of integration in (7)

Formula (7) can be understood as the result of variables separation. Contour γ encircles the singularities of g on the positive half-axis corresponding to the spectrum of the operator $\tilde{\Delta}-1/4$ in the complement of $\tilde{\Omega}$ with corresponding boundary conditions.

3 Ray asymptotics of the spherical Green's function

To simplify formula (7) we should make some statements related to the asymptotics of the Green's function g as a function of ν. For this, consider the process, developing in time, i.e. study the wave equation on the sphere

$$\left(\tilde{\Delta} - \frac{\partial^2}{\partial t^2}\right) w = 0. \tag{9}$$

Let the field be generated by a delta pulse emitted by a point source located at the point ω_0. It is quite clear that the pulse travels with a unit velocity along the sphere. The pulse first travels from the source directly, then it hits the scatterer (the contour $\delta\tilde{\Omega}$) and is scattered for the first time. As a result, the reflected wave and the edge wave scattered by one of the edges is produced. Then this wave hits another edge producing the secondary diffracted wave etc. Let the whole signal $w(t)$ be "recorded" at the spherical observation point ω.

The function $g(\omega, \omega_0, \nu)$ is the Fourier transform of $w(t)$ taken at the point $\sqrt{\nu^2 - 1/4}$.

According to this concept, the function g has the following asymptotics

$$g(\omega, \omega_0, \nu) \sim \sum_m g_m(\omega, \omega_0, \nu) = \sum_m e^{i\kappa(\nu)\theta_m} G_m(\nu), \tag{10}$$

where the summation is held over all diffraction trajectories going from ω_0 to ω, θ_m is the length of the m-th trajectory,

$$\kappa(\nu) = \begin{cases} \nu, & \mathrm{Im}[\nu] > 0, \\ -\nu, & \mathrm{Im}[\nu] < 0, \end{cases} \tag{11}$$

$G_m(\nu)$ are slowly varying functions of ν (comparatively to the exponentials). The terms are sorted according to the rule $\theta_{m+1} \geq \theta_m$. The first term (we assign index $m = 0$ to it) is the direct wave coming along the shortest trajectory.

We will see below that the terms obeying the relation $\theta_m < \pi$ correspond to the scattered field components of the initial conical problem.

Let us compute several first terms of the series (10) in some cases.

The zeroth term and the reflected wave. The direct wave has form

$$g_0(\omega, \omega_0, \nu) = -\frac{e^{i\pi/4}}{2} \frac{e^{i\kappa\theta}}{\sqrt{2\pi\kappa \sin\theta}}, \tag{12}$$

where θ is the angular distance between ω_0 and ω. This expression was obtained by standard ray technique. Namely, the ray Ansatz $e^{i\kappa\theta}$ was matched with the outer (far field) asymptotics of the Green's function of an entire plane.

The asymptotics of the reflected ray is as follows

$$g_1(\omega, \omega_0, \nu) = -R\frac{e^{i\pi/4}}{2} \frac{e^{i\kappa\theta_1}}{\sqrt{2\pi\kappa \sin\theta_1}}. \tag{13}$$

Here R is the reflection coefficient for the surface ($R = 1$ for the Neumann boundary and $R = -1$ for the Dirichlet boundary), θ_1 is the length of the reflected ray.

This asymptotics is not valid when $\theta_1 \approx \pi$.

Diffracted edge wave. This is the ray diffracted a single time by an edge of the flat cone. To compute this term we use the locality principle: we assume that the wave approaching the edge has approximately the structure of a plane wave on a plane. Then we study the process of diffraction of a plane wave by a half-line. Finally, we perform matching between the diffracted cylindrical wave and the ray Ansatz on the sphere. The result is as follows:

$$g_2(\omega, \omega_0, \nu) = \frac{i e^{i\kappa(\nu)(\theta_{21}+\theta_{21})} T_{D,N}(\phi_1, \phi_2)}{4\pi\kappa(\nu)\sqrt{\sin(\theta_{21})\sin(\theta_{22})}} \tag{14}$$

where θ_{11} and θ_{12} are distances between the source and the edge, and between the receiver and the edge. The value $T_{D,N}(\phi_1, \phi_2)$ is the trigonometrical factor. The angles ϕ_1, ϕ_2 describe locally the scattering process. These angles are shown in Fig. 3. The values $T_{D,N}$ are taken from the planar theory of diffraction by a half-line, and they are equal to

$$T_D(\phi_1, \phi_2) = \frac{\cos(\phi_1/2)}{\sin((\phi_2 + \pi)/2) - \sin(\phi_1/2)} - \frac{\cos(\phi_1/2)}{\sin((\phi_2 - \pi)/2) - \sin(\phi_1/2)}, \tag{15}$$

$$T_N(\phi_1, \phi_2) = \frac{\cos(\phi_2/2)}{\sin((\phi_2 + \pi)/2) - \sin(\phi_1/2)} - \frac{\cos(\phi_2/2)}{\sin((\phi_2 - \pi)/2) - \sin(\phi_1/2)}, \tag{16}$$

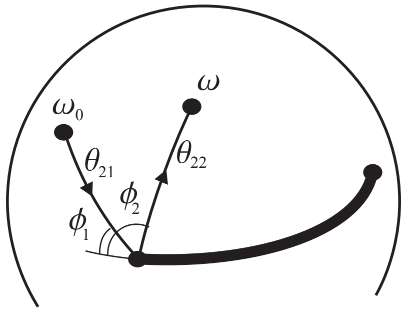

Figure 3: Diffraction by an edge

Secondary diffraction. Consider the wave going along the trajectory shown in Fig. 4. The wave goes from ω_0 to the first edge of the scatterer, then goes along the scatterer, approaches the second edge and goes to ω. In fact, the ray travels along two different trajectories. The first trajectory goes along the upper face of the scatterer, and the second one goes along the lower face.

For the Neumann case the representation is as follows:

$$g_3(\omega, \omega_0, \nu) = \frac{e^{-i\pi/4}}{4} \times [T_N(\pi, \phi_1)T_N(\phi_2, \pi) + T_N(-\pi, \phi_1)T_N(\phi_2, -\pi)] \times \frac{e^{i\kappa(\theta_{31}+\Phi+\theta_{32})}}{(2\pi\kappa)^{3/2}(\sin(\theta_{31})\sin(\Phi)\sin(\theta_{32}))^{1/2}}, \tag{17}$$

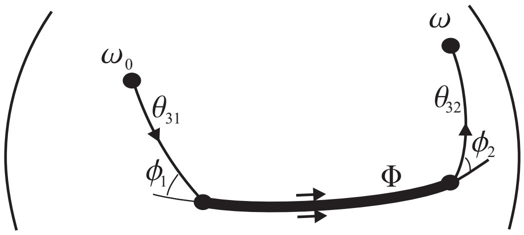

Figure 4: Secondary edge diffraction

where $\kappa = \kappa(\nu)$.

In the Dirichlet case the formula is a bit more complicated.

Penumbral zone. The expansion (14) is valid only when the geodesic arc connecting ω_0 and ω pass far enough from both edges of the scatterer. We are also interested in the case when this ray passes quite close to this point. In this case a penumbral field can be observed on the sphere. Using the technique of phase integrals we obtain the following representation of the penumbral field:

$$g_{\text{pen}}(\omega, \omega_0, \nu) = -\frac{e^{i\kappa\theta + i\pi/4}}{4\sqrt{2\pi\kappa\sin(\theta)}} I(\pm\sqrt{\kappa\Delta\theta}). \quad (18)$$

Here $\kappa = \kappa(\nu)$,

$$I(z) = \frac{2e^{-i\pi/4}}{\sqrt{\pi}} \int_z^\infty e^{i\tau^2} d\tau, \quad (19)$$

$\Delta\theta$ is the difference between the lengths of the ray going from ω_0 to ω through the edge, and the direct ray from ω_0 to ω. θ is the length of the direct ray. Sign $+$ is chosen if the direct ray is obstructed by the scatterer, and sign $-$ is chosen when the direct ray is not obstructed.

The asymptotics (19) is valid if the point ω is far from the point opposite to ω_0, i.e. θ is not close to π.

Focal point proximity. Finally, consider the case when θ is close to π, and thus ω is close to the point ω_0' opposite to ω_0. Introduce local polar coordinates near the point ω_0'. Let the point ω have coordinates (θ, ϕ) in this system. A set of geometrical rays goes from ω_0 to ω_0'. Some of the rays are obstructed by the scatterer. Let the rays which are not obstructed occupy the angles in the set (ϕ_1, ϕ_2) (see Fig. 5).

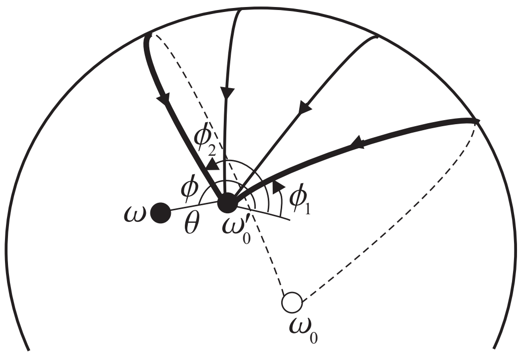

Figure 5: Observation point in the proximity of the focal point

The field in this case is found using the formula

$$g_{\text{foc}}(\omega, \omega_0, \nu) = -\frac{e^{i\pi\kappa}}{4\pi} \int_{\phi_1}^{\phi_2} \exp\{-i\kappa\,\theta\cos(\tau - \phi)\} d\tau. \quad (20)$$

4 Analysis of the integral (7)

To simplify (7) substitute Bessel's function by Sonin's integral. As the result, get

$$u(r, \omega) = e^{3\pi\, i/4}\sqrt{\frac{2}{\pi k r}} \times$$
$$\int_\Gamma e^{ikr\cos\tau} \int_{\gamma_1+\gamma_2} e^{i\nu\tau - i\nu\pi} g(\omega, \omega_0, \nu)\, \nu d\nu\, d\tau. \quad (21)$$

where contour Γ is shown in Fig. 6.

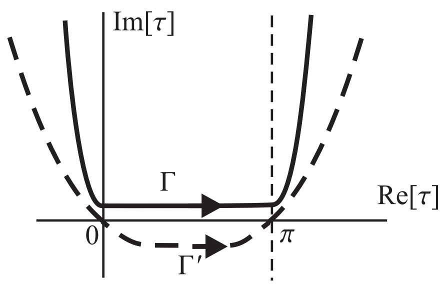

Figure 6: Contour of integration for the Bessel function

Here we split contour γ into two parts: γ_1 is a part of contour going from infinity to 0 below the real axis, and γ_2 goes from 0 to infinity along above the real axis.

Split the inner integral into two integrals I_1 and I_2, corresponding to integration along γ_1 and γ_2, respectively. According to the asymptotic estimation performed above, we can expect that the function I_1 has singularities at the points

$$\tau_m = \pi + \theta_m, \tag{22}$$

where the θ_m are the ray lengths from (10). Respectively, I_2 should have singularities at the points

$$\tau_m = \pi - \theta_m. \tag{23}$$

First consider the integral of the form (21) containing only I_1, i.e. with integration over γ_1. Since there are no singular points on the segment $(0, \pi)$, one can deform contour Γ into a contour Γ' lying in the areas of the exponential decay of the function $\exp\{ikr \cos t\}$ (see Fig. 6). Thus, the integral describes a linear combination of the functions behaving as e^{ikr} and e^{-ikr} for large r. It is well known that the function behaving as e^{-ikr} disappears when both components I_1 and I_2 are taken into account. Thus, it can be ignored. The component behaving as e^{ikr} (taken from $I_1 + I_2$) is the spherical wave diffracted by the tip of the cone. This wave is well studied by Smyshlyaev and coauthors, so we do not study it here (we remind that here we study all waves except the spherical wave). This means that the part with I_1 contains no components we are interested in.

Now consider the integral with I_2. Since some of the values θ_m can be located between 0 and π, we cannot deform Γ into Γ'. Instead, we put the straight part of the contour onto the segment $(0, \pi)$ of the real axis of τ bypassing above the singularities.

We base our consideration on the assumption that for large r the integral (21) can be estimated as a sum of contributions of the singular point belonging to the segment $(0, \pi)$, and each contribution is provided by a small proximity of the singular points, i.e.

$$u(r, \omega) = \sum_m u_m(r, \omega). \tag{24}$$

The terms u_m correspond to the terms of the spherical Green's function g_m, for which $0 < \theta_m < \pi$, i.e. there is finite number of terms in the sum (24).

Consider a vicinity of the singularity $\tau_m = \pi - \theta_m$ belonging to the segment $(0, \pi)$. Near this point approximate the cosine function by its Taylor series up to the quadratic term:

$$\exp\{ikr \cos \tau\} \approx$$

$$\exp\left\{ikr\left(-\cos\theta_m - \xi \sin\theta_m + \frac{\xi^2}{2}\cos\theta_m\right)\right\}, \tag{25}$$

$\xi = \tau - \tau_m$. Substituting this approximation into the integral and changing the order of integration, obtain a representation for the terms of (24):

$$u_m(r, \omega) = 2e^{3\pi\, i/4}\sqrt{\frac{2\pi}{kr}}\, e^{-ikr\cos\theta_m} \times$$

$$\int_0^\infty K(kr, \nu, \theta_m) e^{-i\nu\theta_m} g_m^+(\omega, \omega_0, \nu)\nu\, d\nu \tag{26}$$

where

$$K(z, \nu, \theta) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{i(\nu - z\sin\theta)\xi + \frac{i}{2}z\xi^2\cos\theta_m\right\} d\xi, \tag{27}$$

and g_m^+ is the asymptotics of g_m taken in the upper half-plane (i.e. with $\kappa(\nu) = \nu$). Performing the integration, obtain that

$$K(z, \nu, \theta) = \frac{e^{-i\pi/4}}{\sqrt{-2\pi z\cos\theta}} \exp\left\{-i\frac{(\nu - z\sin\theta)^2}{2z\cos\theta}\right\}. \tag{28}$$

The integral formula (26) with the kernel (28) is the main result of the paper. We claim that together with the asymptotics g_m^+ obtained above this formula describes uniformly the most interesting field components.

5 Some properties of formulas (26), (28)

Ray asymptotics. The function K defined by (28) and treated as a function of ν is a complex (oscillating) Gaussian centered at $z\sin\theta$ and having width $\sim \sqrt{z\cos\theta}$. We remind that according to (26) $z = kr$, thus the width of the Gaussian grows as $k \to \infty$, however, the center position grows faster. If $\sin\theta$ is not close to zero and if g_m^+ behaves as

$$\exp\{i\nu\theta_m\}\nu^{\alpha_m} \qquad \text{as} \quad \nu \to \infty, \tag{29}$$

then for kr large enough one can neglect variation of $g_m^+(\nu)\nu$ in the area covered by the width of the Gaussian and substitute the function in the integral (26) by its value at $\nu = kr\sin\theta$. In this case K acts as a delta-function $\delta(\nu - z\sin\theta)$, giving the following approximation

$$u_m(r, \omega) \approx 2e^{3\pi i/4}\sqrt{2\pi kr}\, e^{-ikr\cos\theta_m} \times$$

$$g_m^+(\omega, \omega_0, kr \sin\theta_m) \sin\theta_m. \tag{30}$$

This formula is very important. It transfers the ray (or sometimes penumbral) asymptotics on the unit sphere into asymptotics in the 3D space. A necessary condition of validity of this formula is as follows:

$$\sqrt{kr} \sin\theta_m \gg \sqrt{|\cos\theta_m|} \tag{31}$$

This relation for large kr denotes that the point is far from a *singular ray* corresponding to the conical penumbra, i.e. to the set of points where the spherical diffracted wave touches some other wave front (namely, the front of wave corresponding to the m-th diffracted wave).

Beside the relation (31), one should study the behavior of the function $g_m^+(\nu)$ in each particular case to find whether the function is smooth comparatively to the scale $\sqrt{kr \sin\theta_m}$.

Connection with the parabolic cylinder functions. If g_m^+ is a power function then the integral (26) is closely connected with the function of parabolic cylinder. Namely, if

$$g_m^+ = A(\omega, \omega_0) e^{i\nu\theta_m} \nu^s \tag{32}$$

then

$$u_m = -A \frac{2i\pi^{i\pi s/4}(kr)^{s/2}(-\cos\theta_m)^{(s+1)/2}}{(s+2)\Gamma(-s-2)\sin(\pi s)} \times$$
$$\exp\left\{-ikr\left(\cos\theta_m + \frac{\sin^2\theta_m}{2\cos\theta_m}\right)\right\} \times$$
$$D_{-s-2}\left(e^{3\pi i/4} \sin\theta_m \sqrt{\frac{kr}{-\cos\theta_m}}\right). \tag{33}$$

Thus, the asymptotics built here is closely connected with the conical penumbral asymptotics constructed and studied in [1, 2, 3]

6 Some examples of asymptotics obtained with the new formula

Wave diffracted by edge. Conical penumbra. Consider the conical penumbra. Use formula (14) for the g_m^+ component and (26) without any additional simplifications. As the result, get the expression in terms of the Fresnel integral:

$$u = -\frac{T_{D,N}(\phi_1, \phi_2) e^{ikr + i\pi/4}}{2\sqrt{2\pi kr}\sqrt{\sin\theta_{21}\sin\theta_{22}}} I\left(\pm\sqrt{\frac{kr}{2}} \sin\theta\right). \tag{34}$$

where I is defined by (19).

Proximity of the shadow of cone tip. Consider the particular case of ω_0 located at the South pole, i.e. having coordinate $\Theta = \pi$. Look for the field in the proximity of the North pole, i.e. for the point $\omega = (\Theta, \phi)$ with $\Theta \approx 0$. Use formula (20) for g_m^+ and substitute the result into (26). The result is as follows:

$$u(\Theta, \phi) = -\frac{e^{i\pi/2} e^{ikr}}{2\pi kr} \times$$
$$\int_{\phi_1}^{\phi_2} \int_0^\infty \exp\left\{i \frac{\nu^2 + (kr\Theta)^2}{2kr} - i\nu\Theta\cos(\tau - \phi)\right\} \nu \, d\nu \, d\tau. \tag{35}$$

This expression matches the Fresnel integral written for the screen with a quarter-plane cut from it.

7 Conclusions

Approximate formula (26) is the main formula connecting the spherical Green's function and the asymptotics of the components of the field diffracted by a cone. In many cases (the major exception is the conical penumbra) this formula can be reduced to a simpler expression (30).

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