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Removing false singular points as a method of solving ordinary differential equations

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A general formalism is described whereby some regular singular points are effectively removed and substantial simplifications ensue for a class of Fuchsian ordinary differential equations, and related confluent equations. These simplifications follow provided the exponents at the singular points satisfy certain relations; explicit, illustrative examples are constructed to demonstrate the ideas.

1 Introduction

Solutions to Fuchsian differential equations, and their confluent subcases, can, in principle, be utilised to solve a wide variety of problems (Ronveaux, 1995) in: mathematical physics (Levai & Williams, 1993), probability theory (Valent, 1986), acoustics (Sleeman, 1967; Arscott & Darai, 1981; Shanin, 2001a,b), conformal mapping (Nehari, 1952; Craster, 1996), and notably, various free boundary problems using a technique initiated by Polubarinova-Kochina (1939a,1939b); modern applications include Hele-Shaw flows, semi-conductor etching and solidification (Howison & King, 1989; Cummings, 1999; Hoang *et al*., 1998; Craster & Hoang, 1998). The method becomes unwieldy when more than three singular points are involved, provided none are removable in the classical sense.

One of the authors Craster (1997) has proposed a method for solving Heun's equation, that is, the Fuchsian equation with four singular points, in one specific case. If the equation has exponents $(0, 2)$ at one of the singular points, and no logarithmic terms appear in the local expansion of the solution at this point then the solution is a linear combination of functions; these are the solutions of two different Fuchsian equations with three singular points (the hypergeometric equation), with the same singular points as the initial Heun's equation, bar the point that had exponents $(0, 2)$ which is now absent, therefore this point is effectively removed from the Heun equation. The advantage of this is that solutions to the hypergeometric equation are well-known. Surprisingly, similar ideas emerge in the study of functional equations (Shanin, 2001a).

The removal of a singular point from the basic formulation provides a significant simplification; however, such points are not removable in the classical sense. The underlying theory of such equations is described in Ince (1927), and apart from Lamé's equation, the few other known solutions are summarized in Polubarinova-Kochina (1991) and Ronveaux (1995). Some hypergeometric expansion solutions to Heun equations primarily by Erdélyi (1942, 1944) are summarized in Ronveaux (1995), however it turns out that our method is distinct and leads to different classes of solutions; we briefly discuss this in § 6.

In this paper, our purpose is to justify and extend this removal of false points procedure thereby establishing the class of problems to which the method is applicable.

2 Removing a false point: an example

2.1 False singular points

An arbitrary Fuchsian equation of second order is

$$\frac{d^2Y}{d\zeta^2} = f(\zeta)\frac{dY}{d\zeta} + g(\zeta)Y, \tag{2.1}$$

where $f(\zeta)$ and $g(\zeta)$ are known rational functions, and we recall that Fuchsian equations have only regular singular points; irregular singular points occur in confluent cases when two regular singular points coalesce in a particular limiting process. We assume equation (2.1) has regular singular points (i.e. the poles of the coefficients f and g) a_i, $i = 1 \ldots v$, and that a local expansion at each singular point yields a pair of exponents (α_i, β_i) that characterise the local behaviour there (see below).

We call a singular point a_j *false* if both exponents α_j and β_j are non-negative integers and there are no logarithmic terms in the local expansions near the singular points. It is well known that such logarithmic terms generally appear in the case when the difference of the exponents is an integer, so a specific restriction must be imposed on the coefficients of the equation. The false singular point is a singularity of the coefficients, but not of the solutions of the equation.

Using an obvious transformation, one can reduce a singular point with the difference between the exponents $\alpha_j - \beta_j$ equal to an integer and without logarithmic terms to a false singular point, or even to a regular point.

There are restrictions on the coefficients of equation (2.1) so that the singular point a_j is false. Considering the simplest false point with exponents equal to 0 and 2, then, from the general theory of Fuchsian equations, the exponents follow from a characteristic equation, and local to a_j:

$$f(\zeta) = \frac{1}{\zeta - a_j} + f_0 + O(\zeta - a_j), \quad g(\zeta) = \frac{g_{-1}}{\zeta - a_j} + g_0 + O(\zeta - a_j), \tag{2.2}$$

for some constants f_0, g_{-1}, g_0. The solution corresponding to the exponent zero can be written in the form

$$Y(\zeta) = \hat{y}_0 + \hat{y}_1(\zeta - a_j) + \hat{y}_2(\zeta - a_j)^2 + O((\zeta - a_j)^3), \tag{2.3}$$

for some constants $\hat{y}_0, \hat{y}_1, \hat{y}_2$. Substituting (2.3) into the governing equation (2.1), we obtain recursive equations for the coefficients at different orders of $(\zeta - a_j)$ and this leads to the following relation:

$$g_{-1}f_0 - g_0 + (g_{-1})^2 = 0, \tag{2.4}$$

and thus to the absence of logarithmic terms local to a_j. For a single false point this relation is a quadratic equation for the parameters of the coefficients of the equation (2.1). In more difficult cases, the relation of the type (2.4) forms a complicated system of algebraic equations.

All other singular points of the equation (i.e. when the difference between the exponents is not an integer or when there are logarithmic terms) will be called *strong*. These points can be of two types. The first type has exponents α_j and β_j with $\alpha_j - \beta_j$ not equal to an integer. In this case two *fundamental solutions* can be chosen for this singular point, such that

$$Y_{j,1}(\zeta) = (\zeta - a_j)^{\alpha_j} y_1(\zeta), \qquad Y_{j,2}(\zeta) = (\zeta - a_j)^{\beta_j} y_2(\zeta), \tag{2.5}$$

where $y_1(\zeta)$ and $y_2(\zeta)$ are regular functions at $\zeta = a_j$ not equal to zero at a_j. If the singular point a_j is at infinity, then the corresponding expansions are

$$Y_{j,1}(\zeta) = \zeta^{-\alpha_j} y_1(1/\zeta), \qquad Y_{j,2}(\zeta) = \zeta^{-\beta_j} y_2(1/\zeta); \tag{2.6}$$

the functions $y_1(\zeta)$ and $y_2(\zeta)$ are regular, and non-zero, at $\zeta = 0$.

The singular points of the second type possess logarithmic behaviour; there are two exponents α_j and β_j, such that $\beta_j - \alpha_j$ is equal to a non-negative integer and the following representation of two fundamental solutions is valid:

$$Y_{j,1}(\zeta) = (\zeta - a_j)^{\alpha_j} y_1(\zeta) + \log(\zeta - a_j) Y_{j,2}(\zeta), \qquad Y_{j,2}(\zeta) = (\zeta - a_j)^{\beta_j} y_2(\zeta), \tag{2.7}$$

where again $y_1(\zeta)$ and $y_2(\zeta)$ are regular at a_j and not equal to zero at this point. If the singular point is located at infinity, then the ansatz is

$$Y_{j,1}(\zeta) = \zeta^{-\alpha_j} y_1(1/\zeta) + \log(\zeta) Y_{j,2}(\zeta), \qquad Y_{j,2}(\zeta) = \zeta^{-\beta_j} y_2(1/\zeta). \tag{2.8}$$

If both exponents are integers, but not both non-negative, and there are no logarithmic terms (i.e. the singular point is formally not false), then we name such a singular point *trivial*. As we mentioned above, an obvious change of variables reduce the trivial singular point to a false singular point.

We also consider irregular singular points of the simplest kind. These points are located at infinity and have fundamental solutions:

$$Y_1(\zeta) = e^{\lambda_1 \zeta} \zeta^{-\mu_1} \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad Y_2(\zeta) = e^{\lambda_2 \zeta} \zeta^{-\mu_2} \sum_{n=0}^{\infty} d_n \zeta^{-n}, \tag{2.9}$$

where the series are considered to be asymptotic in the appropriate sectors as $\zeta \to \infty$, $c_0 \neq 0$, $d_0 \neq 0$, see Olver (1974).

We study two types of differential equations: Fuchsian ones, that have only regular singular points (either with or without logarithmic terms) and confluent Fuchsian ones, which have only regular singular points, bar one at infinity which is irregular as described above. Differential equations of both types have coefficients $f(\zeta), g(\zeta)$ that are rational in ζ. The difference is that the coefficients of confluent Fuchsian equations are allowed to have stronger growth at infinity. In any case, the solution of the equation is a linear combination of two fundamental solutions. So, all solutions belong to a linear space of dimension two and the pair of fundamental solutions at each singular point form the basis of this space.

2.2 A solution for Heun's equation with a false singular point

Consider the Fuchsian equation with four regular singular points

$$\frac{d^2Y}{d\zeta^2} + \left(\frac{\lambda}{\zeta} + \frac{\mu}{\zeta - 1} + \frac{\nu}{\zeta - a}\right)\frac{dY}{d\zeta} + \frac{(\sigma\tau\zeta - q)Y}{\zeta(\zeta - 1)(\zeta - a)} = 0, \tag{2.10}$$

often called Heun's equation. The parameters λ, μ, ν, τ and σ are given, but the accessory parameter q and free point a are, in general, not; the parameters also must satisfy the Fuchsian constraint $\lambda + \mu + \nu - \tau - \sigma = 1$. As earlier, we denote the singular points as $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, $a_4 = a$. The point $\zeta = a$ is the false singularity in our classification; the exponents at this point are equal to 0 and 2 and thus $\nu = -1$. Using the constraint on the sum of the exponents we now have that $\mu = 2 - \lambda + \tau + \sigma$.

Posing a local expansion in the neighbourhood of a, the restriction (2.4) must be satisfied for logarithmic behaviour to be excluded. In practical terms this leads to a relation between q and a, the unknown accessory parameter and singular point position, as the quadratic

$$q^2 - q(a(2\sigma\tau + \lambda + \mu - 2) + (1 - \lambda)) + \sigma\tau a(a(\sigma\tau - 1 + \lambda + \mu) - \lambda) = 0. \tag{2.11}$$

Earlier we found that solutions of equation (2.10) are linear combinations of hypergeometric functions (Craster, 1997). Straightforward calculations based on the properties of the hypergeometric functions show that the function

$$Y_1(\zeta) = (1 - a)(\lambda - 1)\,{}_2F_1(\sigma, \tau, \lambda - 1; \zeta) + (q - a[1 + \sigma + \tau + \sigma\tau - \lambda])\,{}_2F_1(\sigma, \tau, \lambda; \zeta). \tag{2.12}$$

satisfies equation (2.10). The second linearly independent solution can also be obtained and is also a linear combination of hypergeometric functions.

The hypergeometric functions in (2.12) are solutions of Fuchsian equations of similar form to (2.10). The natural question that is initiated by considering the solution (2.12) is whether the method is justifiable, or generalizable, using general properties of Fuchsian equations?

Our studies in different applied areas indicate that this trick can be effectively generalized. Additionally, equations beyond Heun's can be solved, and moreover, the simplification can be achieved even if the explicit form of the solution is not known. The sections below are devoted to this generalization.

3 Theoretical background

Our main aim is to determine: *when can the false singularity be removed*, i.e. when can solutions of (2.1) be represented via the solutions of equations similar to (2.1), but with simpler coefficients (not containing the false singular point).

In this form the problem is slightly artificial. Typically the Fuchsian equation (2.1) is not found directly, together with additional restrictions of the form (2.4). In fact, the authors faced such problems in slightly different formulations, where the ordinary differential equation is disguised and lurks behind the scenes. Or where the problem is not in solving the particular differential equation, but in finding the set of functions possessing some definite properties. In some cases such problems can eventually be reduced to solving the Fuchsian equation of the form (2.1). Sometimes the coefficients of the equation are not all known, but there are some additional restrictions imposed upon the solution. Usually, the set of strong singularities is known, so all other points must be either regular points or false singular points. This means the relation (2.4), or a similar one, must be valid at each singular point, except at the prescribed set of strong singular points.

It is for this reason that we find it necessary to study specific linear spaces of functions rather than the Fuchsian equations directly themselves. As we shall show later, the functional spaces under consideration are closely connected with Fuchsian equations.

3.1 Fuchsian and confluent Fuchsian spaces of functions

Let us introduce a definition of a Fuchsian, or confluent Fuchsian, linear space of functions as one that imitates the behaviour of the solutions of Fuchsian and confluent Fuchsian equation. Namely, let U be a linear space of functions. If the conditions:

(i) this space has dimension two over $\mathbb{C}$, i.e. any three elements of U are linearly dependent with complex constant coefficients;
(ii) all elements of U are regular functions of ζ everywhere except at a finite set of singular points $a_1 \dots a_v$;
(iii) for each singular point of this set, say a_j, two fundamental functions, $U_{j,1}$ and $U_{j,2}$, can be chosen among the elements of U, such that they obey the ansatz (2.5) or (2.7) ((2.6) or (2.8) for the singular point at infinity),

are satisfied then we call the space U a *Fuchsian* space.

If all these conditions are satisfied, except that at infinity, there are fundamental functions of the form (2.9), and this is then a *confluent Fuchsian* space. The solutions of each (confluent) Fuchsian equation form a (confluent) Fuchsian linear space.

Given some point in the complex plane, it falls into one of two categories: either it is one of the a_j, in which case it is singular, or it is a regular point. Only strong singular points of the equation become singular points of the space of its solutions. The false singular points of the equation are actually regular points of the space. If the exponents of a strong singular point a_j are integer, and there are no logarithmic terms in the corresponding asymptotic expansion, then this point is trivial.

3.2 Isomonodromy mappings of the Fuchsian spaces

Consider two distinct Fuchsian, or confluent Fuchsian, spaces U and V. The invertible linear mapping

$$\mathrm{U} \xrightarrow{\varphi} \mathrm{V}$$

is an *isomonodromy* if

(i) the sets of non-trivial singular points $a_1 \dots a_r$ for both spaces coincide,
(ii) the exponents α_j^*, β_j^* of U and the exponents α_j^{**}, β_j^{**} of V are such that all $\alpha_j^* - \alpha_j^{**}$ and $\beta_j^* - \beta_j^{**}$ are integers; if infinity is the irregular point, then $\lambda_1^* = \lambda_1^{**}$, $\lambda_2^* = \lambda_2^{**}$, the values $\mu_1^* - \mu_1^{**}$ and $\mu_2^* - \mu_2^{**}$ are integers,
(iii) for each singular point a_j, the image of the fundamental function $U_{j,1}$ with exponent α_j^* is the fundamental function $V_{j,1}$ with exponent α_j^{**}; the image of the fundamental

function $U_{j,2}$ with exponent β_j^* is the fundamental function $V_{j,2}$ with exponent β_j^{**}. If the singular point has logarithmic behaviour, then $U_{j,1}$ is mapped onto $V_{j,1}$, $U_{j,2}$ is mapped onto $V_{j,2}$ according to (2.6). If the singular point is irregular, then also U_1 is mapped into V_1, U_2 is mapped into V_2 according to (2.9). In other words, the fundamental functions are always mapped into corresponding fundamental functions.

It is useful to be able to confirm that φ is an isomonodromy mapping. The mapping φ is defined if the images of two linearly independent elements of U are known, i.e. we must know the functions $U_1(\zeta), U_2(\zeta)$, which form the basis of U and

$$V_1 = \varphi(U_1), \qquad V_2 = \varphi(U_2), \tag{3.1}$$

which form the basis of V. For each singular point a_j we have two pairs of complex numbers $A_{j,1}, A_{j,2}$ and $B_{j,1}, B_{j,2}$, such that the linear combinations $A_{j,1}U_1 + A_{j,2}U_2$ and $B_{j,1}U_1 + B_{j,2}U_2$ are fundamental functions at a_j. If, for each a_j, $A_{j,1}V_1 + A_{j,2}V_2$ and $B_{j,1}V_1 + B_{j,2}V_2$ with the same A and B are the fundamental functions of V at a_j, then the mapping φ defined by (3.1) is an isomonodromy.

In the examples we construct, the elements of U and V are (confluent or non-confluent) hypergeometric functions; the explicit coefficients A, B are in, say, Abramowitz & Stegun (1969).

There are two simple, but important, cases of the isomonodromy mappings. First, take the space U and multiply all its elements by a rational function $R(\zeta)$. Obviously, the mapping $\mathrm{U} \to R\mathrm{U}$ defined as $U(\zeta) \to R(\zeta)U(\zeta)$ is an isomonodromy. Secondly, construct the space U′ consisting of the derivatives of the elements of U. If among the elements of U there are no functions identically equal to a constant, then the mapping $U(\zeta) \to U'(\zeta)$ is the isomonodromy (prime denotes the differentiation with respect to ζ).

An obvious property of the isomonodromy mappings is that the linear combination of the isomonodromy mappings is also an isomonodromy mapping, i.e. if the mappings $\mathrm{U} \to \varphi(\mathrm{U})$ and $\mathrm{U} \to \psi(\mathrm{U})$ are some isomonodromies, then the mapping $U \to (a\varphi(U) + b\psi(U))$ is an isomonodromy (of course, if the space $a\varphi(\mathrm{U}) + b\psi(\mathrm{U})$ has dimension not less than 2).

Multiplication by a rational function, differentiation and taking a linear combination are, as shown in the next section, enough to construct all spaces isomonodromical to a chosen space U.

3.3 Explicit formulae for the isomonodromy mappings

The current subsection is the key point of the paper and theorem 1 forms the basis for our further results.

Theorem 1 *Let* U *be a Fuchsian, or confluent Fuchsian, space with basis* (U_1, U_2) *and let* $\mathrm{U} \to \varphi(\mathrm{U})$ *and* $\mathrm{U} \to \psi(\mathrm{U})$ *be some isomonodromy mappings. We assume the determinant*

$$D(\zeta) = \begin{vmatrix} \varphi(U_1) & \psi(U_1) \\ \varphi(U_2) & \psi(U_2) \end{vmatrix} \tag{3.2}$$

is not identically equal to zero. Then there exist two rational functions $R_1(\zeta)$ *and* $R_2(\zeta)$, *such that for any* $U \in \mathrm{U}$

$$U = R_1\varphi(U) + R_2\psi(U). \tag{3.3}$$

From linearity, it is necessary to prove formula (3.3) only for U_1 and U_2. Consider the equations

$$U_1 = R_1\varphi(U_1) + R_2\psi(U_1), \quad U_2 = R_1\varphi(U_2) + R_2\psi(U_2). \tag{3.4}$$

This system can be formally solved with respect to R_1 and R_2:

$$R_1(\zeta) = \frac{D_1(\zeta)}{D(\zeta)}, \qquad R_2(\zeta) = \frac{D_2(\zeta)}{D(\zeta)},$$

where

$$D_1(\zeta) = \begin{vmatrix} U_1 & \psi(U_1) \\ U_2 & \psi(U_2) \end{vmatrix}, \qquad D_2(\zeta) = \begin{vmatrix} \varphi(U_1) & U_1 \\ \varphi(U_2) & U_2 \end{vmatrix}.$$

There are three determinants, namely $D(\zeta)$, $D_1(\zeta)$, $D_2(\zeta)$ having similar properties. First, consider the determinant $D(\zeta)$ as a function of ζ. It is a regular function everywhere except the singular points of the space U, namely $a_1 \ldots a_m$.

Consider any singular point a_j ($\neq \infty$). According to the definition of a Fuchsian space, a pair of fundamental functions

$$U_{j,1}(\zeta) = A_{j,1}U_1(\zeta) + A_{j,2}U_2(\zeta), \qquad U_{j,2}(\zeta) = B_{j,1}U_1(\zeta) + B_{j,2}U_2(\zeta) \tag{3.5}$$

can be found with constant complex coefficients $A_{j,i}$ and $B_{j,i}$ local to a_j; the functions $U_{j,1}, U_{j,2}$ have the form (2.5) near a_j. Utilizing this, we rewrite $D(\zeta)$, in the neighbourhood of a_j, in the form

$$D(\zeta) = \begin{vmatrix} \varphi(U_{j,1}) & \psi(U_{j,1}) \\ \varphi(U_{j,2}) & \psi(U_{j,2}) \end{vmatrix} \Big/ \begin{vmatrix} A_{j,1} & A_{j,2} \\ B_{j,1} & B_{j,2} \end{vmatrix}.$$

Near a_j, the determinant $D(\zeta)$ can be represented as a product of the functions, $(\zeta - a_j)^{\alpha_j+\beta_j-n_j}$ and a function regular near a_j. The parameter n_j is an integer (positive or negative) depending on the exponents of the spaces $\varphi(\mathrm{U})$ and $\psi(\mathrm{U})$ near a_j. Any logarithmic terms are eliminated according to the properties of the determinant.

We thus consider the function

$$D_c(\zeta) = D(\zeta)\prod_i(\zeta - a_i)^{n_i-(\alpha_i+\beta_i)},$$

where the product is taken over all values of index i for which a_i is finite; this function is regular in the whole complex ζ-plane.

Consider the behaviour of the function $D(\zeta)$ at infinity. If infinity is a regular, or a regular singular, point then the functions D and D_c have algebraic growth determined by the exponents at infinity. If the space U is confluent Fuchsian and, therefore, infinity is an irregular singular point, then the expansion (2.9) should be taken into account. The determinant $D(\zeta)$ grows at infinity as $e^{(\lambda_1+\lambda_2)\zeta}$ multiplied by some power of ζ. The function D_c is then defined as above, but now multiplied by $e^{-(\lambda_1+\lambda_2)\zeta}$, so, in any case, D_c is regular in the whole complex plane and grows algebraically at infinity and thus, from Liouville's theorem, it is a polynomial.

Similar arguments can be used for the determinants D_1 and D_2. Obviously, the coefficients R_1 and R_2, which are the ratios of the corresponding determinants, are then rational functions as required.

3.4 Connection between the Fuchsian spaces and Fuchsian equations

Theorem 1 has an important corollary: consider the isomonodromy mappings U″ → U′ and U″ → U, then, from Theorem 1, if

$$D(\zeta) = \begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix} \tag{3.6}$$

is not identically zero there exist two rational functions $f(\zeta)$ and $g(\zeta)$, such that for any $U \in$ U the relation

$$U''(\zeta) = f(\zeta)U'(\zeta) + g(\zeta)U(\zeta) \tag{3.7}$$

is valid. It means that the Fuchsian (or confluent Fuchsian) space is the space of solutions for a Fuchsian (or a confluent Fuchsian) differential equation.

The explicit form for the coefficients is

$$f(\zeta) = \begin{vmatrix} U_1'' & U_1 \\ U_2'' & U_2 \end{vmatrix} \Big/ \begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix}, \qquad g(\zeta) = \begin{vmatrix} U_1' & U_1'' \\ U_2' & U_2'' \end{vmatrix} \Big/ \begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix}. \tag{3.8}$$

Under the condition $D \neq 0$ identically, for any isomonodromy mapping φ : V → U there exist two functions, namely $J(\zeta)$ and $H(\zeta)$, such that

$$V(\zeta) = J(\zeta)U(\zeta) + H(\zeta)U'(\zeta). \tag{3.9}$$

This can be proved by considering the mappings V → U and V → U′. Conversely, for any rational functions J and H the mapping between U and V generated by formula (3.9) is an isomonodromy. Therefore, (3.9) describes all spaces that form an isomonodromy to U.

Let us find the Fuchsian equation for V defined as (3.9). We assume that this equation has the form

$$V''(\zeta) = f^*(\zeta)V'(\zeta) + g^*(\zeta)V(\zeta), \tag{3.10}$$

for some new rational functions $f^*(\zeta)$ and $g^*(\zeta)$.

Let the coefficients f and g of the equation (3.7) for U be known, then for any $U \in$ U and $V = \varphi(U)$

$$V'(\zeta) = I(\zeta)U(z) + Z(\zeta)U'(\zeta), \tag{3.11}$$

where

$$I(\zeta) = J'(\zeta) + g(\zeta)H(\zeta), \qquad Z(\zeta) = J(\zeta) + H'(\zeta) + f(\zeta)H(\zeta). \tag{3.12}$$

Differentiation of (3.11) shows that

$$V''(\zeta) = [I'(\zeta) + g(\zeta)Z(\zeta)]U(\zeta) + [I(\zeta) + Z'(\zeta) + f(\zeta)Z(\zeta)]U'(\zeta).$$

Combining these relations with (3.9), we find that

$$f^*(\zeta) = \frac{Z'J - I'H - gZH + JI + fJZ}{JZ - IH}, \; g^*(\zeta) = \frac{I'Z - Z'I + gZ^2 - I^2 + fIZ}{JZ - IH}. \tag{3.13}$$

Let us find the transformation inverse to (3.9). Using (3.11) we can write the transformation (3.9) in the matrix form

$$\begin{pmatrix} V \\ V' \end{pmatrix} = \begin{pmatrix} J & H \\ I & Z \end{pmatrix} \begin{pmatrix} U \\ U' \end{pmatrix}, \tag{3.14}$$

where I and Z are the functions defined by (3.12). Using the inverse matrix, we obtain the formula for the inverse transformation

$$U = \frac{Z}{JZ - IH}V - \frac{H}{JZ - IH}V'. \tag{3.15}$$

The transformations studied in this subsection are similar to Schlesinger transformations of the Fuchsian equations (Jimbo *et al.*, 1981).

There are simple degenerate cases: if the determinant $D(\zeta)$, defined by (3.6), is identically equal to zero, then the elements of U obey a Fuchsian, or confluent Fuchsian, equation of order 1. If the determinant $D(\zeta)$ defined by (3.2) is identically equal to zero, then there exists a rational function $R(\zeta)$ such that for any $U \in$ U the relation $\psi(U) = R\varphi(U)$ is valid.

3.5 Reducing the number of false singular points using isomonodromy

A natural measure of the complexity of a Fuchsian equation is the number of its singular points. Consider Fuchsian equations corresponding to different Fuchsian spaces each forming an isomonodromy to each other. The nontrivial strong singular points of all equations should coincide, but the number of trivial singular points and false singular points can be different, so some equations can, on a practical level, be simpler to solve than the others. Our aim is to find the simplest one.

To calculate the number of singular points of a Fuchsian equation corresponding to a Fuchsian space U we note that the false singular points of the equation are the roots of the determinant $D(\zeta)$ as defined in (3.6).

Without loss of generality, we consider the case when infinity is a regular point of U and also that infinity is not a root of D. The absence of root at infinity means that the growth of D at infinity is of the form $D \sim \zeta^{-2}$.

We also assume there are no trivial singular points. The singular points are $a_1 \ldots a_r$, with corresponding pairs of the exponents as $(\alpha_1, \beta_1) \ldots (\alpha_r, \beta_r)$.

From the previous subsection, the function $D(\zeta)$ is equal to

$$D(\zeta) = P_\epsilon(\zeta) \prod_{i=1}^{r} (\zeta - a_i)^{\alpha_i + \beta_i - 1}, \quad \text{where } \epsilon = -\sum_{i=1}^{r} (\alpha_i + \beta_i - 1) - 2. \tag{3.16}$$

and P_ϵ is a polynomial of order ϵ.

If all the roots of P_ϵ are simple then they correspond to the false singular points of the Fuchsian equation for U. These singular points have exponents $(0, 2)$ and the number of these points is equal to ϵ. When multiple roots of P_ϵ appear, they correspond to false singular points having a larger sum of exponents, for example the pair $(0, N)$, $N > 2$.

If infinity is a regular singular point of U, then the number of false singular points is still defined by ϵ in (3.16). Turning now to the confluent case where infinity is an irregular singular point of a confluent Fuchsian space U, and $a_1 \ldots a_r$ are the regular singular points, then the number of removable singular points, corresponding to simple roots of D is equal to

$$\epsilon^* = -\sum_{i=1}^{r} (\alpha_i + \beta_i - 1) - \mu_1 - \mu_2, \tag{3.17}$$

where μ_1 and μ_2 are defined from the asymptotic behaviour in (2.9).

The value of ϵ (or ϵ^*) is an important parameter of the Fuchsian (confluent Fuchsian) space.

One possible procedure of simplifying Fuchsian equations is the following: let us imagine that we are given a Fuchsian equation in the form (3.7), U has no trivial strong singular points and all false singular points have exponents $(0, 2)$. The space of its solutions is Fuchsian, or confluent Fuchsian. We must then find an isomonodromy mapping $\mathrm{U} \to \mathrm{V}$, such that the parameter ϵ of V is lower than the parameter ϵ of U. We should construct the equation for V with the coefficients (3.13). According to this scheme, the equation for V has fewer singular points than the equation for U. If the simplified equation is then solved and the basis of V is found, then the solutions of the initial equations (the basis of U) can be found using the formulae for the inverse mapping $\mathrm{V} \to \mathrm{U}$ (3.15). Below we describe how to find the mapping $\mathrm{U} \to \mathrm{V}$ that simplifies the equation, and under what conditions this can be done.

An alternative scheme is to find two auxiliary Fuchsian spaces V and W isomonodromical to U (and to each other) and satisfying simpler equations. In this case U can be represented as the linear combination of V and W according to the formula (3.3).

However, the former scheme is more general. To find the mapping, let U be a Fuchsian space, corresponding to the equation (3.7). Furthermore this equation is taken to have r strong nontrivial points $a_1 \dots a_r$, ϵ false points $b_1 \dots b_\epsilon$ with exponents $(0, 2)$ and no trivial strong singular points.

We are looking for the isomonodromy mapping $\mathrm{U} \xrightarrow{\psi} \mathrm{V}$ having the following property: V has no trivial strong singular points and the singularities of V are not stronger than the singularities of U, that is, the exponents of V are not less than the corresponding exponents of U. The mapping ψ given by the explicit formula (3.9) possesses this property if and only if the following conditions imposed on the rational functions J and H are satisfied:

(i) $J(\zeta)$ grows at infinity no faster than a constant, $H(\zeta)$ grows no faster than a quadratic function.
(ii) $J(\zeta)$ and $H(\zeta)$ are regular functions everywhere except at the false and strong singular points of U.
(iii) $J(\zeta)$ and $H(\zeta)$ can have simple poles at the points $b_1 \dots b_\epsilon$, which are the false singular points of the space U. The residues of $J(\zeta)$ and $H(\zeta)$ at these points must obey the relations

$$\text{Residue}[J(\zeta), \zeta = b_j] = \text{Residue}[H(\zeta), \zeta = b_j]\text{Residue}[g(\zeta), \zeta = b_j], \tag{3.18}$$

where $g(\zeta)$ is the coefficient of the equation (2.1) for U.
(iv) $J(\zeta)$ is regular at the strong singular points $a_1 \dots a_r$ and $H(\zeta)$ is equal to zero at each strong singular point a_i.

This class of mappings ψ might appear too narrow, since there are cases when some exponents become greater and some smaller. However, a detailed study shows that the properties of such mappings, namely the number of the degrees of freedom, is the same.

Conditions (i)–(iii) guarantee that there are no strong points of V except at $a_1 \dots a_r$. With regard to condition (iv), the poles at $b_1 \dots b_\epsilon$ do not lead to trivial strong points of

V, because for each element $U(\zeta) \in$ U and for each b_j the following relation is valid:

$$\text{Residue}[g(\zeta), \zeta = b_j]U(b_j) + U'(b_j) = 0.$$

This relation is easily confirmed using the representations (3.8).

If the restrictions (i)–(iv) are satisfied, then the Fuchsian equation for V has the same strong singular points as does U, and the number of false singular points of V is no greater than the number of false singular points of U.

Analyzing the relations (i)–(iv), we conclude that functions $H(\zeta)$ and $J(\zeta)$ can be written in the form

$$H(\zeta) = P_\delta(\zeta)\prod_{i=1}^{r}(\zeta - a_i) \Big/ \prod_{i=1}^{\epsilon}(\zeta - b_i), \tag{3.19}$$

$$J(\zeta) = \sum_{i=1}^{\epsilon}\left(\frac{\text{Residue}[g(\zeta), \zeta = b_i]\text{Residue}[H(\zeta), \zeta = b_i]}{\zeta - b_i}\right) + c, \tag{3.20}$$

where c is an arbitrary constant, and

$$\delta = 2 + \epsilon - r, \tag{3.21}$$

this parameter is equal to 2 plus the number of false points minus the number of strong points. If $\delta \geqslant 0$, then $P_\delta(\zeta)$ is an arbitrary polynomial of ζ of degree δ. If $\delta < 0$, then only trivial isomonodromy transformations obeying the conditions (i)–(iv) exist.

If infinity is a regular strong singular point of a Fuchsian equation, say $a_1 = \infty$, then $H(\zeta)$ has the form

$$H(\zeta) = P_\delta(\zeta)\prod_{i=2}^{r}(\zeta - a_i) \Big/ \prod_{i=1}^{\epsilon}(\zeta - b_i), \tag{3.22}$$

where δ is defined by (3.21). The confluent case is identical except that the degree of the polynomial becomes $\delta^* = \epsilon - r$.

If not all the roots of the polynomial P_ϵ (see (3.16)) are distinct, then the product in the denominator $H(\zeta)$ can just be replaced with the polynomial P_ϵ itself.

We consider $\delta \geqslant 0$, then the transformation (3.9) depends upon at least one free parameter c, and we find the values of this parameter that correspond to simplifications of the initial equation. Namely, we take a_j to be one of the strong singular points of the space U with exponents $(0, \beta_j)$. If c is chosen such that

$$J(a_j) = 0, \tag{3.23}$$

then the exponents of U at a_j are $(1, \beta_j)$. If the constant is chosen such that

$$J(a_j) + \frac{dH(\alpha_j)}{d\zeta} + \lim_{\zeta \to a_j}(H(\zeta)f(\zeta)) = 0, \tag{3.24}$$

then the exponents are equal to $(0, \beta_j + 1)$. In both cases the parameter ϵ of V (the number of false singular points in the simplest case) is less than the number of false singular points of the equation for U, so the space V defined by (3.9), and the equation for V is simpler than that for U.

The procedure we advocate has a simple geometrical interpretation in that the position of the strong singular points is fixed, but the position of removable singularities $b_1 \dots b_\epsilon$ of the equation corresponding to V are functions of $\delta + 1$ free parameters (namely, the coefficients of the polynomial P_δ and the parameter c). If by varying the free parameters we make one of the false singularities coalesce with a strong singularity, then the total number of singular points is reduced. In general, $\delta+1$ false singular points can be removed this way.

If the equation for U has more than $\delta + 1$ false singular points with exponents $(0, 2)$, then the remainder cannot be removed, except maybe in some particular cases, as they are required to provide the necessary number of parameters for the monodromy group of the equation.

The procedure of finding an appropriate isomonodromy mapping described above is quite complicated, in the relatively simple examples that we construct we can pose an ansatz for the form of the mapping to within several unknown parameters; these are found by direct substitution. However, one result of the current subsection is important: in the general case, when $\delta \geqslant 0$ (or $\delta^* \geqslant 0$) it is possible to construct an isomonodromy mapping to a space having a lower value of ϵ (or ϵ^*).

4 Removing false singular points

In this section we explicitly simplify and solve Fuchsian or confluent Fuchsian equations using an isomonodromy mapping. The examples are based on Heun's, and a related confluent, equation.

It is not always necessary to carry out the procedure outlined in the previous section in all details. In particular the formulae (3.13) and (3.15) are not simple and in the present examples we can avoid using them. Instead, we use explicit relations for the hypergeometric and confluent hypergeometric functions, which, as we shall see below, have themselves the form of the isomonodromy mappings.

It is convenient to introduce the following notation for the Fuchsian (P_F) and confluent Fuchsian ($P_{F\infty}$) spaces:

$$P_F \begin{pmatrix} a_1 & a_2 & \dots & a_r & \\ \alpha_1 & \alpha_2 & \dots & \alpha_r & \zeta \\ \beta_1 & \beta_2 & \dots & \beta_r & \end{pmatrix}, \quad P_{F\infty} \begin{pmatrix} a_1 & a_2 & \dots & a_r & \infty & \\ \alpha_1 & \alpha_2 & \dots & \alpha_r & \lambda_1, \mu_1 & \zeta \\ \beta_1 & \beta_2 & \dots & \beta_r & \lambda_2, \mu_2 & \end{pmatrix}, \tag{4.1}$$

and this notation is similar to that for the Riemann P-function (Whittaker & Watson, 1927): The upper row denotes the positions of the singular points, and the columns beneath give the exponents. Note that unlike the P-function notation of a Fuchsian equation, which is similar to (4.1) but includes all the singular points, in our case the sum of all the exponents must not necessarily be equal to the number of indicated singular points minus two.

Generally, the notations in (4.1) do not contain all the necessary information to reconstruct the spaces uniquely, so we use these notations only to make clear what behaviour is expected at the singular points. These notations enable one to calculate the important parameters ϵ and δ of the space.

4.1 Heun's equation

We now interpret, in terms of the isomonodromy mappings, the method proposed in Craster (1996, 1997) for solving Heun's equation with a false singular point that has a pair of exponents (0, 2). We return to Heun's equation, (2.10), as considered in § 2.2.

To solve Heun's equation we now try to find a Fuchsian space Y consisting of solutions to (2.10). This space corresponds to the notation:

$$P_F\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma & \zeta \\ 1-\lambda & 1-\mu & \tau & \end{pmatrix} \tag{4.2}$$

The order of the polynomial, described in § 3.5 is such that $\epsilon = 1$; this corresponds to appearance of a single false point, namely, this point is $\zeta = a$.

To solve the equation we use the formula (3.3) and take two auxiliary Fuchsian spaces U and V that correspond to the notations

$$P_F\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma & \zeta \\ 2-\lambda & 1-\mu & \tau & \end{pmatrix} \quad \text{and} \quad P_F\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma & \zeta \\ 1-\lambda & 2-\mu & \tau & \end{pmatrix}, \tag{4.3}$$

respectively, and represent Y as a linear combination of U and V. To actually be of any use in constructing the full solution, the spaces U and V must be connected with an isomonodromy mapping φ: U $\xrightarrow{\varphi}$ V. The parameter ϵ is equal to zero for both auxiliary spaces, so the Fuchsian equations for these auxiliary spaces do not contain additional false singularities. As these Fuchsian equations have three strong singular points, one then concludes that the basis of the auxiliary spaces are solutions of hypergeometric equations. One noteworthy point, to which we shall return later, is that the decomposition in (4.3) is, at first sight, completely different from that used in Craster (1997). Using the standard notation for hypergeometric functions (Abramowitz & Stegun, 1969) we take the basis of U and V in the form

$$U_1(\zeta) = {}_2F_1(\sigma, \tau, \lambda - 1; \zeta), \tag{4.4}$$

$$U_2(\zeta) = \frac{\Gamma(\lambda-2)\zeta^{2-\lambda}{}_2F_1(2+\sigma-\lambda, 2+\tau-\lambda, 3-\lambda; \zeta)}{\Gamma(2-\lambda)\Gamma(\lambda-\sigma-1)\Gamma(\lambda-\tau-1)}, \tag{4.5}$$

and

$$V_1(\zeta) = {}_2F_1(\sigma, \tau, \lambda; \zeta), \tag{4.6}$$

$$V_2(\zeta) = \frac{\Gamma(\lambda-1)\zeta^{1-\lambda}{}_2F_1(1+\sigma-\lambda, 1+\tau-\lambda, 2-\lambda; \zeta)}{\Gamma(1-\lambda)\Gamma(\lambda-\sigma)\Gamma(\lambda-\tau)}. \tag{4.7}$$

Direct calculations based on the properties of hypergeometric functions show that the mapping generated by $U_1 \to V_1$, $U_2 \to V_2$ is an isomonodromy mapping.

Each linear combination of the form U + CV, for some constant C, is a Fuchsian space, moreover, the study of the exponents shows that the Fuchsian equation for this space contains no more than 1 false singularity. In general, this equation has one false point, that is, this equation has the form (2.10) with some a and q satisfying (2.11), but for several values of C it is just a hypergeometric equation.

Our next task is to find the value of C such that the space U + CV gives the solution to the Heun equation (2.10) with some particular values a and q. The simplest way to do this is to substitute the linear combination, say $U_1 + CV_1$ directly into the Heun equation (2.10) and then use known recursion formulae for hypergeometric functions and their derivatives. Tedious calculations show that

$$C(1-a)(\lambda-1) + a[1+\sigma+\tau+\sigma\tau-\lambda] - q = 0$$

$$C(1-\lambda)[a\sigma\tau - q + (1-a)(1-\lambda)] + a[\lambda-1-\sigma][\lambda-1-\tau] = 0.$$

Eliminating C yields the constraint (2.4) relating q and a again. The resulting explicit solution to the Heun equation, arbitrary to within a constant multiplicative factor, is that the function Y_1 given by (2.12) is the solution of the equation (2.10). Together with another linearly independent solution that utilizes U_2 and V_2 we obtain the solution to (2.10). There are two different Heun's equations both with a false point at a, they follow from the two values of q obtained as solutions of the constraint (2.4); and (2.12) is the solution to both.

It is noteworthy that the choice of the auxiliary spaces U and V is not unique. Recalling the geometrical interpretation of the isomonodromy mapping given in the previous section, we conclude that the false singular point can coalesce with any of the three strong singularities of (2.10), moreover, for each singularity there are two cases, each corresponding to one of the exponents being weakened. So, there are actually 6 auxiliary spaces associated with 6 hypergeometric equations of which the spaces (4.4)–(4.7) are only two of them. This modifies the heuristic arguments presented in Craster (1997), there the whole issue of choosing what are called here auxiliary spaces was performed using an approach based upon solving a related mixed boundary value problem (see also § 5.2). The choice of which exponents to weaken was based upon an unnecessary physical argument. All of these six spaces are connected with each other by the isomonodromy mappings belonging to the class studied in § 3.4. These mappings are well known in the theory of special functions. For example, the following relations between the so-called *contiguous* functions (see Whittaker & Watson, 1927, section 14.7) can be interpreted as (3.3) and (3.9):

$$c\,{}_2F_1(a,b,c;\zeta) - (c-a)\,{}_2F_1(a,b,c+1;\zeta) - a\,{}_2F_1(a+1,b,c+1;\zeta) = 0, \qquad (4.8)$$

$$\zeta\frac{d}{d\zeta}F(a,b,c;\zeta) = (c-1)[{}_2F_1(a,b;c,\zeta) - {}_2F_1(a,b;c-1;\zeta)]. \qquad (4.9)$$

4.2 False points with exponents $(0, N)$

If the polynomial $P_\epsilon(\zeta)$ of § 3.5 has multiple roots this corresponds to problems involving false points with exponents $(0, N)$. To see how one deals with this situation we now investigate Heun's equation (2.10), when the exponents at $\zeta = a$ are now $(0, N)$ (N is an integer $\geqslant 1$), and the local expansion around that point contains no logarithmic terms in its development.

The Heun's equation having a false point $(0, N)$ emerges in the following applied problems. The linearly independent solutions of Heun's equation give the mapping functions for the upper half of the complex ζ–plane into a curvilinear quadrangle with vertex angles $\pi(1-\lambda), \pi(1-\mu), \pi(1-\nu), \pi(\sigma-\tau)$ (or $\pi(\tau-\sigma)$). Curvilinear polygons often occur in the hodograph plane visualization of free boundary problems. The vertices map to the points $0, 1, a$ and ∞ respectively. For the cases that concern us here one vertex angle is $N\pi$; this corresponds to an exponent $(0, N)$ and without loss of generality we take this to occur at $\zeta = a$ and hence $\nu = 1 - N$. The conformal mapping here is potentially non-schlicht. If we consider the problem as a conformal mapping (Nehari 1952) then our fundamental problem is how one relates the accessory parameter and free point (q and a), and after doing so how one then relates this to the geometrical configuration, that is, determining the constants that connect the auxiliary solutions.

In our current terminology, a false singular point having exponents $(0, N)$ can be interpreted as $N - 1$ coincident false points with exponents $(0, 2)$; it is a multiple root of the polynomial $P_\epsilon(\zeta)$. Earlier we concentrated on the case when the polynomial $P_\epsilon(\zeta)$ had only distinct simple roots. So, the theory developed earlier must be generalized a little. For example, the restriction (2.4) is valid only for the points with exponents $(0, 2)$ and it must be modified.

As in the previous subsection, there are 3 strong regular singularities, so we can represent the solution as a linear combination of solutions of hypergeometric functions.

For one false point with exponents $(0, N)$, the constraint (2.11) relating q and a must be generalized as follows: We proceed by taking a local expansion around $\zeta = a$ so $Y(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - a)^n$ and then we substitute back into the Fuchsian equation. Examining each consecutive coefficient of $\zeta - a$ one finds N simultaneous equations for just $\hat{y}_0, .., \hat{y}_{N-1}$; the equation arising from $(\zeta - a)^{N-1}$ has no term involving $\hat{y}_N$ (or $\hat{y}_{N+m}$ $m = 1, \ldots$). The coefficient of $O[(\zeta - a)^M]$ is

$$\hat{y}_{M+1}[a(a-1)(M+1)(M+1-N)] + \hat{y}_{M-1}[\sigma\tau + (M-1)(\lambda + \mu + 1 - N + M - 2)]$$
$$+\hat{y}_M[M((2a-1)(M-1) + (\lambda(a-1) + a\mu + (1-N)(2a-1))) + \sigma\tau a - q]. \tag{4.10}$$

Hence solving these explicitly connects a and q as an Nth order polynomial in q; the relation is not pretty, but it is explicit.

To solve Heun's equation, as earlier, we try to find a Fuchsian space Y, having the basis (Y_1, Y_2), possessing the required properties. This space can be denoted by

$$P_F \begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma & \zeta \\ 1-\lambda & 1-\mu & \tau & \end{pmatrix} \quad \text{or} \quad P \begin{pmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \sigma & \zeta \\ 1-\lambda & 1-\mu & N & \tau & \end{pmatrix}, \tag{4.11}$$

the latter notation being the Riemann P-function notation. The order of the polynomial, described in § 3.5 is such that $\epsilon = N - 1$, and hence we have false points.

The solution is sought as a linear combination of N auxiliary Fuchsian spaces $\mathrm{U}^{(m)}$ with the basis $(U_1^{(m)}, U_2^{(m)})$, $m = 1, \ldots, N$, that are the solutions of

$$P_F \begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma & \zeta \\ N+1-\lambda-m & m-\mu & \tau & \end{pmatrix} \quad \text{for} \quad m = 1, \ldots, N. \tag{4.12}$$

These all share the same monodromy data and strong singular points as the original equation. As these auxiliary Fuchsian spaces again only involve three regular singular points they have the following bases

$$U_1^{(m)}(\zeta) = {}_2F_1(\sigma, \tau, \lambda_m; \zeta), \tag{4.13}$$

$$U_2^{(m)}(\zeta) = \frac{\Gamma(\lambda_m - 1)\zeta^{1-\lambda_m}{}_2F_1(1+\sigma-\lambda_m, 1+\tau-\lambda_m, 2-\lambda_m; \zeta)}{\Gamma(1-\lambda_m)\Gamma(\lambda_m-\sigma)\Gamma(\lambda_m-\tau)}, \tag{4.14}$$

where $\lambda_m = \lambda + m - N$ for $m = 1 \dots N$. The isomonodromy mappings are defined by the correspondence between the elements of the basis.

Unfortunately the algebra becomes slightly unwieldy for practical examples with large N, nonetheless explicit solutions can be constructed, e.g. for $N = 3$ a solution, to within an arbitrary multiplicative constant, is simply:

$$Y(\zeta) = {}_2F_1(\sigma, \tau; \lambda; \zeta) + C_1\ {}_2F_1(\sigma, \tau; \lambda - 1; \zeta) + C_2\ {}_2F_1(\sigma, \tau; \lambda - 2; \zeta), \tag{4.15}$$

with constants related via

$$C_1(1-a)(\lambda-2) + 2a(2-\lambda+\sigma+\tau) + \sigma\tau a - q = 0,$$

$$C_2(\lambda-1)[2(1-a)(\lambda-1) - \sigma\tau q + q] = C_1 a(\lambda-1-\sigma)(\lambda-1-\tau)$$

and the other linearly independent solution follows using (4.14). The parameters q and a are related via the cubic

$$(\sigma\tau a - q)((\sigma\tau + 2 + 2\sigma + 2\tau)a - q + 2 - 2\lambda)(\sigma\tau a - q - 4a + 2 + (3 + \sigma + \tau)a - \lambda)$$
$$+2a(a-1)(\sigma\tau + 1 + \sigma + \tau)(\sigma\tau a - q)$$
$$+2((\sigma\tau + 2 + 2\sigma + 2\tau)a - q + 2 - 2\lambda)a\sigma\tau(a-1) = 0. \tag{4.16}$$

Evidently we can remove a singular point from a Fuchsian equation if the equation has exponents $(0, N)$ and no logarithmic terms at that singular point, the analysis and general theory follow our earlier sections, except that now we require N auxiliary Fuchsian spaces, rather than just two.

4.3 Confluent equations

Confluence is an important special case, it allows us to evaluate possible changes that an irregular singular point might introduce, and also they are intimately related to several equations in mathematical physics i.e. Schrödinger's equation and in diffraction theory (Komarev *et al.*, 1976; Leaver, 1986; Shanin, 2001a).

We consider confluence where the regular singular point at a merges with that at infinity to create an irregular singular point there. Explicitly we take the limit as $a \to \infty$ in conjunction with $\tau = v \to a\hat{a}$ and $q \to a(c\hat{a} + \hat{q})$. To make the resulting equation closer to Kummer's confluent hypergeometric equation, which we utilize later, we then rescale by setting $\zeta = \hat{a}\hat{\zeta}$ this leads to a confluent Heun equation in the form:

$$\frac{d^2Y}{d\hat{\zeta}^2} + \left(\frac{\lambda}{\hat{\zeta}} - 1 + \frac{1-N}{\hat{\zeta}-\hat{a}}\right)\frac{dY}{d\hat{\zeta}} + \left(-\frac{c}{\hat{\zeta}} + \frac{\hat{q}}{\hat{\zeta}(\hat{\zeta}-\hat{a})}\right)Y = 0. \tag{4.17}$$

The exponents at the point $\hat{\zeta} = \hat{a}$ are $0, N$. That is, if we pose a local Taylor expansion of the solutions local to this point there are no logarithmic terms a relation emerges connecting $\hat{q}$ and $\hat{a}$; in general this is a polynomial in q of degree N. If $N = 2$ this is just $\hat{q}^2 + (\lambda - \hat{a} - 1)\hat{q} - c\hat{a} = 0$.

The solutions of equation (4.17) belong to the confluent Fuchsian space Y corresponding to

$$P_{F\infty}\left(\begin{matrix} 0 & \infty & \\ 0 & 0,\ c & \zeta \\ 1-\lambda & 1,\ \lambda - c + 1 - N & \end{matrix}\right). \tag{4.18}$$

For this notation the parameter $\epsilon^* = N - 1$.

As described earlier we aim to find the solution in terms of a linear combination of Fuchsian spaces

$$P_{F\infty}\left(\begin{matrix} 0 & \infty & \\ 0 & 0,\ c & \zeta \\ m-\lambda & 1,\ \lambda - c + 1 - m & \end{matrix}\right), \tag{4.19}$$

for $m = 1, \ldots, N$. The solutions to the $P_{F\infty}$-functions are found from Kummer's equation. The linearly independent solutions are

$$U_1^{(m)}(\hat{\zeta}) = \mathscr{M}(c, \lambda + 1 - m, \hat{\zeta}), \tag{4.20}$$

$$U_2^{(m)}(\hat{\zeta}) = \frac{\hat{\zeta}^{m-\lambda}\Gamma(m + c - \lambda)\Gamma(\lambda + 1 - m)\mathscr{M}(m + c - \lambda, 1 - \lambda + m, \hat{\zeta})}{\Gamma(c)\Gamma(1 - \lambda + m)}, \tag{4.21}$$

for $m = 1 \ldots N$, where $\mathscr{M}(a, b, z)$ is Kummer's function (Abramowitz & Stegun, 1969, Ch. 13). The coefficients are chosen in such a way that the mappings generated by the correspondence between the basis are isomonodromies. This is not a unique choice, we could reorder these, or take linear combinations of them, if we so desired, but that would not alter the fundamentals of what follows.

Consider the case $N = 2$. A direct substitution of a linear combination of $U_1^{(1)}$ and $U_1^{(2)}$ into the equation (4.17) shows that the solution is

$$Y(\zeta) = (\hat{q} - \hat{a})\mathscr{M}(c, \lambda, \hat{\zeta}) + (\lambda - 1)\mathscr{M}(c, \lambda - 1, \hat{\zeta}); \tag{4.22}$$

the other linearly independent solution following by use of equation (4.21). The solution is a linear combination of the solutions of two different confluent hypergeometric equations; this mirrors our earlier approach to Heun's equation.

Increasing values of N lead to gradually more complicated algebraic relations, but with the simple basic form that we expect: for instance, after some algebra, for $N = 3$ one has

$$\begin{aligned} Y(\zeta) &= a(2a - q)(\lambda - 1 - c)\mathscr{M}(c, \lambda, \hat{\zeta}) \\ &-(\lambda - 1)(2[\lambda - 1] + q)[(2a - q)\mathscr{M}(c, \lambda - 1, \hat{\zeta}) - (\lambda - 2)\mathscr{M}(c, \lambda - 2, \hat{\zeta})] \end{aligned} \tag{4.23}$$

with q determined from the cubic

$$[q(q + \lambda - a - 2) - 2ac][2(\lambda - a - 1) + q] - 2a(c + 1)q = 0. \tag{4.24}$$

5 Applications

We now describe applications of the theory in applied areas.

5.1 Diffraction theory and embedding formulae

Consider the problem of diffraction of a plane wave by an ideal strip with Dirichlet boundary conditions. As shown in Noble (1958), this problem can be reformulated as a Wiener–Hopf equation of the form

$$e^{\mathrm{i}kd}\Phi_+(k) + \frac{\Phi_1(k)}{\sqrt{k_0^2 - k^2}} + e^{-\mathrm{i}kd}\Phi_-(k) = 2\frac{\sin(k - k_*)d}{k - k_*}, \tag{5.1}$$

where k_0 is the wavenumber of the waves in the medium, $2d$ is the width of the strip, $\Phi_\pm$ are unknown functions regular in the upper ($+$) and lower ($-$) half-planes, and growing no faster than $k^{-3/2}$ there, Φ_1 is an entire function growing no faster than $k^{-1/2}e^{-\mathrm{i}kd}$ in the upper half-plane ($k^{-1/2}e^{\mathrm{i}kd}$ in the lower half-plane). The parameter k_* is the projection of the wavenumber of the incident wave on the strip line.

Consider the following functions:

$$U_\pm(k) = (k - k_*)e^{\pm\mathrm{i}kd}\Phi_\pm(k) \pm \mathrm{i}e^{\pm\mathrm{i}(k-k_*)d}, \quad U_1(k) = \frac{(k - k_*)\Phi_1(k)}{\sqrt{k_0^2 - k^2}}. \tag{5.2}$$

Using these functions we rewrite (5.3) as

$$U_+(k) + U_1(k) + U_-(k) = 0. \tag{5.3}$$

Relation (5.3) can be used for analytic continuation of U_+ (U_-) into the lower (upper) half-planes. Using analytic continuation, we see that the functions $U_\pm$ and U_1 are elements of a confluent Fuchsian space U of dimension 2, having the following regular singular points, exponents and corresponding fundamental functions:

$$\pm k_0: \quad (0, -1/2), \quad (U_\pm, U_1).$$

At infinity there is an irregular singular point with asymptotic behaviour $k^{-1/2}e^{\pm\mathrm{i}kd}$, where the "$\pm$" correspond to $U_\pm$. There are additional restrictions that reflect the dependence of the solution on the angle of incidence.

$$U_1(k_*) = 0, \qquad U_+(k_*) = \mathrm{i}. \tag{5.4}$$

First, let us ignore the relations (5.4). The diffraction problem is reduced to finding a confluent Fuchsian space. Using the methodology constructed earlier, the functions $U_\pm$, and U_1, are solutions of a confluent Fuchsian equation. For brevity we omit the explicit equation which is given in Shanin (2001a). For our current purposes all necessary information is provided by the known behaviour of the solutions at the singular points.

Consider the exponents at the singular points: the parameter ϵ^* is equal to 2, therefore, it has two false singular points. The parameter δ^* is equal to 0, therefore one singular point can be removed, thereby simplifying the problem.

To proceed we want to find auxiliary confluent Fuchsian spaces U^1 and U^2, having the following properties: The bases of the spaces are (U_+^1, U_-^1), (U_+^2, U_-^2), moreover, there are functions $U_1^1 = -(U_+^1 + U_-^1)$ and $U_1^2 = -(U_+^2 + U_-^2)$. The points $\pm k_0$ are regular singular points of the spaces; the exponents and the pairs of the fundamental solutions are similar to that of U. Infinity is an irregular singular point. The behaviour there is different from that of U, namely,

$$U_+^1 \sim k^{-3/2}e^{ikd}, \quad U_1^1 \sim k^{-3/2}e^{-ikd} \qquad U_+^2 \sim k^{-1/2}e^{ikd}, \quad U_1^2 \sim k^{-1/2}e^{-ikd}$$

in the upper half-plane,

$$U_-^1 \sim k^{-1/2}e^{-ikd}, \quad U_1^1 \sim k^{-1/2}e^{ikd} \qquad U_-^2 \sim k^{-3/2}e^{-ikd}, \quad U_1^2 \sim k^{-3/2}e^{ikd}$$

in the lower half-plane. The exponents of the auxiliary spaces at infinity are weakened in comparison with U, in agreement with the method of removing false singular points that we have described.

Each auxiliary problem has a simple physical interpretation. It corresponds to a diffraction problem with a line source located at one of the edges of the strip. Note that these solutions do not depend upon the parameter k_*. The auxiliary spaces are solutions of auxiliary confluent Fuchsian equations having just one false singular point. Unfortunately, the parameter δ^* is equal to -1 and this remaining false point cannot be removed. Therefore, the auxiliary solutions are not well-known special functions from, say, Abramowitz & Stegun (1969).

By construction, the auxiliary spaces are isomonodromical to U, therefore, one can seek U in the form

$$\mathrm{U} = C_1\mathrm{U}_1 + C_2\mathrm{U}_2.$$

The constants C_1 and C_2 are taken such that the relations (5.4) are fulfilled. Indicating explicitly the dependence of all values on the parameter k_*, one can write, finally, that

$$\Phi(k, k_*) = \frac{\sqrt{k_0^2 - k^2}}{k - k_*}[C_1(k_*)U_1^1(k) + C_2(k_*)U_1^2(k)]. \tag{5.5}$$

This is the embedding formula, that is, it represents the full solution in terms of simpler solutions independent of k_*. It is closely related to embedding formulae that emerge from integral equation representations (Williams, 1982; Biggs *et al.,* 2000, 2001). The result is important not only because $U_1^{1,2}$ are simpler than U_1, but because $\Phi(k, k_*)$ in (5.5), dependant on two variables, becomes expressed through several functions depending on a single variable. In practical terms this is a considerable simplification.

5.2 Riemann–Hilbert methods

The original motivation behind Polubarinova-Kochina's approach of solving free boundary problems was to avoid a matrix Riemann–Hilbert formulation which appears slightly disguised her book, Polubarinova-Kochina (1962), and explicitly in Tsitskishvili (1976), amongst others.

Often functions in the upper half of a complex plane are defined via conditions connecting their imaginary parts along the real axis. If these conditions are linear and alter at 0, 1 and ∞, we can then analytically continue a function, say, $W(\zeta)$ into the lower half of the complex ζ plane as

$$W(\zeta) = W(\zeta) \text{ for } \mathrm{Im}(\zeta) > 0, \quad W(\zeta) = \overline{W(\overline{\zeta})} \text{ for } \mathrm{Im}(\zeta) < 0. \tag{5.6}$$

Using $\pm$ to denote limiting values as we approach the upper/ lower edges of $\mathrm{Im}(\zeta) = 0$, then for two functions, a system of piecewise continuous homogeneous matrix Riemann–Hilbert problems appears,

$$\begin{pmatrix} W_+ \\ Z_+ \end{pmatrix} = G \begin{pmatrix} W_- \\ Z_- \end{pmatrix} \tag{5.7}$$

where the matrix G is, say,

$$G = \begin{pmatrix} \cos 2\pi\mu_2 & -i\sin 2\pi\mu_2 \\ -i\sin 2\pi\mu_2 & \cos 2\pi\mu_2 \end{pmatrix} \quad \mathrm{Re}(\zeta) < 0, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 0 < \mathrm{Re}(\zeta) < 1,$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi\mu_1} \end{pmatrix} \quad \mathrm{Re}(\zeta) > 1. \tag{5.8}$$

with μ_1, μ_2 constants.

To solve this Riemann–Hilbert problem, using Fuchsian equations via an appropriate Fuchsian space of functions, the functions $W(\zeta)$ and $Z(\zeta)$ are taken to form a basis of the space Y. This space has three strong singular points: 0, 1 and ∞.

Asymptotic behaviour of $W(\zeta), Z(\zeta)$ local to the strong points is required, and for the example above the space has the notation

$$P_F \begin{pmatrix} 0 & 1 & \infty & \\ m_1 - \mu_2 & m_3 & m_5 - (\mu_1 + \mu_3)/2 - 1 & \zeta \\ m_2 - 1 + \mu_2 & m_4 - \mu_1 & m_6 - (\mu_1 - \mu_3)/2 - 1 & \end{pmatrix}$$

for some integers $m_i, i = 1 \dots 6$ which follow from the local behaviour of W and Z (μ_3 is defined as $\cos \pi\mu_3 = \cos 2\pi\mu_2 \cos \pi\mu_1$ for the example above).

Additionally, the formulae for G enable us to find fundamental solutions near 0 and 1. Namely, at the singular point $\zeta = 0$ the fundamental solutions corresponding to the exponents $m_1 - \mu_2$ and $m_2 - 1 + \mu_2$ respectively are

$$Y_{1,1}(\zeta) = W(\zeta) + Z(\zeta), \quad Y_{1,2}(\zeta) = W(\zeta) - Z(\zeta), \tag{5.9}$$

(cf equation (3.5) of § 3.3), with similar equations at 1 and ∞.

Given integer constants $m_1 \dots m_6$ from the local behaviour, we must find a Fuchsian space with prescribed exponents and restrictions on the monodromy data. Using our technique, the solution comes from a Fuchsian differential equation having three strong singularities and, dependant upon the m_i's, some false singularities, it is in the form of a linear combination of hypergeometric functions multiplied by elementary functions (see Craster (1997) and Craster & Hoang (1998)), although those articles use direct methods rather than that advocated here. Importantly, the methodology described in this paper tells us how many false points can be removed, and provides a route to do so.

Perhaps surprisingly, there is an intimate connection between these matrix Riemann–Hilbert problems, related Wiener–Hopf formulations, and Fuchsian equations; we advocate this Fuchsian approach as a useful line of attack.

There is potential for this approach to generalize, if the matrices are all diagonal or upper, or lower, triangular (the monodromy data is then trivial); one can reformulate the original boundary value problem in, for instance, free boundary problems to utilize a potential plane, Zhukovsky function, or the method of inversion. Additional singular points in this context can be introduced and for four unremovable points Heun polynomial solutions are found; perhaps confusingly, they are not actually polynomials, but of the form $\zeta^{\alpha_1}(\zeta-1)^{\alpha_2}(\zeta-a)^{\alpha_3}P(\zeta)$, with $P(\zeta)$ polynomial.

6 Concluding remarks

This study confirms, and explores in depth, the approach by which some points are removed from a Fuchsian equation. We find that some points can be removed, specifically if these points have exponents $(0, N)$ and no logarithmic behaviour in the local expansion; this is reasonably common in applications.

Moreover, we generate a notation and some useful theorems, the main results of which are the following: the solution of the Fuchsian equation having r strong points and ϵ false points of the exponents $(0, 2)$ can be represented in the form of a linear combination of the solutions of several different equations having r strong points and $r-3$ false points; an irregular singular point is equivalent to two strong points; and the false point with exponents $(0, N)$ is equivalent to $N-1$ false points with exponents $(0, 2)$.

This identification of false and strong points allows us to proceed rapidly when faced with the problems such as those illustrated in section 4. Several specific details emerge that are useful, first when dealing with exponents $(0, 2)$ the heuristic argument used by Craster (1997) to identify, what we call here, auxiliary spaces can be sidestepped and there are in fact several auxiliary spaces that could be used, not just the specific pair used there. This non-unique choice of auxiliary spaces does not mean that the final result is non-unique, but simply that there are many interrelations between these auxiliary spaces and that any single space can be written in terms of any other. Second the analysis generalizes naturally to exponents $(0, N)$, and finally that the Wiener–Hopf and Mellin transform approach is now redundant as the theory now fits into that of Fuchsian equations in a consistent manner. As the method is now systematic it is much easier to apply.

It is worthwhile to briefly contrast our approach with that of Erdélyi (1942, 1944), these papers are more general in some sense as the point a is not necessarily false and solutions for general Heun's equations are sought as hypergeometric series in the form

$$P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \lambda+m & \zeta \\ 1-\gamma & 1-\delta & \mu-m & \end{pmatrix}, \tag{6.1}$$

via a parameterization in m, that is, the behaviour of two neighbouring singular points is fixed and an expansion proceeds by varying the exponents at the remaining fixed point, either at infinity, or at the origin. In some circumstances, which depend upon specific values of the exponents of the Heun equation at $0, 1, \infty$ the series terminates and degenerate cases emerge that are Heun polynomials, see Ronveaux (1995). The method we utilize is focused on a specific type of behaviour at a and fixes the behaviour at only one of the remaining singular points, whilst systematically changing both the others (not one as in Erdélyi). So, if viewed as an expansion our Ansatz is different. The solutions we find are, on one hand, restricted to a being a false singular point, but on the other hand we generate explicit solutions that are combinations of hypergeometric functions that need not be degenerate and which would not be found using Erdélyi's approach.

It is also worth noting that there are connections with Painlevé equations, and generalizations of the Darboux–Halphens system, and these connections would be interesting to explore further.

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References

[1] Abramowitz, M. & Stegun, I. A. (1969) *Handbook of Mathematical Functions*. Dover.

[2] Arscott, F. M. & Darai, A. (1981) Curvilinear co-ordinate systems in which the Helmholtz equation separates. *IMA J. Appl. Math.* **27**, 33–70.

[3] Biggs, N. R. T., Porter, D. & Stirling, D. S. G. (2000) Wave diffraction through a perforated breakwater *Q. J. Mech. Appl. Math.* **53**, 375–391.

[4] Biggs, N. R. T. & Porter, D. (2001) Wave diffraction through a perforated barrier of non-zero thickness *Q. Jl. Mech. Appl. Math.* **54**, 523–547.

[5] Craster, R. V. (1996) Conformal mappings involving curvilinear quadrangles. *IMA J. Appl. Math.* **57**, 181–191.

[6] Craster, R. V., (1997) The solution of a class of free boundary problems. *Proc. Roy. Soc. Lond. A* **453**, 607–630.

[7] Craster, R. V. & Hoang, V. H., (1998) Applications of Fuchsian differential equations to free boundary problems. *Proc. Roy. Soc. Lond. A* **454**, 1241–1252.

[8] Cummings, L. J., (1999) Flow around a wedge of arbitrary angle in a Hele-Shaw cell. *Euro. J. Appl. Math.* **10**, 547–560.

[9] Erdélyi, A. (1942) The Fuchsian equation of second order with four singularities. *Duke Math. J.* **9**, 48–58.

[10] Erdélyi, A. (1944) Certain expansions of the solutions of the Heun equation. *Quart. J. Math. Oxford Ser.* **15**, 62–69.

[11] Hoang, H. V., Hill, J. M. & Dewynne, J. N. (1998) Pseudo steady-state solutions for solidification in a wedge. *IMA J. Appl. Math.* **60**, 109–121.

[12] Howison, S. D. & King, J. R. (1989) Explicit solutions to six free-boundary problems in fluid flow and diffusion. *IMA J. Appl. Math* **42**, 155–175.

[13] Ince, E. L. (1927) *Ordinary Differential Equations*. Longman.

[14] Jimbo, M., Miwa, T. & Ueno, K. (1981) Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I. General theory and tau function. *Physica D*, **2**, 306–352.

[15] Komarev, I. V., Ponomarev, L. I. & Slavyanov, S. Y. (1976) *Spheroidal and Coulomb Spheroidal Functions*. Nauka, Moscow.

[16] Leaver, E. W. (1986) Solutions to a generalized spheroidal wave equation: Teukolsky's equations in general relativity, and the two-center problem in molecular quantum mechanics. *J. Maths. Phys.* **27**, 1238–1265.

[17] Levai, G. & Williams, B. W. (1993) The generalized Coulomb problem: An exactly solvable model. *J. Phys. A: Math. Gen.* **26**, 3301–3306.

[18] Nehari, Z. (1952) *Conformal Mapping*. McGraw-Hill.

[19] Noble, B. (1958) *Methods Based Upon the Wiener–Hopf Technique*. Pergamon.

[20] Olver, F. (1974) *Asymptotics and Special Functions*. Academic Press.

[21] Polubarinova-Kochina, P. Y. (1939a) An application of the theory of linear differential equations to some problems of groundwater motion (three singular points). *Izv. A.N. CCCP*, **3**, 329–350.

[22] Polubarinova-Kochina, P. Y. (1939b) An application of the theory of linear differential equations to some problems of groundwater motion (number of singular points greater than three). *Izv. A.N. CCCP*, **5**, 579–602.

[23] Polubarinova-Kochina, P. Y. (1991) Accessory parameters in circular quadrangles. *Prikl. Matem. Mekhan.* **55**, 222–227.

[24] Ronveaux, A. (1995) *Heun's Differential Equations*. Oxford University Press.

[25] Shanin, A. V. (2001a) Three theorems concerning diffraction by a strip or a slit. *Q. J. Math. Appl. Mech.* **54**, 107–137.

[26] Shanin, A. V. (2001b) Diffraction on a slit: Some properties of Schwarzchild's series. *Int. Sem. 'Days on Diffraction', May 29–June 3, St. Petersburg: Zapiski Seminarov, 275, 258–285* (In Russian).

[27] Sleeman, B. D. (1967) The scalar scattering of a plane wave by an ellipsoid. *J. Inst. Math. Applic.* **3**, 4–15.

[28] Tsitskishvili, A. R. (1976) Conformal mapping of a half plane to a curvilinear pentagon with a cut. *Differentsial'nye Uravneniya*, **12**, 2044–2051.

[29] Valent, G. (1986) An integral transform involving Heun functions and a related eigenvalue problem. *SIAM J. Math. Anal.* **17**, 688–703.

[30] Whittaker, E. T. & Watson, G. N. (1927) *A Course of Modern Analysis*. Cambridge University Press.

[31] Williams, M. H. (1982) Diffraction by a finite strip. *Q. J. Math. Appl. Mech.* **35**, 103–124.