# Diffraction by two ideal strips

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October 30, 2001

**Abstract**

The problem of scattering of a plane wave by two ideal strips lying in one plane is studied. The Wiener-Hopf functional equation is formulated and studied. The following results are obtained. 1) The embedding formula is derived. This formula enables to express the far-field diagram, depending on two variables (the angle of incidence and the angle of scattering) as the combination of 4 functions depending on one variable. 2) The ordinary differential equation with respect to the spectral variable is derived for the components of the far-field diagram. 3) The evolution equation describing the dependence of the far-field diagram on the parameters of the problem (such as the coordinates of the edges of the scatterer) are derived.

## 1 Formulation of the problem

Let the 2D Helmholtz equation

$$\Delta u + k_0^2 u = 0 \tag{1.1}$$

be valid on the $(x, y)$ plane. The cross-sections of two strips coincide with the segments $(a_1, a_2)$ and $(a_3, a_4)$ of the x-axis (see Fig. 1). We imply that the time dependence everywhere has the form $e^{-\mathrm{i}\omega t}$. Sometimes it would also be useful to imply that $k_0$ has a small positive imaginary part corresponding to small dissipation in the media.

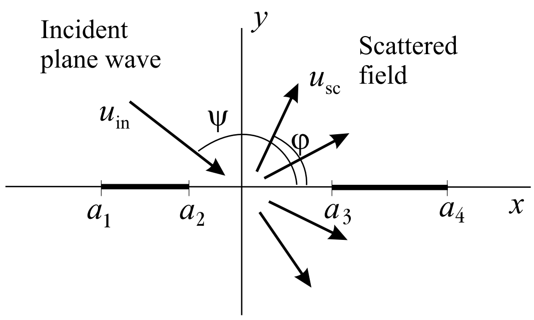


Figure 1: Geometry of the problem

Let the Dirichlet boundary conditions

$$u = 0 \tag{1.2}$$

be valid on the strips.

The incident field is a plane wave coming from the upper half-plane

$$u^{\mathrm{in}} = e^{-\mathrm{i}k_* x - \mathrm{i}\sqrt{k_0^2 - k_*^2}\, y}, \tag{1.3}$$

where $k_* = k_0 \cos\psi$; $\psi$ is the angle of incidence.

Due to the obvious symmetry of the scattered field, a problem with mixed boundary conditions can be formulated for it:

$$u^{\rm sc}(x,0) = -e^{-{\rm i}k_* x} \qquad \text{for} \quad x \in (a_1,a_2)\cup(a_3,a_4), \tag{1.4}$$

$$\frac{\partial}{\partial y}u^{\rm sc}(x,0) = 0 \qquad \text{for} \quad x \in (-\infty,a_1)\cup(a_2,a_3)\cup(a_4,\infty). \tag{1.5}$$

It is clear that one can consider the field only in the half-plane $y > 0$.

The radiation conditions are satisfied at infinity. The scattered field should not contain the components coming from infinity or growing at infinity. These conditions will be taken into account when constructing the integral representation of the scattered field.

Meixner edge conditions should be taken into account. Here we demand that the total field should have the asymptotic near the edges that is similar to the asymptotics for the static problem, namely

$$u \sim r^{1/2}, \tag{1.6}$$

where $r$ is the distance between the observation point and the nearest edge. The normal derivative on the strips, therefore, behaves like $r^{-1/2}$.

## 2 Derivation of the functional equation

Here we follow [1]. Represent the scattered field for $y > 0$ in the integral form

$$u^{\rm sc}(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} A(k) e^{-{\rm i}kx + {\rm i}\sqrt{k_0^2-k^2}y} dk. \tag{2.1}$$

Here the branch of the square root is chosen such that the field satisfies the radiation condition. Namely, it corresponds to the path of integration bypassing below the point $k = k_0$ and above the point $k = -k_0$.

Perform the Fourier transformation. The spectral function $A(k)$ can be expressed in terms of the values of $u(x,0)$:

$$A(k) = \int\limits_{-\infty}^{\infty} u^{\rm sc}(x,+0) e^{{\rm i}kx} dx. \tag{2.2}$$

Taking into account the boundary conditions (1.4), (1.5), we rewrite (2.2) in the form

$$A(k) = \left(\int\limits_{-\infty}^{a_1} + \int\limits_{a_2}^{a_3} + \int\limits_{a_4}^{\infty}\right) u^{\rm sc}(x,0)e^{{\rm i}kx}dx + {\rm i}\frac{-e^{{\rm i}(k-k_*)a_1} + e^{{\rm i}(k-k_*)a_2} - e^{{\rm i}(k-k_*)a_3} + e^{{\rm i}(k-k_*)a_4}}{k-k_*}. \tag{2.3}$$

On the other hand, the spectral function $A(k)$ can be calculated through the normal derivative of the field:

$${\rm i}\sqrt{k_0^2-k^2}A(k) = \int\limits_{-\infty}^{\infty} \frac{\partial u^{\rm sc}(x,+0)}{\partial y} e^{{\rm i}kx}dx = \left(\int\limits_{a_1}^{a_2} + \int\limits_{a_3}^{a_4}\right) \frac{\partial u^{\rm sc}(x,+0)}{\partial y} e^{{\rm i}kx}dx. \tag{2.4}$$

Comparing (2.3) with (2.4) we obtain the *functional equation*:

$$U_0(k) + U_1(k) + U_2(k) + U_3(k) + U_4(k) = 0 \tag{2.5}$$

for all real $k$, where

$$
\begin{aligned}
U_0(k) &= \int\limits_{-\infty}^{a_1} u^{\rm sc}(x,0)e^{{\rm i}kx}dx - \frac{{\rm i}e^{{\rm i}(k-k_*)a_1}}{k-k_*}, \\
U_1(k) &= \frac{\rm i}{\sqrt{k_0^2-k^2}}\int\limits_{a_1}^{a_2} \frac{\partial u^{\rm sc}(x,0)}{\partial y}e^{{\rm i}kx}dx. \\
U_2(k) &= \int\limits_{a_2}^{a_3} u^{\rm sc}(x,0)e^{{\rm i}kx}dx + \frac{{\rm i}e^{{\rm i}(k-k_*)a_2}}{k-k_*} - \frac{{\rm i}e^{{\rm i}(k-k_*)a_3}}{k-k_*}, \\
U_3(k) &= \frac{\rm i}{\sqrt{k_0^2-k^2}}\int\limits_{a_3}^{a_4} \frac{\partial u^{\rm sc}(x,0)}{\partial y}e^{{\rm i}kx}dx. \\
U_4(k) &= \int\limits_{a_4}^{\infty} u^{\rm sc}(x,0)e^{{\rm i}kx}dx + \frac{{\rm i}e^{{\rm i}(k-k_*)a_4}}{k-k_*}.
\end{aligned}
\tag{2.6}
$$

It is clear that the spectral functions $U_0 \dots U_4$ are connected with the field defined on the fragments, in which the $x$-axis is decomposed by the points $a_1 \dots a_4$.

Note that the spectrum of the field $A(k)$ is equal to

$$
A(k) = -\left(U_1(k) + U_3(k)\right).
$$

We introduce the *far-field diagram* as

$$
f(k,k_*) \equiv -\sqrt{k_0^2-k^2}A(k) = \sqrt{k_0^2-k^2}\left(U_1(k)+U_3(k)\right). \tag{2.7}
$$

It is common to express $f$ as a function of the angular variables $\varphi$ and $\psi$ (the angles of scattering and incidence, respectively). The angular variables are connected with the wavenumber variables $k$, $k_*$ via the relations

$$
k = -k_0\cos\varphi, \qquad k_* = k_0\cos\psi. \tag{2.8}
$$

We shall use below the obvious corollary of the functional equation (2.5). In the set of five unknown functions $U(k)$ there are only 4 linearly independent, i.e. any unknown function can be written as a linear combination of 4 other functions.

# 3 Analytic properties of the unknown functions

Functional equation (2.5) contains 5 unknown functions. To make it possible to solve this equation, one should impose some additional restrictions on the unknown functions.

Consider the definitions of the unknown functions (2.6). These formulas are close to the Fourier (Laplace) transforms of the functions defined on a half-line or a segment. The properties of such transforms are well known. For example, the Fourier transform of a function defined on a positive half-axis (such as $U_4(k)$ for $a_4 = 0$) is a regular function in the upper half-plane of the complex variable $k$, moreover, its growth in the upper half-plane is determined by the asymptotics of function $u^{\rm sc}(x,0)$ near the edge $x = a_4$, known from the Meixner's conditions.

Applying known properties of Fourier transformation to the definitions (2.6), we obtain that:

- the function $U_0$ is regular in the lower half-plane (maybe with the exception of the point $k = k_*$, where there is a pole with the known residue);
- the function $U_4$ is regular in the upper half-plane (maybe with the exception of the point $k = k_*$, where there is a pole with the known residue);

- the functions $U_2$, $\sqrt{k_0^2 - k^2}U_1$, $\sqrt{k_0^2 - k^2}U_3$ are entire on the whole $k$ plane.

Actually, as it will be clear later, these functions can have only the following singularities: $k = \infty$, $k_*$, $\pm k_0$. The singularity $k = k_*$ causes no problems: only simple poles can exist in it. The behaviour of the unknown functions at $k = \pm k_0$ is more complicated, since the functions can have branch points there.

Note that $k_0$ belongs to the upper half-plane and $-k_0$ belongs to the lower half-plane. It means that the properties mentioned above bring some useful information concerning the behaviour of the unknown functions at the branch points.

Besides, the following estimations of growth based on the Meixner's conditions are valid as $|k| \to \infty$:

$$\begin{array}{llll} U_0(k) \sim k^{-3/2} e^{\mathrm{i}ka_1} \quad \text{for} \quad \mathrm{Im}[k] < 0, & & U_4(k) \sim k^{-3/2} e^{\mathrm{i}ka_4} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ U_1(k) \sim k^{-3/2} e^{\mathrm{i}ka_2} \quad \text{for} \quad \mathrm{Im}[k] < 0, & & U_1(k) \sim k^{-3/2} e^{\mathrm{i}ka_1} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ U_2(k) \sim k^{-3/2} e^{\mathrm{i}ka_3} \quad \text{for} \quad \mathrm{Im}[k] < 0, & & U_2(k) \sim k^{-3/2} e^{\mathrm{i}ka_2} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ U_3(k) \sim k^{-3/2} e^{\mathrm{i}ka_4} \quad \text{for} \quad \mathrm{Im}[k] < 0, & & U_3(k) \sim k^{-3/2} e^{\mathrm{i}ka_3} \quad \text{for} \quad \mathrm{Im}[k] > 0, \end{array} \tag{3.1}$$

This fact is not easy to prove, but equation (2.5) with the analytic restrictions listed in this section is enough to determine the unknown functions. We shall call the set of the functional equation, conditions of analyticity and the estimations of growth having the form (3.1) a *functional problem.* Below we shall formulate some similar functional problems with slightly different asymptotics.

Following [1], we can derive a confluent Fuchsian ordinary differential equation (ODE) of order 4 for the functions $U$. The coefficients of this ODE are known up to several constant parameters. There are several monodromy restrictions that enable to determine these parameters.

However, the number of unknown parameters in this case is rather big and the numerical procedure is rather complicated. Also the research of [1] does not utilize the benefits of the embedding formula. That is why here we develop another technique (which is, certainly, is closely connected to that of [1]).

# 4 Auxiliary solutions, auxiliary functional problems and the embedding formula

The functions $U_0$, $U_1$, $U_2$, $U_3$, $U_4$ depend on two variables: $k$ and $k_*$. In the previous section we discussed the properties of these functions taking into account their dependence only on the first variable (the second variable was considered as a parameter of the problem). Here we shall discuss the dependence on the second variable. It seems surprising, but it is possible to extract the dependence on the angle of incidence.

First, consider the estimations of growth (3.1). It seems to be clear that each asymptotics is connected with one of 4 edges $a_1 \dots a_4$.

Introduce 4 auxiliary functional problems similar to the one formulated above. Namely, consider 4 sets of functions indexed by $m$ (note that $m$ is the upper index, not a power) $\{U_0^m(k), U_1^m(k), U_2^m(k), U_3^m(k), U_4^m\}$, where $m = 1 \dots 4$. For each $m$ formulate a functional equation similar to (2.5):

$$U_0^m(k) + U_1^m(k) + U_2^m(k) + U_3^m(k) + U_4^m(k) = 0 \tag{4.1}$$

Formulate the conditions of analyticity similar to those formulated in the previous section:

- the function $U_0^m$ is regular in the lower half-plane;
- the function $U_4^m$ is regular in the upper half-plane;
- the functions $U_2^m$, $\sqrt{k_0^2 - k^2}U_1^m$, $\sqrt{k_0^2 - k^2}U_3^m$ are entire on the whole $k$ plane.

Note that now the possibility of the pole at $k = k_*$ is excluded.

Finally, for each $m$ define a set of asymptotics of the form (3.1). However, it is necessary to weaken the restrictions related to the edge $a_m$, namely we replace the order of growth $k^{-3/2}$ with $k^{-1/2}$, i.e. we allow correspondent functions to grow faster.

For example, for $m = 2$ we find in (3.1) the asymptotics associated having $a_2$ in the exponent and weaken them. The result is:

$$\begin{array}{llllll} U_0^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_1} & \text{for} & \mathrm{Im}[k] < 0, & U_4^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_4} & \text{for} & \mathrm{Im}[k] > 0, \\ U_1^2(k) \sim k^{-1/2} e^{\mathrm{i}ka_2} & \text{for} & \mathrm{Im}[k] < 0, & U_1^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_1} & \text{for} & \mathrm{Im}[k] > 0, \\ U_2^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_3} & \text{for} & \mathrm{Im}[k] < 0, & U_2^2(k) \sim k^{-1/2} e^{\mathrm{i}ka_2} & \text{for} & \mathrm{Im}[k] > 0, \\ U_3^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_4} & \text{for} & \mathrm{Im}[k] < 0, & U_3^2(k) \sim k^{-3/2} e^{\mathrm{i}ka_3} & \text{for} & \mathrm{Im}[k] > 0. \end{array} \tag{4.2}$$

as $|k| \to \infty$. (Note the asymptotics for $U_1^2$, $\mathrm{Im}[k] < 0$ and $U_2^2$, $\mathrm{Im}[k] > 0$). Similarly we obtain the sets of asymptotic restrictions for other 3 values of $m$.

Let us discuss briefly the physics related to the auxiliary functional problems formulated here. Consider a particular value of $m$. Instead of a plane incident wave, take a point source located near the edge $x = a_m, y = 0$. Since there is no possibility to study the point source located just at the edge, let the source be located at the distance $r$ from the edge ($r$ is small), and let the limit $r \to 0$ be taken. One can see that the amplitude of the source should be proportional to $r^{-1/2}$ providing the existence of finite limit for this procedure.

The sets of unknown functions are determined by the auxiliary functional problems up to $m$ constant factors. To fix this uncertainty we declare that as $|k| \to \infty$

$$\begin{array}{lcll} U_1^1 & = & k^{-1/2} e^{\mathrm{i}ka_1} + O(k^{-3/2} e^{\mathrm{i}ka_1}) & \text{for } \mathrm{Im}[k] > 0, \\ U_2^2 & = & k^{-1/2} e^{\mathrm{i}ka_2} + O(k^{-3/2} e^{\mathrm{i}ka_2}) & \text{for } \mathrm{Im}[k] > 0, \\ U_3^3 & = & k^{-1/2} e^{\mathrm{i}ka_3} + O(k^{-3/2} e^{\mathrm{i}ka_3}) & \text{for } \mathrm{Im}[k] > 0, \\ U_4^4 & = & k^{-1/2} e^{\mathrm{i}ka_4} + O(k^{-3/2} e^{\mathrm{i}ka_4}) & \text{for } \mathrm{Im}[k] > 0, \end{array} \tag{4.3}$$

i.e. we demand that the point sources near the edges have the unit strength in some sense. Chosen branch of the square root is real on the real positive half-axis.

The emebedding formula expresses the relation between the solution of initial diffraction problem (with the plane-wave incidence) and the auxiliary problems. First we "guess" the form of this formula, then we study the coefficients and prove that they are have a very simple structure.

Let us find the coefficients $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, depending on $k$ and $k_*$, such that

$$\begin{array}{lcl} U_1(k, k_*) & = & \alpha_1 U_1^1(k) + \alpha_2 U_1^2(k) + \alpha_3 U_1^3(k) + \alpha_4 U_1^4(k), \\ U_2(k, k_*) & = & \alpha_1 U_2^1(k) + \alpha_2 U_2^2(k) + \alpha_3 U_2^3(k) + \alpha_4 U_2^4(k), \\ U_3(k, k_*) & = & \alpha_1 U_3^1(k) + \alpha_2 U_3^2(k) + \alpha_3 U_3^3(k) + \alpha_4 U_3^4(k), \\ U_4(k, k_*) & = & \alpha_1 U_4^1(k) + \alpha_2 U_4^2(k) + \alpha_3 U_4^3(k) + \alpha_4 U_4^4(k). \end{array} \tag{4.4}$$

Note that (4.4) and the functional equations yield also

$$U_0(k, k_*) = \alpha_1 U_0^1(k) + \alpha_2 U_0^2(k) + \alpha_3 U_0^3(k) + \alpha_4 U_0^4(k). \tag{4.5}$$

The equations (4.4) can be solved as a system of linear algebraic equations with respect to $\alpha_m$ using the Cramer's rule. The solution can be found immediately:

$$\alpha_1 = \frac{D_1}{D}, \qquad \alpha_2 = \frac{D_2}{D}, \qquad \alpha_3 = \frac{D_3}{D}, \qquad \alpha_4 = \frac{D_4}{D}, \tag{4.6}$$

where $D_1 \dots D_4$ and $D$ are the determinants:

$$D = \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 & U_1^4 \\ U_2^1 & U_2^2 & U_2^3 & U_2^4 \\ U_3^1 & U_3^2 & U_3^3 & U_3^4 \\ U_4^1 & U_4^2 & U_4^3 & U_4^4 \end{vmatrix}, \qquad D_1 = \begin{vmatrix} U_1 & U_1^2 & U_1^3 & U_1^4 \\ U_2 & U_2^2 & U_2^3 & U_2^4 \\ U_3 & U_3^2 & U_3^3 & U_3^4 \\ U_4 & U_4^2 & U_4^3 & U_4^4 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} U_1^1 & U_1 & U_1^3 & U_1^4 \\ U_2^1 & U_2 & U_2^3 & U_2^4 \\ U_3^1 & U_3 & U_3^3 & U_3^4 \\ U_4^1 & U_4 & U_4^3 & U_4^4 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} U_1^1 & U_1^2 & U_1 & U_1^4 \\ U_2^1 & U_2^2 & U_2 & U_2^4 \\ U_3^1 & U_3^2 & U_3 & U_3^4 \\ U_4^1 & U_4^2 & U_4 & U_4^4 \end{vmatrix}, \qquad D_4 = \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 & U_1 \\ U_2^1 & U_2^2 & U_2^3 & U_2 \\ U_3^1 & U_3^2 & U_3^3 & U_3 \\ U_4^1 & U_4^2 & U_4^3 & U_4 \end{vmatrix} \tag{4.7}$$

These determinants have very interesting properties. Consider $D$ as an example.

Due to the representation (4.7), we have the regularity restriction on the behaviour of all functions in the upper half-plane of the variable $k$, namely each element of this determinant is either a regular function in the upper half-plane or can be represented as a product of a regular function multiplied by $(k_0^2 - k^2)^{-1/2}$. Beside that, we have the estimation of growth in the upper half-plane for each function. Collecting all this information together, we conclude that the product $(k^2 - k_0^2)D(k)$ is regular in the upper half-plane and $D$ grows in the upper half-plane like $k^{-2}e^{\mathrm{i}k(a_1+a_2+a_3+a_4)}$.

The function $(k^2 - k_0^2)e^{-\mathrm{i}k(a_1+a_2+a_3+a_4)}D(k)$ is regular in the upper half-plane and grows there no faster than a constant.

Due to the functional equations and the properties of the determinants, the same determinant $D$ can be written in another form:

$$D = - \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 & U_1^4 \\ U_2^1 & U_2^2 & U_2^3 & U_2^4 \\ U_3^1 & U_3^2 & U_3^3 & U_3^4 \\ U_0^1 & U_0^2 & U_0^3 & U_0^4 \end{vmatrix} \tag{4.8}$$

Using this representation we can study the properties of $D$ in the lower half-plane. Namely, we can prove that the function $(k^2 - k_0^2)e^{-\mathrm{i}k(a_1+a_2+a_3+a_4)}D(k)$ is regular in the lower half-plane, and it grows there no faster than a constant.

Apply the Louville theorem to the function $(k^2 - k_0^2)e^{-\mathrm{i}k(a_1+a_2+a_3+a_4)}D(k)$. According to the properties mentioned above, this function should be identically equal to a constant. Taking into account the asymptotics (4.3), we find that:

$$D(k) = \frac{e^{\mathrm{i}k(a_1+a_2+a_3+a_4)}}{k^2 - k_0^2}. \tag{4.9}$$

Using the same technique, we find the expressions for other determinants:

$$D_m(k) = \frac{C_m(k_*)e^{\mathrm{i}k(a_1+a_2+a_3+a_4)}}{(k^2 - k_0^2)(k - k_*)}, \qquad m = 1 \dots 4, \tag{4.10}$$

where the values $C_m$ do not depend on the variable $k$.

Therefore,

$$U_\nu(k, k_*) = \frac{1}{k - k_*} \sum_{m=1}^{4} C_m(k_*) U_\nu^m(k), \tag{4.11}$$

where $\nu = 0 \dots 4$, and this is the weakest form of the embedding formula.

The constants can be taken from the asymptotics of the functions $U_\nu$. Namely, let the following asymptotics be valid:

$$\begin{aligned} U_1(k, k_*) &= c_1(k_*)k^{-3/2}e^{\mathrm{i}ka_1} + O(k^{-5/2}e^{\mathrm{i}ka_1}) \\ U_2(k, k_*) &= c_2(k_*)k^{-3/2}e^{\mathrm{i}ka_2} + O(k^{-5/2}e^{\mathrm{i}ka_2}) \\ U_3(k, k_*) &= c_3(k_*)k^{-3/2}e^{\mathrm{i}ka_3} + O(k^{-5/2}e^{\mathrm{i}ka_3}) \\ U_4(k, k_*) &= c_3(k_*)k^{-3/2}e^{\mathrm{i}ka_4} + O(k^{-5/2}e^{\mathrm{i}ka_4}) \end{aligned} \tag{4.12}$$

as $k \to \mathrm{i}\infty$. Substitute these asymptotics into (4.8). By calculating the asymptotics of the determinants $D_m$ in the upper half-plane, we obtain the expression (4.10) with $C_m(k_*) = c_m(k_*)$.

The physical meaning of the constant $c_1 \dots c_4$ is the following. They are connected with the edge asymptotics of the field, caused by the incident plane wave falling at the angle $\psi$. It follows from the reciprocity theorem that this problem can be reversed, i.e. one can put the source near one of the edges and find the amplitude of the outgoing wave at the angle $\psi$. We remind that the auxiliary functions are the spectral functions related to the source near one of the edge, The detailed calculations show that

$$C_m(k_*) = \mathrm{i}\sqrt{k_0^2 - k_*^2}[U_1^m(-k_*) + U_3^m(-k_*)]. \tag{4.13}$$

(This formula can be derived in terms of the theory of functional problems developed here, however, the proof is rather complicated and here we do not describe it.)

Finally, the far-field diagram of the field with a plane wave incidence becomes expressed in the form

$$\boxed{f(k,k_*) = \frac{\mathrm{i}}{k-k_*}\sum_{m=1}^{4} f_m(k) f_m(-k_*),} \tag{4.14}$$

where

$$f_m(k) = \sqrt{k_0^2 - k^2}\,(U_1^m(k) + U_3^m(k))\,.$$

Representation (4.14) is the *embedding formula* for our problem. It is obvious how this formula can be generalized on the case of several strips located in one plane: the summation should be performed over all edges of the system.

Earlier the embedding formula was obtained in [2] for a single strip and in [3] for several strips, so this result is basically not new. However, the formula given in [3] has much more complicated form.

According to the embedding formula, one needs to find the auxiliary functions, which depends on a single variable $k$, to have the possibility to calculate the far-field diagram for each $k$ and $k_*$.

# 5 Spectral ordinary differential equations for the auxiliary functions

Unfortunately, there is no compact representation for $U_\nu^m$. However, in this section we prove that these functions obey an ordinary differential equation (ODE) with respect to the variable $k$; the coefficients of this equation are rational functions of $k$ having rather simple form. Thus, the auxiliary functions $U_\nu^m(k)$ can be found numerically by solving the corresponding equation with appropriate boundary conditions.

The scheme of constructing the spectral ODE here is similar to that of constructing the embedding formula. Namely, we guess the form of the equation, solve a system of linear algebraic equations to find the coefficients of the ODE and prove that the coefficients are rational functions of $k$ by using the Liouville theorem. We shall not perform the analysis in all details, however the calculations are very simple. Some discussion of this procedure can be found in [1].

Introduce the matrix notation for the auxiliary functions:

$$\hat{U}(k) = \begin{pmatrix} U_1^1 & U_2^1 & U_3^1 & U_4^1 \\ U_1^2 & U_2^2 & U_3^2 & U_4^2 \\ U_1^3 & U_2^3 & U_3^3 & U_4^3 \\ U_1^4 & U_2^4 & U_3^4 & U_4^4 \end{pmatrix}.$$

Denote the differentiation with respect to $k$ by prime. Let us find the matrix $\hat{K}$ of the dimension $4\times 4$, such that the following equation is valid:

$$\boxed{\hat{U}' = \hat{K}\hat{U}.} \tag{5.1}$$

This is the spectral ODE. It follows from the functional equations (4.1) that the similar ODE should be valid for the remaining column of the unknown functions:

$$\begin{pmatrix} U_0^1 \\ U_0^2 \\ U_0^3 \\ U_0^4 \end{pmatrix}' = \hat{K}\begin{pmatrix} U_0^1 \\ U_0^2 \\ U_0^3 \\ U_0^4 \end{pmatrix}.$$

The equation (5.1) can be formally solved with respect to the matrix $\hat{K}$. The elements of this matrix become expressed as the ratios of the determinants:

$$\hat{K}_{mn} = D_{mn}(k)/D(k), \tag{5.2}$$

where the determinant $D$ is given by (4.7) and each determinant $D_{mn}$ is obtained by substituting of the $n$-th column of $D$ by the derivative of the $m$-th column. For example,

$$D_{11} = \begin{vmatrix} (U_1^1)' & U_1^2 & U_1^3 & U_1^4 \\ (U_2^1)' & U_2^2 & U_2^3 & U_2^4 \\ (U_3^1)' & U_3^2 & U_3^3 & U_3^4 \\ (U_4^1)' & U_4^2 & U_4^3 & U_4^4 \end{vmatrix}.$$

Note that for each determinant $D_{mn}$ there exists an alternative representation similar to (4.8) expressing this determinant in terms of the functions $U_1^m, U_2^m, U_3^m, U_0^m$ . Using two possible representations of each determinant we can study the behaviour of the determinants in the lower and upper half-planes of the variable $k$. After applying the Liouville theorem, we obtain that each element of $\hat{K}$ can be represented as $P_2(k)/(k_0^2 - k^2)$, where $P_2$ is a quadratic polynom.

After a more detailed study of growth of each element of $\hat{K}$ we can present the coefficients in the form:

$$\boxed{\hat{K}(k) = \begin{pmatrix} \mathrm{i}a_1 & 0 & 0 & 0 \\ 0 & \mathrm{i}a_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}a_3 & 0 \\ 0 & 0 & 0 & \mathrm{i}a_4 \end{pmatrix} + \frac{1}{k - k_0}\hat{K}^+ + \frac{1}{k + k_0}\hat{K}^-,} \tag{5.3}$$

where $\hat{K}^+$ and $\hat{K}^-$ are some matrices, depending on $k_0$ and $a_1 \dots a_4$, but not depending on $k$.

Now we know the coefficients of the equation (5.1) up to 32 constant parameters. Using the analyticity and growth restrictions for the functions $U_n^m$, one can formulate the link problem for finding the unknown constants. One can show that the number of links is equal to the number of unknown constants (a similar result was obtained in [1]). Moreover, the same analyticity and growth restrictions can be used to specify the functions $U_n^m$ among the other solutions of the equation (5.1). However, this method seems to be too complicated for practical calculations, since the presence of global monodromy restrictions leads to a very complicated eigenvalue problem. In [4] a simple and effective approximate technique based on the diffraction series is developed for a single strip problem. The same technique can be developed for the case of several strips for practical calculations.

Equation (5.1) with the coefficient (5.3) is the most important equation related to the problem of diffraction by several strips. Two main reasons exist for saying that this equation is a generalization of the Wiener-Hopf method. First, we found the functions, to which the Liouville theorem is applicable. These functions are the Wronsky-type determinants $D$ and $D_{mn}$. The application of the Liouville theorem is the key point of the Wiener-Hopf method. Second, the method developed here can be applied to the classical Sommerfeld problem of diffraction by an ideal half-plane. It leads to a degenerated ODE of order 1. The coefficient $K$ in this case is equal to the logarithmic derivative of the unknown spectral function. The solution coincides with that obtained by the Wiener-Hopf method.

# 6 Evolution equations

In this section we shall study the dependence of the auxiliary functions $U_n^m$ on the parameters $a_1 \dots a_4$. We shall call the equations describing this dependence the *evolution equations*. Another important issue stued in this section is the dependence of the matrices $\hat{K}^+$, $\hat{K}^-$ (containing undetermined parameters) on $a_1 \dots a_4$.

Consider the derivatives of the functions $U_n^m$ with respect to $a_j$, $j = 1 \dots 4$. Namely, let us find the matrices $\hat{A}^1 \dots \hat{A}^4$ such that

$$\boxed{\frac{\partial}{\partial a_j}\hat{U} = \hat{A}^j \hat{U},} \tag{6.1}$$

We again follow the procedure used above for the derivation of the embedding formula and the spectral ODE. The equation (6.1) can be formally solved with respect to the matrix $\hat{A}^j$. Each element of the matrix $\hat{A}^j$ can be represented as a ratio of the determinants

$$A_{mn}^j = D_{mn}^j(k)/D(k),$$

where $D^j_{mn}$ is the determinant obtained from $D$ by substituting of the $n$-th column by the derivative of the $m$-th column with respect to $a_j$. For example,

$$D^j_{11} = \begin{vmatrix} (U^1_1)_{,a_j} & U^2_1 & U^3_1 & U^4_1 \\ (U^1_2)_{,a_j} & U^2_2 & U^3_2 & U^4_2 \\ (U^1_3)_{,a_j} & U^2_3 & U^3_3 & U^4_3 \\ (U^1_4)_{,a_j} & U^2_4 & U^3_4 & U^4_4 \end{vmatrix}.$$

Using the functional equations (4.1), we can construct an alternative representation for each $D^j_{mn}$ in terms of the functions $U^m_1, U^m_2, U^m_3, U^m_0$. Two representations describe the behaviour of each element in the upper and lower half-planes. We can apply Liouville theorem and find that all elements of $\hat{A}^j$ are the polynomials of $k$ having degree no more than 1 (i.e. they are either constants or linear functions of $k$).

Introduce the constants $C^m_n$ as the following coefficients in the asymptotics:

$$U^m_n = \delta_{nm} k^{-1/2} e^{\mathrm{i}ka_m} + C^m_n k^{-3/2} e^{\mathrm{i}ka_m} + O(k^{-5/2} e^{\mathrm{i}ka_m}), \quad n = 1 \dots 4, \tag{6.2}$$

as $k \to \mathrm{i}\infty$; $\delta_{mn}$ is the Kronecker's delta.

Now we can substitute these asymptotics into the determinants related to the elements of $\hat{A}^j$. The result is:

$$A^j_{mn}(k) = \mathrm{i}k\delta_{jm}\delta_{mn} + \mathrm{i}C^m_n(\delta_{jn} - \delta_{jm}). \tag{6.3}$$

Let us represent the coefficients $C^m_n$ in terms of the elements of the matrices $\hat{K}^\pm$ (see (5.3)). Substitute the asymptotics (6.2) into the determinants $D(k)$ and $D_{mn}(k)$. Consider first the determinant $D$. From one hand, it has the form (4.9). From the other hand, the asymptotics yield

$$D(k) = k^{-2} + (C^1_1 + C^2_2 + C^3_3 + C^4_4)k^{-3} + O(k^{-4}).$$

Comparing these representations we conclude that

$$C^1_1 + C^2_2 + C^3_3 + C^4_4 = 0 \tag{6.4}$$

Now consider the determinants $D_{mn}$ (see (5.2)). The diagonal elements have the asymptotics

$$D_{mm}(k) = \mathrm{i}a_m k^{-2} + (C^1_1 + C^2_2 + C^3_3 + C^4_4 - 1/2)k^{-3} + O(k^{-4})$$

Comparing this representation with (5.3) and taking into account (6.4), we conclude that

$$K^+_{mm} + K^-_{mm} = -1/2 \tag{6.5}$$

The non-diagonal elements have the asymptotics

$$D_{mn} = \mathrm{i}C^m_n(a_n - a_m)k^{-3} + O(k^{-4}).$$

Therefore

$$K^+_{mn} + K^-_{mn} = \mathrm{i}(a_n - a_m)C^m_n \qquad \text{for } m \neq n. \tag{6.6}$$

Now we can represent the coefficients of the equations (6.1) in the form

$$\boxed{A^j_{mn}(k) = \mathrm{i}k\delta_{jm}\delta_{mn} + \frac{(K^+_{mn} + K^-_{mn})(\delta_{jn} - \delta_{jm})}{a_n - a_m},} \tag{6.7}$$

where we imply that the second term is equal to zero if $a_n = a_m$. Now the coefficients of the evolution equation (6.1) become expressed in terms of the coefficients of the spectral ODE (5.1).

Equations (6.1) describe the evolution of the auxiliary functions $U^m_n$ when the parameters of the problems (namely, the coordinates of the edges) are changed. However, the coefficients of the evolution equations (6.1) also depend on the coordinates of the edges. Now we are going to derive the equations that describe the changes of the elements of $\hat{K}^\pm$ when the parameters $a_1 \dots a_4$ are changed.

The following derivative can be calculated by two different methods:

$$\frac{\partial^2 \hat{U}}{\partial k \partial a_j} = \left( \frac{\partial \hat{K}}{\partial a_j} + \hat{K}\hat{A}^j \right) \hat{U} = \frac{\partial^2 \hat{U}}{\partial a_j \partial k} = \left( \frac{\partial \hat{A}^j}{\partial k} + \hat{A}^j \hat{K} \right) \hat{U}.$$

Multiply this equation by $\hat{U}^{-1}$. Note that the determinant of this matrix is not zero almost everywhere. This yields

$$\boxed{\frac{\partial \hat{K}}{\partial a_j} = \frac{\partial \hat{A}^j}{\partial k} + [\hat{A}^j, \hat{K}].} \tag{6.8}$$

Substitute (5.3) and (6.7) into (6.8). Note that the equation (6.8) must be fulfilled identically for all values of $k$. So we can expand (6.8) into partial fractions and write separate equations for each denominator. The straightforward calculations show that (6.8) is equivalent to

$$\frac{\partial K^{\pm}_{mn}}{\partial a_j} = \pm \mathrm{i} k_0 K^{\pm}_{mn}(\delta_{mj} - \delta_{nj}) + \frac{(K^{+}_{mj} + K^{-}_{mj})K^{\pm}_{jn}}{a_j - a_m} + \frac{(K^{+}_{jn} + K^{-}_{jn})K^{\pm}_{mj}}{a_n - a_j} +$$

$$\sum_{l \neq j} \frac{(K^{+}_{jl} + K^{-}_{jl})K^{\pm}_{ln}}{a_j - a_l} + \sum_{l \neq j} \frac{(K^{+}_{lj} + K^{-}_{lj})K^{\pm}_{ml}}{a_l - a_j}. \tag{6.9}$$

We imply that the second term in the r.-h.s. of (6.9) is equal to zero if $m = j$, and the third term is equal to zero if $n = j$.

Thus we obtained a system of nonlinear evolution equations for the coefficients of (5.1). Now we can outline the scheme of the numerical procedure for finding the far-field diagram for some incident angle.

1. Choose any arbitrary set of values $a_1 \dots a_4$ and calculate the elements of the matrix $\hat{K}$. One can choose most symmetrical configuration. Moreover, one can choose the widths of the strips and the slit between them to be big comparatively to the wavelength. In this case the elements of $\hat{K}$ can be found approximately using the diffraction series method.

2. Using the equation (6.9) and taking the matrix $\hat{K}$ found on the previous step as the initial values for it, find numerically the coefficients of the matrix $\hat{K}$ for any given configuration $a_1 \dots a_4$.

3. Solve the equation (5.1). The boundary data for this equation is composed of the analyticity restrictions, growth restrictions and the asymptotics (4.3). The result is the set of the functions $U^m_n$.

4. Using the embedding formula (4.14) construct the far-field diagram.

# Acknowledgements

The current work is supported by the RFBR grants NN 00-15-96530, 01-02-06119.

# References

[1] Shanin A.V. *Three theorems concerning diffraction by a strip or a slit* , Quart. Journ. of Mech. and Appl. Math., V.54, N.1, pp.107–137 (2001).

[2] Williams, M.H., *Diffraction by a finite strip*, Quart. Journ. of Mech. and Appl. Math., V.35, N.1, pp.103–124 (1982).

[3] Biggs N. R. T., Porter, D., Stirling, D.S.G. *Wave diffraction through a perforated breakwater*, Quart. Journ. of Mech. and Appl. Math., V.53, N.3, pp.375–391 (2000).

[4] Shanin A.V., *To the problem of diffraction by a slit: Some properties of the Schwarzschild's series*, Zapiski seminarov POMI, V 275, 2001 (In Russian).