

Further progress in the coordinate equations theory

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Abstract

A new method is proposed for solving diffraction problems having piecewise linear ideal boundaries. According to this method, the initial Helmholtz equation is replaced by a system of matrix equations of order 1. The coefficients of these equations are rational matrix functions of the coordinates. The properties of the coordinate matrix system are close to that of the ordinary differential equation, therefore the new method can be treated as a generalization of the separation of variables.

1 Introduction

Separation of variables (SV) is the most powerful technique for finding exact solutions of the problems related to partial differential equations. However, this method has some weak points. Two of them are the following. First, this method is applicable to a very restricted set of geometries. All these geometries, say for the Helmholtz equation, are classified by group theory analysis; the detailed discussion of this matter can be found, for example, in [1]. Second, even a simple diffraction problem leads to a complicated Fourier series in special functions. As a rule, the structure of the field cannot be easily found from the series. A special technique, such as Watson's transform, is required to study the exact solution obtained using SV. Sadly, such a transform is not known for many interesting cases.

In the present paper we propose a technique that can help to overcome the disadvantages mentioned above, i.e. it is applicable to a wider class of problems and the solution is not represented as a Fourier series. The main idea of the method is to reduce the diffraction problem, which involves the Helmholtz equation, boundary, edge and radiation conditions, to an ordinary differential equation (ODE) of a rather simple form. The coefficients of the equation are rational functions of the coordinates.

Here we describe the main idea of the SV generalization. Consider the simplest case of SV: the wave field in a rectangular area. Consider the Helmholtz equation

$$\Delta u + k_0^2 u = 0 \quad (1.1)$$

in a rectangle $0 < x < L$, $0 < y < M$ with Dirichlet conditions at the boundaries, $u = 0$. The solution is the product

$$u = \sin kx \sin \gamma y, \quad k = \frac{\pi m}{L}, \quad \gamma = \sqrt{k_0^2 - k^2} = \frac{\pi n}{M}, \quad m, n \in Z. \quad (1.2)$$

Construct the vector $\mathbf{U} = (u^1, u^2, u^3, u^4)^T$ of dimension 4 with the following components:

$$u^1 = u, \quad u^2 = \frac{\partial u}{\partial x}, \quad u^3 = \frac{\partial u}{\partial y}, \quad u^4 = \frac{\partial^2 u}{\partial x \partial y}.$$

Note that all components of the vector are solutions of the Helmholtz equation in the rectangle, but the boundary conditions for these solutions should be chosen differently.

One can check, that the vector \mathbf{U} obeys the differential equations

$$\frac{\partial \mathbf{U}}{\partial x} = \mathbf{XU}, \quad \frac{\partial \mathbf{U}}{\partial y} = \mathbf{YU}, \quad (1.3)$$

with the matrix coefficients

$$\mathbf{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k^2 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma^2 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 \end{pmatrix}. \quad (1.4)$$

The situation is similar for each classical SV. Let η and ξ be the coordinates, in which the variables are separated. Then an eigenfunction can be written as a product $u = T(\eta)S(\xi)$, and the functions T and S obey differential equations of order 2:

$$\begin{aligned} T''(\eta) - f_1(\eta)T'(\eta) - g_1(\eta)T(\eta) &= 0, \\ S''(\xi) - f_2(\xi)S'(\xi) - g_2(\xi)S(\xi) &= 0, \end{aligned}$$

One can see that the following equations are valid (we denote the derivatives by the indices here):

$$\frac{\partial}{\partial \eta} \begin{pmatrix} u \\ u_{,\eta} \\ u_{,\xi} \\ u_{,\eta\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g_1(\eta) & f_1(\eta) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g_1(\eta) & f_1(\eta) \end{pmatrix} \begin{pmatrix} u \\ u_{,\eta} \\ u_{,\xi} \\ u_{,\eta\xi} \end{pmatrix}, \quad (1.5)$$

$$\frac{\partial}{\partial \xi} \begin{pmatrix} u \\ u, \eta \\ u, \xi \\ u, \eta \xi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g_2(\xi) & 0 & f_2(\xi) & 0 \\ 0 & g_2(\xi) & 0 & f_2(\xi) \end{pmatrix} \begin{pmatrix} u \\ u, \eta \\ u, \xi \\ u, \eta \xi \end{pmatrix}. \quad (1.6)$$

The equations (1.3) will now be called the *coordinate equations*. These equations formally are in partial derivatives, but their properties are similar to that of an ordinary differential equation. Namely, the solution of a partial differential equation can be found when its values are known on a contour in the (x, y) -plane. Conversely, the solution of the system (1.3) is specified by the value of the vector \mathbf{U} at just one arbitrary point P . Any other point P' can be connected with P by a line, and an ordinary differential equation describing the behaviour of \mathbf{U} on this line can be constructed as a restriction of (1.3) on this line. The value $\mathbf{U}(P)$ should be used as the initial conditions for this equation.

One can see that the coefficients of the equations (1.5) and (1.6) have a very strict structure. Some coefficients should be equal to zero, and the other ones should depend on a single variable. The idea of generalization these equations is by allowing the unknown vector \mathbf{U} and the coefficient matrices to have a more flexible structure. We demand only that the first component of the vector is the solution of the given diffraction problem, all other components can be chosen arbitrarily. The vector can have an arbitrary dimension, and the elements of the coefficient matrices can be arbitrary functions of the coordinates. Obviously, the structure of the coefficients should be much simpler than the structure of the solutions, otherwise the equations are senseless.

Surprisingly, equations of the form (1.3) with relatively simple coefficients can be written for a wide class of diffraction problems.

Of course, some price is paid for this generalization of the SV method. While the order of the ODEs in the classical SV applied to the Helmholtz equation is normally equal to 2, the order of the ODEs emerging in our method is usually higher than 2. The more complicated scatterer leads to equation of higher order. The result is a situation, in which the analytical solution of the ODE is not known. However, the equations obtained here belong to a well-studied class: they are confluent Fuchsian. A powerful theory has been developed for these equations, so the situation is not completely hopeless.

Our paper has a strong relation to the works [2], [3], [4] devoted to the strip problem. The ODEs obtained in these works for the field on the scatterer or for the far-field diagram can be found as particular cases of the coordinate equations obtained here. The procedure proposed below is close to that of [3]. We should mention here also the work [5], where the structure of the eigenfunction series for the several strips problem was constructed. However, the relation of these series to

our method still remains questionable.

The paper is organized as follows. In Section 2 the coordinate equations are derived for the problem of diffraction of a plane wave by two Dirichlet or Neumann strips. The coefficients and the initial conditions are known up to several numerical constants, which should be found either by solving the eigenvalue problem or using some other technique. In Section 3 some special issues related to the coordinate equations are discussed. The solvability of the equations and obeying the Helmholtz equation and the boundary conditions by the solutions are these issues. The equations for the directivities are also derived in this section. In Section 4 we represent the analytical solution of the coordinate equations for the case of diffraction by a half-plane and the numerical solution of the coordinate equation for the case of two strips.

2 Problem formulation

Let the Helmholtz equation (1.1) be fulfilled in the (x, y) -plane. The time dependence is assumed to have the form $e^{-i\omega t}$ and is omitted everywhere. The incident plane wave

$$u_{\text{in}} = e^{-ik_*x - i\sqrt{k_0^2 - k_*^2}y}, \quad k_* = k_0 \cos \psi$$

(ψ is the angle of incidence) illuminates the scatterer which is a set of two strips (see Fig. 1) with Dirichlet boundary conditions $u^{\text{d}} = 0$.

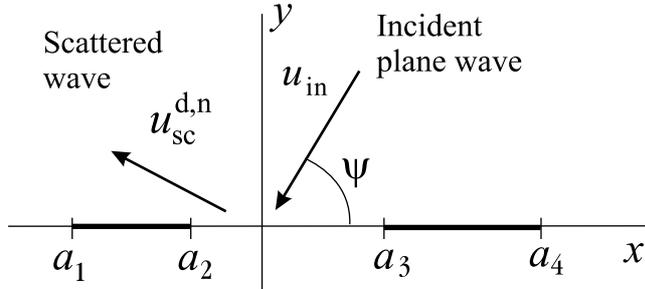


Figure 1: Geometry of the problem

The problem is effectively two-dimensional. The (x, y) -cross-section of the strips is the set of the segments (a_1, a_2) , (a_3, a_4) of the x -axis. All the techniques presented below can be obviously generalized to the case of an arbitrary number of the strips located in one plane and bearing the Dirichlet boundary conditions, The case of 2 strips is chosen just because it is the simplest case, for which the classical SV is not

known. Note that our method gives nontrivial results even for the cases of one strip and a half-plane, for which the classical SV is applicable.

The total field u^d can be split into the incident and the scattered field:

$$u^d = u_{\text{in}} + u_{\text{sc}}^d.$$

The scattered field is symmetrical. The boundary conditions for u_{sc}^d are as follows.

$$u_{\text{sc}}^d(x, \pm 0) = -e^{-ik_*x} \quad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \quad (2.1)$$

$$\partial_y u_{\text{sc}}^d(x, 0) = 0 \quad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty). \quad (2.2)$$

We assume that the radiation condition is fulfilled for the scattered field. Moreover, Meixner's conditions are fulfilled in the vicinity of the *edges* of the scatterer, i.e. near the points $(a_m, 0)$. In our case Meixner's conditions demand that the field has the asymptotics

$$u \sim A + B(r^{1/2}) + O(r)$$

near the edges.

We formulate the problem of the diffraction by a set of Neumann strips as well. Let the geometry of the strips be the same as for the Dirichlet problem. Let the Neumann boundary conditions be fulfilled by the field u^n on the strips:

$$\partial_y u^n = 0 \quad \text{for} \quad y = \pm 0, \quad x \in (a_1, a_2) \cup (a_3, a_4).$$

The total field, again, can be represented as the sum

$$u^n = u_{\text{in}} + u_{\text{sc}}^n,$$

u_{sc}^n is the scattered field, which is antisymmetric and satisfies the following boundary conditions:

$$\partial_y u_{\text{sc}}^n(x, \pm 0) = i\sqrt{k_0^2 - k_*^2} e^{-ik_*x} \quad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \quad (2.3)$$

$$u_{\text{sc}}^n(x, 0) = 0 \quad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty). \quad (2.4)$$

The radiation and the edge conditions are fulfilled by u_{sc}^n .

3 Auxiliary solutions for the strip and the slit problem

The derivation of the coordinate equations for $u_{sc}^{d,n}$ is performed in two steps. Firstly, the auxiliary (oversingular) functions are introduced and the coordinate equations for them are derived. Secondly, the coordinate equations for the $u_{sc}^{d,n}$ are derived using the equations for the auxiliary functions.

Introduce the local cylindrical coordinates (ρ_m, θ_m) near the edges of the scatterer (see Fig. 2).

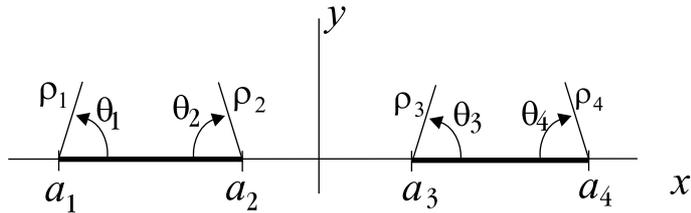


Figure 2: Local cylindrical coordinates

Consider the diffraction problem with the same Dirichlet strips as described above, but with the excitation by a line source rather than by a plane wave. Let the source be located near one of the edges $(a_m, 0)$. Namely, let v^m , $m = 1 \dots 4$ be the limit of the solution of the inhomogeneous Helmholtz equation

$$\Delta v^m + k_0^2 v^m = \pi^{1/2} \epsilon^{-3/2} \delta(\rho_m - \epsilon) \delta(\theta_m - \pi), \quad (3.1)$$

Here δ is the Dirac's delta-function; $\epsilon \rightarrow 0$. The strength of the source is equal to $\epsilon^{-1/2}$. Note that the dependence of strength of the source on ϵ is chosen such that there exists the non-zero limit of the solution. The boundary condition for v^m have the form

$$v^m(x, \pm 0) = 0 \quad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \quad (3.2)$$

Obviously, the function v^m is symmetrical, hence the following boundary condition is valid:

$$\partial_y v^m(x, 0) = 0 \quad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty). \quad (3.3)$$

Also, the radiation condition is fulfilled by v^m .

Meixner's conditions are assumed to be satisfied for each non-zero ϵ , but the behaviour of the limiting function v^m near the edge $(a_m, 0)$ appears to violate Meixner's condition just because the presence of the source near this edge. This behaviour can

be studied by the standard technique. Taking the limit, we obtain that the function behaves like $v^m \sim \rho_m^{-1/2}$. Therefore, instead of taking the limit of the line-source problem, one can find v^m as the solution having the *oversingular* behaviour near the edge a_m .

In more details, the asymptotics of the solutions v^m near the edges $(a_n, 0)$ have the form

$$v^m(\rho_n, \theta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}} \rho_n^{-1/2} \sin \frac{\theta_n}{2} + \frac{2C_n^m}{\sqrt{\pi}} \rho_n^{1/2} \sin \frac{\theta_n}{2} + O(\rho_n^{3/2}), \quad (3.4)$$

where C_n^m are some unknown coefficients; $\delta_{m,n}$ is the Kronecker's delta.

We also introduce the auxiliary oversingular functions w^m , $m = 1, \dots, 4$ for the Neumann strips problem.

The boundary conditions have the form

$$\partial_y w^m(x, \pm 0) = 0 \quad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \quad (3.5)$$

$$w^m(x, 0) = 0 \quad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty). \quad (3.6)$$

The radiation condition is fulfilled. The Meixner's conditions are satisfied for each non-zero ϵ . The functions w^m are characterised by the oversingular terms in their asymptotics:

$$w^m(\rho_n, \theta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}} \rho_n^{-1/2} \cos \frac{\theta_n}{2} + \frac{2E_n^m}{\sqrt{\pi}} \rho_n^{1/2} \cos \frac{\theta_n}{2} + O(\rho_n^{3/2}), \quad (3.7)$$

E_n^m are some constants.

The constants C_n^m and E_n^m play an important role in the subsequent derivation of the coordinate equations. Namely, they are the numerical parameters of the coefficients \mathbf{X} and \mathbf{Y} . At this stage these parameters are unknown, but we assume that they can be calculated by some technique.

4 An important note

We suppose that k_0 belongs neither to the spectrum of the Dirichlet nor of the Neumann strips problem. It means that the functions $u_{\text{sc}}^{\text{d},n}$ are defined uniquely and the following statement is true. If some function $u(x, y)$ obeys the homogeneous Helmholtz equation, radiation condition at infinity, homogeneous boundary conditions (2.4), (2.3) or (3.5), (3.6) on the x -axis and Meixner's conditions at the edges, then $u \equiv 0$. This feature is very important for derivation of the coordinate equations. If k_0 belongs to the spectrum, the method should be modified.

Note that the auxiliary solutions v^m and w^m are allowed to be non-zero only because they violate Meixner's condition at one of the edges.

5 Derivation of the coordinate equations for the auxiliary functions

Construct the vector of the unknowns

$$\mathbf{U} = (v^1, v^2, v^3, v^4, w^1, w^2, w^3, w^4)^T. \quad (5.1)$$

Our purpose here is to construct the equations of the form (1.3) for this vector. We find it convenient also to introduce the vectors of dimension 4:

$$\mathbf{V} = (v^1, v^2, v^3, v^4)^T, \quad \mathbf{W} = (w^1, w^2, w^3, w^4)^T.$$

Construct the derivative

$$v^* \equiv \frac{\partial v^m}{\partial \theta_m} = \left[(x - a_m) \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] v^m,$$

where θ_m is treated as a global cylindrical coordinate. Obviously, v^* obeys the homogeneous Helmholtz equation (1.1) and the radiation condition. Moreover, detailed study shows that v^* satisfies the boundary conditions (3.5) and (3.6) (note that v^m is antisymmetrical and smooth enough on the strips, and that $\partial_y^2 v^m = -(\partial_x^2 + k_0^2) v^m$).

Meixner's conditions are violated by v^* . Using the asymptotics (3.4) one can obtain asymptotic estimations for v^* near the edge a_n :

$$v^m = \left(\frac{(-1)^m \delta_{m,n}}{2} + (a_n - a_m) C_n^m \right) \frac{1}{\sqrt{\pi}} \rho_n^{-1/2} \cos \frac{\theta_n}{2} + O(\rho_n^{1/2}). \quad (5.2)$$

Using these asymptotics, one can construct the combination

$$v^{**} = v^* + \frac{(-1)^m}{2} w^m + \sum_{n=1}^4 (a_n - a_m) C_n^m w^n,$$

which satisfies the Meixner's conditions. Besides, v^{**} satisfies the Helmholtz equation, radiation condition and the boundary conditions (3.5), (3.6). Due to Subsection 4, $v^{**} \equiv 0$. Therefore,

$$\left[(x - a_m) \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] v^m = \mathbf{R}^m \mathbf{W} \equiv \sum_{n=1}^4 R_n^m w^n, \quad m = 1 \dots 4, \quad (5.3)$$

where \mathbf{R}^m is the string with the elements

$$R_n^m = \frac{(-1)^{m-1} \delta_{m,n}}{2} - (a_n - a_m) C_n^m, \quad n = 1 \dots 4. \quad (5.4)$$

Thus, one equation expressing the spatial derivatives of v^m through the components of the vector \mathbf{U} is constructed. We need more such equations to obtain the separate expressions for the derivatives. For this we should find several combinations containing the derivatives of v^m and w^m with respect to x and y and having the singularities at $(a_n, 0)$ not stronger than $\rho_n^{-1/2}$. One can check directly that the combinations

$$\left[(x - a_m) \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] w^m, \quad \frac{\partial v^m}{\partial y} + (-1)^m \frac{\partial w^m}{\partial x}, \quad \frac{\partial w^m}{\partial y} - (-1)^m \frac{\partial v^m}{\partial x}$$

satisfy this condition. Using the technique described above, i.e. studying the asymptotics of these combinations, subtracting the appropriate sums of v^n or w^n and taking into account the uniqueness of the solution, we obtain the following representations:

$$\left[(x - a_m) \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] w^m = \mathbf{Q}^m \mathbf{V} \equiv \sum_{n=1}^4 Q_n^m v^n, \quad m = 1 \dots 4, \quad (5.5)$$

$$\frac{\partial v^m}{\partial y} + (-1)^m \frac{\partial w^m}{\partial x} = \mathbf{S}^m \mathbf{W} \equiv \sum_{n=1}^4 S_n^m w^n, \quad m = 1 \dots 4, \quad (5.6)$$

$$\frac{\partial w^m}{\partial y} - (-1)^m \frac{\partial v^m}{\partial x} = \mathbf{T}^m \mathbf{V} \equiv \sum_{n=1}^4 T_n^m v^n, \quad m = 1 \dots 4, \quad (5.7)$$

where \mathbf{Q}^m , \mathbf{S}^m and \mathbf{T}^m are the rows of dimension 4:

$$Q_n^m = \frac{(-1)^m \delta_{m,n}}{2} - (a_n - a_m) E_n^m, \quad (5.8)$$

$$S_n^m = -C_n^m - (-1)^{m+n} E_n^m, \quad (5.9)$$

$$T_n^m = -E_n^m - (-1)^{m+n} C_n^m \quad (5.10)$$

for $n = 1 \dots 4$.

The equations (5.3), (5.5), (5.6), (5.7) can be solved with respect to the x - and y -derivatives of v^m and w^m for each m . The result can be written in the form of the coordinate equations

$$\frac{\partial \mathbf{U}}{\partial x} = \mathbf{X} \mathbf{U}, \quad \frac{\partial \mathbf{U}}{\partial y} = \mathbf{Y} \mathbf{U}, \quad (5.11)$$

where \mathbf{X} and \mathbf{Y} are the block matrices

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1^1 & \mathbf{X}_2^1 \\ \mathbf{X}_1^2 & \mathbf{X}_2^2 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1^1 & \mathbf{Y}_2^1 \\ \mathbf{Y}_1^2 & \mathbf{Y}_2^2 \end{pmatrix}, \quad (5.12)$$

The elements of the matrices \mathbf{X}_j^i and \mathbf{Y}_j^i are given by the following formulae:

$$\begin{aligned} (X_1^1)_n^m &= \frac{(-1)^{m-1}(x-a_m)(Q_n^m - (x-a_m)T_n^m)}{(x-a_m)^2 + y^2}, & (X_2^1)_n^m &= -\frac{y(R_n^m - (x-a_m)S_n^m)}{(x-a_m)^2 + y^2}, \\ (X_1^2)_n^m &= -\frac{y(Q_n^m - (x-a_m)T_n^m)}{(x-a_m)^2 + y^2}, & (X_2^2)_n^m &= \frac{(-1)^m(x-a_m)(R_n^m - (x-a_m)S_n^m)}{(x-a_m)^2 + y^2}, \\ (Y_1^1)_n^m &= \frac{(-1)^{m-1}y(Q_n^m - (x-a_m)T_n^m)}{(x-a_m)^2 + y^2}, & (Y_2^1)_n^m &= \frac{(x-a_m)R_n^m + y^2S_n^m}{(x-a_m)^2 + y^2}, \\ (Y_1^2)_n^m &= \frac{(x-a_m)Q_n^m + y^2T_n^m}{(x-a_m)^2 + y^2}, & (Y_2^2)_n^m &= \frac{(-1)^m y(R_n^m - (x-a_m)S_n^m)}{(x-a_m)^2 + y^2}. \end{aligned}$$

Thus, the coordinate equations for the auxiliary solutions are derived. Note the following features of these equations.

- The coefficients of the matrices are rational functions of x and y . The denominators have the form $(x-a_m)^2 + y^2$.
- The coefficients contain several unknown constants C_n^m, E_n^m , not depending on x and y . These constants should be found using a special eigenvalue problem or some other technique.
- The auxiliary functions can be found from the system (5.11) without any Fourier series.
- The classical separation of variables cannot be performed for two strips.

6 Derivation of the coordinate equations for the plane wave incidence

Let the functions u^d and u^n have the following asymptotics near the edges a_n :

$$u^d = A_n + \frac{2C_n}{\sqrt{\pi}}\rho^{1/2} \sin \frac{\theta_n}{2} + O(\rho_n), \quad (6.1)$$

$$u^n = B_n + \frac{2E_n}{\sqrt{\pi}}\rho^{1/2} \cos \frac{\theta_n}{2} + O(\rho_n), \quad (6.2)$$

where A_n, B_n, C_n, E_n are some constants.

Consider the following combination:

$$(u_{sc}^d)^* = \frac{\partial u_{sc}^d}{\partial x} + ik_* u_{sc}^d$$

This combination obeys the Helmholtz equation, and it satisfies homogeneous boundary conditions (3.2), (3.3). The function $(u_{sc}^d)^*$ satisfies the radiation condition, but

it does not satisfy the Meixner's edge conditions. However, this can be fixed by calculating the asymptotics of $(u_{\text{sc}}^{\text{d}})^*$ at the edges and subtracting an appropriate combination of the functions v^n . After performing this procedure and applying the statement of Subsection 4, one can prove that

$$\frac{\partial u_{\text{sc}}^{\text{d}}}{\partial x} = -ik_* u_{\text{sc}}^{\text{d}} - \sum_{n=1}^4 (-1)^n C_n v^n. \quad (6.3)$$

Analogously, one can obtain three other equations:

$$\frac{\partial u_{\text{sc}}^{\text{n}}}{\partial x} = -ik_* u_{\text{sc}}^{\text{n}} + \sum_{n=1}^4 (-1)^n E_n w^n, \quad (6.4)$$

$$\frac{\partial u_{\text{sc}}^{\text{d}}}{\partial y} = -i\sqrt{k_0^2 - k_*^2} u_{\text{sc}}^{\text{n}} - \sum_{n=1}^4 C_n w^n, \quad (6.5)$$

$$\frac{\partial u_{\text{sc}}^{\text{n}}}{\partial y} = -i\sqrt{k_0^2 - k_*^2} u_{\text{sc}}^{\text{d}} - \sum_{n=1}^4 E_n v^n. \quad (6.6)$$

These equations together with (5.3), (5.5), (5.3), (5.5) can be rewritten in the matrix form. Namely, introduce the vector

$$\tilde{\mathbf{U}} = (u_{\text{sc}}^{\text{d}}, u_{\text{sc}}^{\text{n}}, v^1, v^2, v^3, v^4, w^1, w^2, w^3, w^4)^T.$$

The coordinate equations for this vector have the form

$$\frac{\partial \tilde{\mathbf{U}}}{\partial x} = \tilde{\mathbf{X}} \tilde{\mathbf{U}}, \quad \frac{\partial \tilde{\mathbf{U}}}{\partial y} = \tilde{\mathbf{Y}} \tilde{\mathbf{U}}, \quad (6.7)$$

where $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are the block matrices:

$$\begin{aligned} \tilde{\mathbf{X}} &= \begin{pmatrix} \mathbf{H}_1 & \mathbf{G}_1 \\ 0 & \mathbf{X} \end{pmatrix}, & \tilde{\mathbf{Y}} &= \begin{pmatrix} \mathbf{H}_2 & \mathbf{G}_2 \\ 0 & \mathbf{Y} \end{pmatrix}, & (6.8) \\ \mathbf{H}_1 &= \begin{pmatrix} -ik_* & 0 \\ 0 & -ik_* \end{pmatrix}, & \mathbf{H}_2 &= \begin{pmatrix} 0 & -i\sqrt{k_0^2 - k_*^2} \\ -i\sqrt{k_0^2 - k_*^2} & 0 \end{pmatrix}, \\ \mathbf{G}_1 &= \begin{pmatrix} C_1 & -C_2 & C_3 & -C_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -E_1 & E_2 & -E_3 & E_4 \end{pmatrix}, \\ \mathbf{G}_2 &= - \begin{pmatrix} 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \\ E_1 & E_2 & E_3 & E_4 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The matrices \mathbf{X} and \mathbf{Y} are given by (5.12). Thus, the coordinate equation involving the scattered field u_{sc} is constructed. This equation has dimension 10.

Note that the scattered field is the solution of a single coordinate equation, and it is not represented by a Fourier series.

Note also that the vector $\tilde{\mathbf{U}}$ can be redefined as follows:

$$\tilde{\mathbf{U}} = (u^d, u^n, v^1, v^2, v^3, v^4, w^1, w^2, w^3, w^4)^T,$$

i.e. the total fields can be considered instead of the scattered ones. The new vector obeys the equations (6.7) with the same coefficients.

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