Wave Diffraction by a Flat Cone

Andrey V. Shanin

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Abstract

New analytical results are presented for the problem of a plane acoustic wave scattering by a flat cone (a quarter plane) with Dirichlet boundary conditions. The results are obtained within a general framework developed by author for the strip/slit diffraction problem. These results include (i) embedding formulae representing the diffraction coefficient in the factorized form through the *edge Green's functions* depending separately on the direction of incidence and scattering, and (ii) the coordinate equations for the auxiliary functions that reduce the partial differential problem to a boundary problem for a system of ordinary differential equations. The new approach can be treated as a generalization of the separation of variables technique.

1 Introduction

Scattering by a cone with a polygonal cross-section is one of the main *canonical* diffraction problems, i.e. it can be used as a building block for the Geometrical Theory of Diffraction (GTD) or some other composite theory. That is why conical problems retain a considerable interest of researchers. The most interesting cases for applications are the flat cone and the trihedral cone. The first one models, say, the edge of an aircraft wing and the last one can be a model of a corner of a building.

Recently, a general approach to conical problems has been developed by the group of Smyshlyaev [1, 2]. This approach is as follows. On the *first step* the problem is formulated in the polar coordinates, and the radial variable is separated from the spherical ones. As the result, one obtains the formula for the diffraction coefficient of the flat cone problem:

$$f(\omega, \omega_0) = \frac{\mathrm{i}}{\pi} \int_\gamma e^{-\mathrm{i}\pi\nu} g(\omega, \omega_0, \nu) \nu \, d\nu \tag{1}$$

(see [1]). Here ω and ω_0 are the direction of incidence and scattering, ν is the separation constant, g is the Green's function of the corresponding problem on a unit sphere.

On the *second step* the spherical Green's function g should be found. Generally it can be computed by solving the boundary integral equations for the spherical problem. Alternatively the function g can be expressed trough the eigenfunctions Φ_j and eigenvalues ν_j of the spherical problem as follows:

$$g(\omega, \omega_0, \nu) = \sum_j \frac{\Phi_j(\omega)\Phi_j(\omega_0)}{\nu^2 - \nu_j^2}. \tag{2}$$

However, in some particular cases (we should note that the flat cone belongs to this set) the eigenfunctions can be found analytically by separating variables in the sphero-conal coordinates (see, e.g. Kraus and Levine [3] for the flat cone). This approach takes into account that the flat cone is a degenerate case of an elliptic cone.

In the current paper we are going to modify this procedure as follows. First, we shall modify the representation (1). Instead of using the Green's function $g(\omega, \omega_0, \nu)$ we shall obtain the integral representation involving the spherical edge Green's functions $v^{1,2}(\omega, \nu)$. Such a function can be treated as a limiting case of the Green's function g as the source location approaches the edge of the scatterer. This function depends on less variables (on 3 rather than on 5), therefore it is more convenient for numerical tabulation. The new representation will be called the *embedding formula*. Second, we prefer to avoid using the representation (2), since it leads to an ineffective numerical procedure. Also it should be mentioned that separation of variables can be performed for no other polygonal cone except the flat one. Solving the boundary integral equations for g could be an alternative, but it leads to lack of elegance and analytical understanding. Instead, we propose a novel technique of the *coordinate equations*, which can be treated as a generalization of separation of variables. This technique enables us to find the edge Green's functions $v^{1,2}$ without series. The new method can be applied to a wide class of problems, e.g. to a trihedral cone problem.

2 Basic relations

2.1 Problem formulation

Let the Helmholtz equation

$$\Delta u + k_0^2 u = 0. \tag{3}$$

be valid in the 3D space (x, y, z). The time dependence of all variables has the form $e^{-\mathrm{i}\Omega t}$ and it is omitted henceforth.

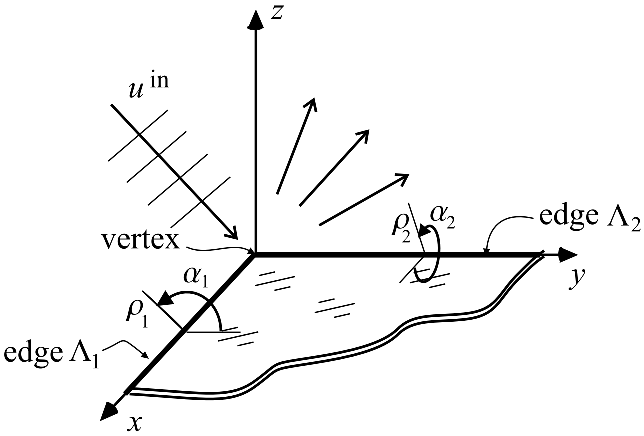

Figure 1: Geometry of the problem

The screen occupies the quarter-plane $z = 0, x > 0, y > 0$. The lines $(x > 0, y = 0, z = 0)$ and $(x = 0, y > 0, z = 0$ will be named the *edges* of the scatterer and denoted by the symbols Λ_1 and Λ_2 (see Figure 1).

The Dirichlet boundary conditions fulfilled on the both sides of the cone.

Let the incident field has the form of a plane wave

$$u^{\rm in} = e^{-{\rm i}(k_x x + k_y y + k_z z)}, \tag{4}$$

where $k_x^2 + k_y^2 + k_z^2 = k_0^2$.

Besides the governing equation and boundary conditions, the radiation, edge and vertex conditions should be imposed to make a proper problem formulation. The radiation condition is not easy to formulate in this case, but its physical meaning is clear: there should not be components of the field coming from infinity except $u^{\rm in}$ and the wave components reflected from the sides and scattered by the edges.

The edge condition follows from the theory of diffraction by an ideal half-plane. The edge is source-free if the field near the edge behaves like

$$u \sim \rho_{1,2}^{1/2} \sin \frac{\alpha_{1,2}}{2}, \tag{5}$$

where ρ_j and α_j are the local cylindrical coordinates near the edge Λ_j.

The vertex conditions can be formulated in the form

$$u = O(1), \quad \nabla u = o(r^{-1/2}) \quad \text{as} \quad r \to 0, \tag{6}$$

where r is the distance from the vertex of the cone.

We shall assume below that the theorem of uniqueness is valid for the problem of diffraction by a cone, i.e. if a field satisfies the Helmhotz equation, Dirichlet boundary conditions, radiation, edge and vertex conditions, then it is identically equal to zero.

2.2 Diffraction coefficient

Below we consider only the spherical component of the field. This component has the form

$$u^{\rm sc}(\omega, r) = 2\pi \frac{e^{{\rm i}k_0 r}}{k_0 r} f(\omega) + O(e^{{\rm i}k_0 r}(k_0 r)^{-2}). \tag{7}$$

Here r is the distance from the vertex, and ω is the point on the unit sphere marking the direction of scattering, and $f(\omega)$ is the *diffraction coefficient*. We prefer to indicate explicitly the dependence of the diffraction coefficient on the direction ω_0 from which the incident plane wave is coming:

$$f = f(\omega; \omega_0).$$

Introduce the spherical coordinates for the points ω and ω_0: $\omega(\theta, \varphi)$, $\omega_0(\theta_0, \varphi_0)$ The positive z-axis corresponds to the direction of $\theta = 0$, and the positive x-axis corresponds to $\theta = \pi/2$, $\varphi = 0$.

Besides, we shall use the "Cartesian" coordinates (ξ, η) and (ξ_0, η_0) for ω and ω_0, respectively:

$$\xi = \sin\theta\cos\varphi, \quad \eta = \sin\theta\sin\varphi.$$

The diffraction coefficient $f(\omega; \omega_0)$ is the main function to be determined within the current research.

3 Embedding formula for the diffraction coefficients

3.1 Edge Green's functions in the 3D space

Introduce the *edge Green's function* G_y having the prescribed oversingular (unphysical) asymptotics at the edge as follows. Consider the inhomogeneous Helmholtz equation

$$\left(\Delta + k_0^2\right)\hat{G}_y(x, y, z; Y, \epsilon) = \sqrt{\frac{\pi}{\epsilon}}\,\delta(x + \epsilon, y - Y, z).$$

Solve this problem taking into account boundary, radiation, vertex, and edge conditions and take the limit

$$G_y(x, y, z; Y) = \lim_{\epsilon \to 0} \hat{G}_y(x, y, z; Y, \epsilon).$$

$G_y(x, y, z; Y)$ is one of the edge Green's functions for our problem.

Since the edge condition physically means the absence of the sources at the edge, and the function G_y does possess the source at the edge, it should violate the edge condition. A detailed local study of the edge behaviour of G_y shows that the following property is valid. If the integral

$$I(x, y, z) = \int_0^\infty h(Y) G_y(x, y, z; Y) dY$$

is constructed for a smooth enough density function h, then the edge asymptotics for I near the edge Λ_2 has the form

$$I(\rho_2, \alpha_2, y) = -\frac{h(y)}{\sqrt{\pi}}\rho_2^{-1/2}\sin\frac{\alpha_2}{2} + O(\rho_2^{1/2}\sin\frac{\alpha_2}{2})$$

in the local cylindrical coordinates.

Analogously, introduce another edge Green's function $G_x(x, y, z; X)$ for the point source located near the edge Λ_1. By symmetry,

$$G_x(x, y, z; X) = G_y(y, x, z; X).$$

Introduce the directivities f_y and f_x of the edge Green's functions as the coefficients of the following asymptotic expansions:

$$G_y(\omega, r; Y) = 2\pi\frac{e^{\mathrm{i}k_0 r}}{k_0 r} f_y(\omega; Y) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}), \tag{8}$$

$$G_x(\omega, r; X) = 2\pi\frac{e^{\mathrm{i}k_0 r}}{k_0 r} f_x(\omega; X) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}). \tag{9}$$

Obviously,

$$f_x(\xi, \eta; X) = f_y(\eta, \xi; X). \tag{10}$$

Finally, define $C_G(x; Y)$ as the coefficient of the edge asymptotics of the edge Green's function G_y (i.e. with the source at Λ_2) observed near the edge Λ_1:

$$G_y(\rho_1, \alpha_1, x; Y) = \frac{2C_G(x; Y)}{\sqrt{\pi}}\rho_1^{1/2}\sin\frac{\alpha_1}{2} + O(\rho_1^{3/2}).$$

Note that a similar coefficient can be defined by taking the source at the edge Λ_1 (i.e. by studying the edge Green's function G_x) and the observation point at the edge Λ_2. However, due to the reciprocity principle the source and the observation point can be interchanged, so it will be the same coefficient. Moreover, due to symmetry

$$C_G(x;y) = C_G(y;x).$$

3.2 Embedding formulae in the 3D domain

Let us prove the following theorem.

Theorem 1 *The following integral representations (the so-called embedding formulae in 3D space) are valid for the diffraction coefficient $f(\omega, \omega_0)$:*

$$f(\omega, \omega_0) = \frac{4\pi^2 \mathrm{i}}{k_0^2(\xi + \xi_0)} \int\limits_0^\infty f_y(\omega; Y)\, f_y(\omega_0; Y)\, dY, \tag{11}$$

$$f(\omega, \omega_0) = \frac{4\pi^2 \mathrm{i}}{k_0^2(\eta + \eta_0)} \int\limits_0^\infty f_x(\omega; X)\, f_x(\omega_0; X)\, dX, \tag{12}$$

$$f(\omega, \omega_0) = \frac{4\pi^2}{k_0^3(\xi + \xi_0)(\eta + \eta_0)} \times$$
$$\iint_0^\infty [f_x(\omega; X)\, f_y(\omega_0; Y) + f_x(\omega_0; X)\, f_y(\omega; Y)]\, C_G(X; Y)\, dX\, dY. \tag{13}$$

The formulae (11), (12), (13) can be proved using the technique developed in [4], i.e. by applying the operators

$$H_x = \frac{\partial}{\partial x} + \mathrm{i}k_0\xi_0, \qquad H_y = \frac{\partial}{\partial y} + \mathrm{i}k_0\eta_0$$

and

$$H_{xy} = \left(\frac{\partial}{\partial x} + \mathrm{i}k_0\xi_0\right)\left(\frac{\partial}{\partial y} + \mathrm{i}k_0\eta_0\right)$$

to the total field and taking into account the theorem of uniqueness.

3.3 Edge Green's functions on a sphere

Let us make some preliminary steps for the subsequent consideration. The embedding formulae (11), (12) and (13) will be used below to modify the representation (1). A natural way for this is to separate the radial variable and to study the spherical problem for each value of the separation constant. That is why we need an analog of the edge Green's function introduced for the problem on a sphere.

Define the Laplace-Beltrami operator in the spherical coordinates:

$$\tilde{\Delta} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

One can formulate an eigenvalue Dirichlet problem on a sphere as follows. Let the Laplace-Beltrami equation

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) v(\omega, \nu) = 0 \tag{14}$$

be valid on a sphere with the cut S corresponding to the cross-section of the flat cone (i.e. S is the line $\theta = \pi/2$, $0 < \varphi < \pi/2$). Let the Dirichlet boundary condition $v = 0$ be valid on S

The edge conditions are formulated for the *edges* L_1 and L_2 of the spherical problem, i.e. for the ends of S, which are the cross-sections of Λ_1 and Λ_2, respectively. Introduce the local spherical coordinates $\zeta_{1,2}$, $\phi_{1,2}$ near the edges as it is shown in Fig. 2. One should demand that the solution near the edges grows no faster than $\zeta_{1,2}^{1/2} \sin(\phi_{1,2}/2)$.

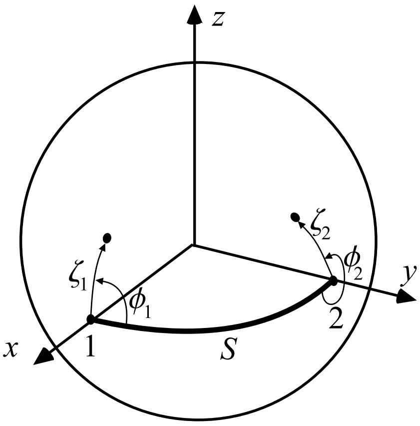

Figure 2: Local spherical coordinates near the edges

The functions satisfying all these conditions exist for a discrete set of real values of ν (eigenvalues of the problem). These values form the *spectrum* of the problem and will be denoted by ν_j. Corresponding eigenfunctions will be denoted by $\Phi_j(\omega)$. One can show that all $\nu_j > 1/2$.

We assume that the eigenfunctions are normalized as follows:

$$\iint \Phi_m(\omega)\Phi_n(\omega)\, d\omega = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

Here the integration is performed over the whole sphere.

Define the *spherical edge Green's function* $v^1(\omega, \nu)$ for values of ν not belonging to the spectrum as follows. First, define the function $\hat{v}^1(\omega, \nu; \kappa)$ as the solution of the spherical problem with a point source located at the point $\omega_\kappa = (\theta = \pi/2, \varphi = -\kappa)$ close to the edge $\varphi = 0$. We assume that the inhomogeneous Laplace-Beltrami equation is valid:

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) \hat{v}^1(\omega, \nu, \kappa) = \sqrt{\frac{\pi}{\kappa}}\, \delta(\theta - \pi/2, \varphi + \kappa).$$

The boundary condition at the cut and the edge conditions are also taken into account. Solve this problem for each κ and take the limit

$$v^1(\omega, \nu) = \lim_{\kappa \to 0} \hat{v}^1(\omega, \nu, \kappa).$$

Function v^1 can be represented trough the eigenfunctions of the spherical problem. Namely, let the edge asymptotics of the eigenfunction Φ_j near L_1 and L_2, respectively, have the form:

$$\Phi_j(\zeta_1, \phi_1) = \frac{2C_j}{\sqrt{\pi}} \zeta_1^{1/2} \sin \frac{\phi_1}{2} + O(\zeta_1^{3/2}), \tag{15}$$

$$\Phi_j(\zeta_2, \phi_2) = \frac{2\tilde{C}_j}{\sqrt{\pi}} \zeta_2^{1/2} \sin \frac{\phi_2}{2} + O(\zeta_2^{3/2}). \tag{16}$$

Here C_j and $\tilde{C}_j$ are some constants.

Using orthogonality and completeness of the eigenfunctions, one can obtain the relation

$$v^1(\omega, \nu) = 2 \sum_j \frac{C_j \Phi_j(\omega)}{\nu^2 - \nu_j^2}. \tag{17}$$

Analogously, taking the source at the point $\tilde{\omega}_\kappa$ with the coordinates $\theta = \pi/2$, $\varphi = \pi/2 + \kappa$, one can introduce the edge Green's function $v^2(\omega, \nu)$. Due to symmetry, $v^2(\theta, \varphi, \nu) = v^1(\theta, \pi/2 - \varphi, \nu)$. Using the asymptotics (16) this function can be written in the form

$$v^2(\omega, \nu) = 2 \sum_j \frac{\tilde{C}_j \Phi_j(\omega)}{\nu^2 - \nu_j^2}. \tag{18}$$

Introduce also the coefficient $C_2^1(\nu)$ describing the asymptotics of v^1 near the edge L_2:

$$C_2^1(\nu) = 2 \sum_j \frac{C_j \tilde{C}_j}{\nu^2 - \nu_j^2} = \lim_{\kappa \to 0} \frac{\sqrt{\pi}}{2\sqrt{\kappa}} v^1(\theta = \pi/2, \varphi = \pi/2 + \kappa) = 2 \sum_j \frac{C_j \tilde{C}_j}{\nu^2 - \nu_j^2}.$$

3.4 Embedding formulae in the spectral domain

Let us formulate the following theorem.

Theorem 2 *The following embedding formulae in the spectral domain are valid:*

$$f(\omega, \omega_0) = \frac{1}{4\pi \mathrm{i}(\eta + \eta_0)} \int_\gamma e^{-\mathrm{i}\pi\nu} [v^1(\omega_0, \nu) v^1(\omega, \nu + 1) + v^1(\omega, \nu) v^1(\omega_0, \nu + 1)] \, d\nu, \tag{19}$$

$$f(\omega, \omega_0) = \frac{1}{4\pi \mathrm{i}(\xi + \xi_0)} \int_\gamma e^{-\mathrm{i}\pi\nu} [v^2(\omega_0, \nu) v^2(\omega, \nu + 1) + v^2(\omega, \nu) v^2(\omega_0, \nu + 1)] \, d\nu, \tag{20}$$

$$f(\omega, \omega_0) = \frac{\mathrm{i}}{8\pi(\xi + \xi_0)(\eta + \eta_0)} \int_\Gamma \frac{e^{-\mathrm{i}\pi\nu}}{\nu} C_2^1(\nu) [B(\omega, \omega_0, \nu) + B(\omega_0, \omega, \nu)] \, d\nu, \tag{21}$$

where

$$B(\omega, \omega_0, \nu) = (v^1(\omega, \nu + 1) - v^1(\omega, \nu - 1))(v^2(\omega_0, \nu + 1) - v^2(\omega_0, \nu - 1)),$$

γ and Γ are the contours of integration shown in the Figure 3 and Figure 4. Contour Γ consists of the infinite loop and two small loops.

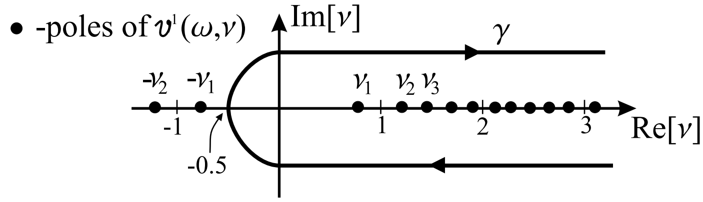

Figure 3: Contour of integration for formula (19)

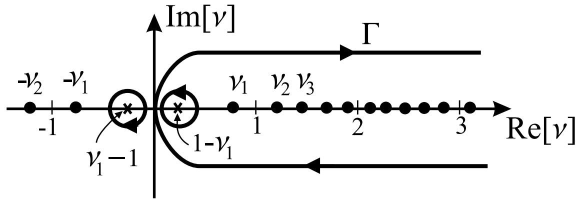

Figure 4: Contour of integration for formula (21)

The procedure of the proof is similar to the one described in [2], [1].

Compare the properties of the embedding formulae with that of (1). There are two features making the embedding formulae preferable.

(i) The functions $v^{1,2}(\omega, \nu)$ depend on three scalar variables, while $g(\omega, \omega_0, \nu)$ depend on five ones. Therefore functions $v^{1,2}$ require less computational efforts, if such tabulation if necessary.

(ii) The integrals (19), (20) and (21) are better than (1) from the point of view of convergence.

The last statement should be explained. The integral (1) is convergent only in the sense of distributions. However, the authors of [1] discuss the possibility of transforming the contour of integration in (1) in such a way that the integrand decays exponentially along the contour. We should remind here that the exponentially decaying integrals are much more attractive for the numerical analysis than the divergent ones.

In application to our case, the main result of [1] is that such a transformation is available only for the points ω and ω_0 satisfying the inequalities

$$\arccos \xi + \arccos \xi_0 > \pi \tag{22}$$

and

$$\arccos \eta + \arccos \eta_0 > \pi. \tag{23}$$

Let us study the growth of the integrands of (19), (20) and (21). Following [1] and [5], we obtain the following estimations:

$$\begin{aligned} v^1(\omega, \nu) &\sim \exp\{-|\mathrm{Im}\, \nu| \arccos \xi\} |\nu|^{-1/2}, \\ v^2(\omega, \nu) &\sim \exp\{-|\mathrm{Im}\, \nu| \arccos \eta\} |\nu|^{-1/2}, \\ C_2^1(\nu) &\sim \exp\{-|\mathrm{Im}\, \nu| \pi/2\} \end{aligned}$$

The integrand of (19) can be estimated as

$$|\nu|^{-1} \exp\{-\mathrm{i}\pi\nu - |\mathrm{Im}\, \nu| (\arccos \xi + \arccos \xi_0)\}.$$

It means that the integral (19) is convergent for all ω, ω_0, and if the inequality (22) is valid, then this integral can be converted into the exponentially convergent one by using the contour γ' instead of γ as it is shown in Figure 5a. Note that the inequality (23) is not necessary in this case.

Similarly, the formula (20) can be considered.

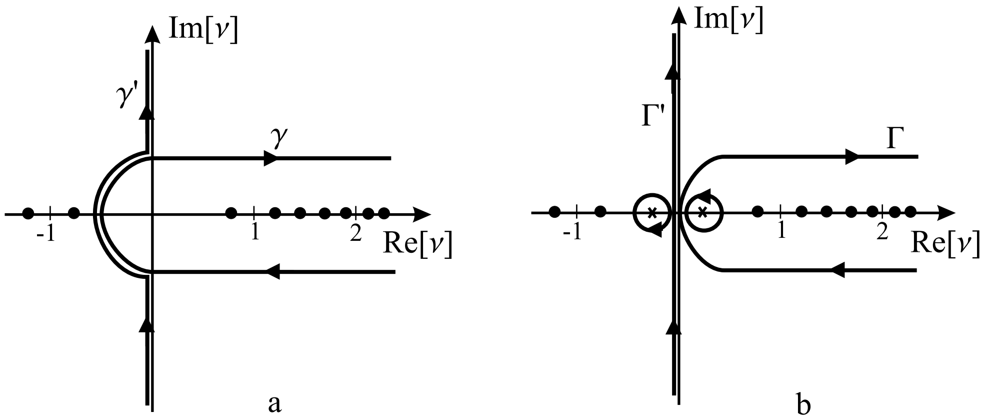

Figure 5: Deformation of the contours of integration

The integrand of (21) can be estimated as

$$|\nu|^{-2}\exp\{-\mathrm{i}\pi\nu\}\left(\exp\{-|\mathrm{Im}\,\nu|(\arccos\xi+\arccos\eta_0+\pi/2)\}+\right.$$
$$\left.\exp\{-|\mathrm{Im}\,\nu|(\arccos\xi_0+\arccos\eta+\pi/2)\}\right).$$

The integral is convergent for almost each ω and ω_0 (when the exponential factor oscillates). If

$$\arccos\xi+\arccos\eta_0>\pi/2 \tag{24}$$

and

$$\arccos\xi_0+\arccos\eta>\pi/2 \tag{25}$$

the open loop of the contour of integration Γ can be deformed into the contour Γ' shown in Fig. 5b, along which the convergence is exponential.

Consider the inequalities (22), (23), (24) and (25). When they stop to be valid it is reasonable to expect the singularities of the diffraction coefficient. All singularities of the diffraction coefficient can be found by simple physical consideration. The cylindrical wave diffracted by the edges of the quarter plane are due to the simple poles of f. These poles are located at the lines $\xi+\xi_0=0$ and $\eta+\eta_0=0$. The intersection of these lines correspond to the reflected plane wave. Beside these sets there can be "secondary" singularities corresponding to the rays diffracted first by one edge and then by another one (see Fig. 6). These rays result into the branch lines corresponding to the sets $\eta=\sqrt{1-\xi_0^2}$ and $\xi=\sqrt{1-\eta_0^2}$. The first singularity appears only if $\xi_0>0$, and the second one appears only if $\eta_0>0$.

According to this consideration, one can see that the embedding formulae enable one to "extract" the singularities (poles) out of the integral into the rational factor. The formulae (19) and (20) extract only one pole each, and the formula (21) extracts both poles. Only the secondary singularities remain in the integral term.

Physically, this can be explained as follows. The operator H_{xy} gives zero when it acts on the incident plane wave, reflected plane wave, and both scattered cylindrical waves.

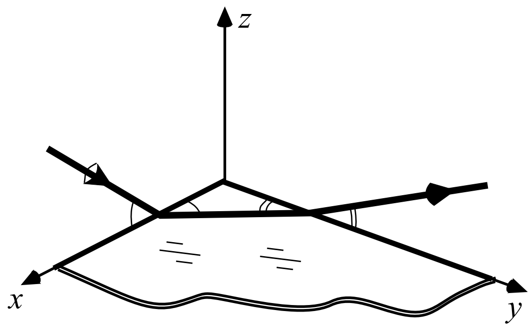

Figure 6: Rays corresponding to the secondary singularities

4 Coordinate equations for the edge Green's functions on the sphere

As one can see, by means of the embedding formulae the diffraction coefficient becomes expressed in terms of the functions $v^{1,2}$ and the function C_2^1 associated with their asymptotics. These functions can be calculated using the series (17),(18) with the eigenfunctions Φ_j found by the classical separation of variables. However, below we propose a completely new method to calculate $v^{1,2}$. This method can be applied to a wider range of the diffraction problems, in particular to those, for which the classical separation of variables is not known. For example, the same method, almost unmodified, can be applied to a sphere with several Dirichlet cuts located on the line $\theta = \pi/2$ or to a sphere with a triangular hole corresponding to a trihedral cone having right angles at the vertex. Both of these problems cannot be treated by any other analytical method.

4.1 Additional unknown functions

Define the functions $w^1(\omega, \nu)$ and $w^2(\omega, \nu)$as the edge Green's functions for the Neumann problem as follows.

For each κ small enough define $\hat{w}^1(\omega, \nu, \kappa)$ as the solution of the inhomogeneous Laplace-Beltrami equation

$$\left[\tilde{\Delta} + \nu^2 - \frac{1}{4}\right] \hat{w}^1(\omega, \nu, \kappa) = \frac{1}{2}\sqrt{\frac{\pi}{\kappa}}\, \delta(\varphi - \kappa) \left[\delta(\theta - \pi/2 + 0) - \delta(\theta - \pi/2 - 0)\right]. \tag{26}$$

Neumann boundary conditions

$$\frac{\partial \hat{w}^1}{\partial \theta} = 0$$

on S and edge conditions ($\hat{w}^1 \sim \zeta_{1,2}^{1/2}$) are satisfied by $\hat{w}^1$. The function w^1 is defined as the limit

$$w^1(\omega, \nu) = \lim_{\kappa \to 0} \hat{w}^1(\omega, \nu, \kappa).$$

The function w^1 is the edge Green's function for the Neumann problem. It corresponds to the problem with the source located at the edge L_1. The structure of the source is chosen such that the field is non-trivial (antisymmetric) and has a non-zero limit as $\kappa \to 0$.

The function w^2 is the edge Green's function corresponding to the source located at the edge L_2. Define this function using the symmetry relation

$$w^2(\theta,\varphi) = w^1(\pi-\theta,\pi/2-\varphi).$$

It is easy to show that the edge asymptotics of the edge Green's functions have the form:

$$v^m(\phi_n,\zeta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}}\zeta_n^{-1/2}\sin\frac{\phi_n}{2} + \frac{2C_n^m}{\sqrt{\pi}}\zeta_n^{1/2}\sin\frac{\phi_n}{2} + O(\zeta_n^{3/2}) \tag{27}$$

and

$$w^m(\phi_n,\zeta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}}\zeta_n^{-1/2}\cos\frac{\zeta_n}{2} + \frac{2E_n^m}{\sqrt{\pi}}\zeta_n^{1/2}\cos\frac{\phi_n}{2} + O(\zeta_n^{3/2}), \tag{28}$$

where $m,n = 1,2$; δ is the Kronecker's delta; C_n^m and E_n^m are some unknown coefficients depending on ν. Note that for $m=1$, $n=2$ the value of $C_n^m = C_2^1$ coincides with $C_2^1(\nu)$ from formula (21).

Due to the obvious symmetry

$$C_2^1 = C_1^2, \quad E_2^1 = E_1^2, \quad C_1^1 = C_2^2, \quad E_1^1 = E_2^2. \tag{29}$$

Here and below we omit the argument ν of the functions $v^{1,2}$, $w^{1,2}$ and of the coefficients C_n^m and E_n^m. We assume that the non-resonant case is considered, i.e. that ν belongs neither to the spectrum of the Dirichlet nor of the Neumann problem. This means that if the field satisfies the equation (14), boundary conditions of the Dirichlet or Neumann type on S and the Meixner's edge conditions (i.e. it grows at the edges no faster than $\zeta_n^{-1/2}$), then the field is identically equal to zero.

4.2 Derivation of the coordinate equations for the edge Green's functions

Let us prove the following theorem.

Theorem 3 *Let the vector* $\mathbf{U}$ *be defined as*

$$\mathbf{U} = (v^1, v^2, w^1, w^2)^T \tag{30}$$

for any value of ν *not belonging to the spectrum of the Dirichlet or Neumann problem. This vector obeys the coordinate equations of the form*

$$\frac{\partial}{\partial\theta}\mathbf{U} = \mathbf{X}\mathbf{U}, \qquad \frac{\partial}{\partial\varphi}\mathbf{U} = \mathbf{Y}\mathbf{U}, \tag{31}$$

with the coefficients $\mathbf{X}$, $\mathbf{Y}$, *whose explicit form is given by the relations (49).*

Proof. Call the function v an *oversingular combination* if it satisfies the Laplace-Beltrami equation (14), boundary conditions either of the Dirichlet or of the Neumann type on S and behaves at the edges like

$$v(\phi_n,\zeta_n) = \frac{C_n}{\sqrt{\pi}}\zeta_n^{-1/2}\sin\frac{\phi_n}{2} + O(\zeta_n^{1/2})$$

in the Dirichlet case or

$$v(\phi_n, \zeta_n) = \frac{E_n}{\sqrt{\pi}} \zeta_n^{-1/2} \cos\frac{\phi_n}{2} + O(\zeta_n^{1/2})$$

in the Neumann case. Note that the oversingular functions do not satisfy the Meixner's edge conditions, so they are not necessarily equal to zero.

In the non-resonant case it is clear that

$$v = -C_1 v^1 - C_2 v^2, \quad \text{or} \quad v = -E_1 w^1 - E_2 w^2. \tag{32}$$

(Note that, for example, the combination $u + C_1 v^1 + C_2 v^2$ in the Dirichlet case satisfies the Laplace-Beltrami equation (14), boundary and edge conditions, therefore this combination should be equal to zero.)

Derive the coordinate equation for the vector $\mathbf{U}$ of (30) as follows. Seek for the combinations of the derivatives of $v^{1,2}$ and $w^{1,2}$ that are oversingular functions. Note that generally the combination of the derivatives of $v^{1,2}$ and $w^{1,2}$ do not satisfy the conditions of the oversingular solution since the functions v and w are oversingular themselves, and therefore their derivatives normally contain the terms of $\zeta_{1,2}^{-3/2}$. So, only several specific combinations can be found, that form the basis of the oversingular differentiations of $v^{1,2}$ and $w^{1,2}$.

Introduce 3 differential operators T_1, T_2 and T_3 as follows. The operator T_3 is simply

$$T_3 = \frac{\partial}{\partial \varphi}, \tag{33}$$

where φ is the spherical coordinate used above. Two other operators are also differentiations with respect to the rotations, but the axes are chosen as the x and y directions, respectively. Thus,

$$T_1 = \frac{\partial}{\partial \phi_1}, \qquad T_2 = \frac{\partial}{\partial \phi_2}, \tag{34}$$

where ϕ_1 and ϕ_2 are considered as the global (rather than local) coordinates. The explicit form of T_1 and T_2 in the coordinates (θ, φ) is as follows:

$$\begin{aligned} T_1 &= -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\varphi}, \\ T_2 &= \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\varphi} \end{aligned}$$

Obviously, if some function v obeys the Laplace-Beltrami equation, then $T_j[v]$ also obeys the same equation. This follows from the fact that the Laplace-Beltrami operator is invariant with respect to any rotation of the sphere.

It is less obvious but easy to prove that if v obeys the Dirichlet boundary conditions on the sides of S, then $T_3[u]$ obeys the Dirichlet conditions, while $T_1[v]$ and $T_2[v]$ obey the Neumann condition on S. Conversely, if v obeys Neumann condition, then $T_3[v]$ obeys Neumann condition, while $T_1[v]$ and $T_2[v]$ obey the Dirichlet condition.

The first four oversingular combinations are the following:

$$T_1[v^1], \quad T_1[w^1], \quad T_2[v^2], \quad T_2[w^2].$$

The other four combinations are more complicated:

$$T_3[v^1] + T_2[w^1], \quad T_2[v^1] - T_3[w^1], \quad T_3[v^2] - T_1[w^2], \quad T_2[v^2] + T_3[w^2].$$

Studying the asymptotics of the oversingular combinations, we obtain eight representations of the form (32):

$$T_1[v^1] = C_2^1 w^2 + \frac{1}{2} w^1, \tag{35}$$
$$T_1[w^1] = E_2^1 v^2 - \frac{1}{2} v^1, \tag{36}$$
$$T_2[v^2] = C_1^2 w^1 + \frac{1}{2} w^2, \tag{37}$$
$$T_2[w^2] = E_1^2 v^1 - \frac{1}{2} v^2, \tag{38}$$

$$T_3[v^1] + T_2[w^1] = (C_1^1 + E_1^1) v^1 - C_2^1 v^2, \tag{39}$$
$$T_2[v^1] - T_3[w^1] = (C_1^1 + E_1^1) w^1 - E_2^1 w^2, \tag{40}$$
$$T_3[v^2] - T_1[w^2] = C_1^2 v^1 - (C_2^2 + E_2^2) v^2, \tag{41}$$
$$T_2[v^2] + T_3[w^2] = E_1^2 w^1 + (C_2^2 + E_2^2) w^2. \tag{42}$$

The system (35)–(42) consists of 8 equations and contains 8 independent derivatives of 4 functions $v^{1,2}$ and $w^{1,2}$ with respect to the coordinates (θ, φ). So, one can express these derivatives separately. The representation of the derivatives has the form of the equations (31) written for the vector $\mathbf{U}$ defined by (30) and the coefficients are defined by the formulae (49) given in Appendix A. These equations are the coordinate equations for the edge Green's functions. □

4.3 Some properties of the coordinate equations

Consider the equations (31), (49). They have the following properties.
(i) The solvability condition should be valid:

$$\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \varphi}\mathbf{U}\right) = \frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \theta}\mathbf{U}\right).$$

A sufficient condition for this is as follows:

$$\mathbf{XY} - \mathbf{YX} + \frac{\partial}{\partial \varphi}\mathbf{X} - \frac{\partial}{\partial \theta}\mathbf{Y} = 0. \tag{43}$$

One can check directly, that the matrices (49) obey this condition identically.
(ii) All components of the vector $\mathbf{U}$ should satisfy the Laplace-Beltrami equation (14). Note that due to (31),

$$\tilde{\Delta}\mathbf{U} = \left[\mathbf{X}^2 + \frac{\cos\theta}{\sin\theta}\mathbf{X} + \frac{\partial}{\partial \theta}\mathbf{X} + \frac{1}{\sin^2\theta}\left(\mathbf{Y}^2 + \frac{\partial}{\partial \varphi}\mathbf{Y}\right)\right]\mathbf{U}. \tag{44}$$

A direct substitution of (49) into (44) shows that

$$\mathbf{X}^2 + \frac{\cos\theta}{\sin\theta}\mathbf{X} + \frac{\partial}{\partial\theta}\mathbf{X} + \frac{1}{\sin^2\theta}\left(\mathbf{Y}^2 + \frac{\partial}{\partial\varphi}\mathbf{Y}\right) = \left(\frac{1}{4} + (C_1^1 + E_1^1)^2 - (C_2^1)^2 - (E_2^1)^2\right)\mathbf{I}, \quad (45)$$

where $\mathbf{I}$ is the 4×4 identity matrix. Comparing (45) with

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right)\mathbf{U} = 0, \quad (46)$$

we conclude that (46) is fulfilled provided that

$$(C_1^1 + E_1^1)^2 = (C_2^1)^2 + (E_2^1)^2 - \nu^2. \quad (47)$$

The last relation makes it possible to express the combination $C_1^1 + E_1^1$ in terms of the parameters C_2^1 and E_2^1. It means that the coefficients of the equations (31), (49) contain only two unknown numerical parameters depending on ν, namely C_2^1 and E_2^1. We remind that the parameter C_2^1 is very important, because it stands in the embedding formula (21). **(iii)** Consider the equations (31) at the cut S. Note that the boundary conditions on S have the form

$$v^1 = 0, \quad v^2 = 0, \quad \frac{\partial}{\partial\theta}w^1 = 0, \quad \frac{\partial}{\partial\theta}w^2 = 0. \quad (48)$$

Due to the form of the matrix $\mathbf{X}$ (see (49)) the last two conditions follow from the first two ones. Due to the form of the matrix $\mathbf{Y}$, if $v^1 = v^2 = 0$ at a single point of S, then the same conditions are valid on the whole cut S. I.e., it is necessary to check the boundary conditions (only two of them!) at a single point of S.

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Appendix. The explicit form of the coordinate equations

The explicit expressions for the coefficients of the coordinate equations (31) are as follows:

$$\mathbf{X} = \begin{pmatrix} X_1^1 & X_2^1 & X_3^1 & X_4^1 \\ X_1^2 & X_2^2 & X_3^2 & X_4^2 \\ X_1^3 & X_2^3 & X_3^3 & X_4^3 \\ X_1^4 & X_2^4 & X_3^4 & X_4^4 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} Y_1^1 & Y_2^1 & Y_3^1 & Y_4^1 \\ Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 \\ Y_1^3 & Y_2^3 & Y_3^3 & Y_4^3 \\ Y_1^4 & Y_2^4 & Y_3^4 & Y_4^4 \end{pmatrix}, \quad (49)$$

where

$$\begin{aligned} X_1^1 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{\cos\varphi - 2(C_1^1 + E_1^1)\sin\varphi}{2(1 - \cos^2\varphi\,\sin^2\theta)}, \\ X_2^1 &= -\cos\varphi\,\cos\theta\,\sin\theta\,\frac{E_2^1\cos\varphi - C_2^1\sin\varphi}{1 - \cos^2\varphi\,\sin^2\theta}, \end{aligned}$$

$$
\begin{aligned}
X_3^1 &= \frac{(C_1^1 + E_1^1)\cos\varphi\,\cos^2\theta - \sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)} \\
X_4^1 &= -\frac{E_2^1\cos\varphi\,\cos^2\theta + C_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
X_1^2 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}, \\
X_2^2 &= -\sin\varphi\,\cos\theta\,\sin\theta\,\frac{2(C_1^1+E_1^1)\cos\varphi - \sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}, \\
X_3^2 &= \frac{C_2^1\cos\varphi + E_2^1\cos^2\theta\,\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta} \\
X_4^2 &= \frac{\cos\varphi - 2(C_1^1+E_1^1)\cos^2\theta\,\sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

$$
\begin{aligned}
X_1^3 &= \frac{2(C_1^1+E_1^1)\cos\varphi\,\cos^2\theta + \sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}, \\
X_2^3 &= -\frac{C_2^1\cos\varphi\,\cos^2\theta + E_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}, \\
X_3^3 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{2(C_1^1+E_1^1)\sin\varphi + \cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)} \\
X_4^3 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
X_1^4 &= \frac{E_2^1\cos\varphi + C_2^1\cos^2\theta\,\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}, \\
X_2^4 &= -\frac{2(C_1^1+E_1^1)\cos^2\theta\,\sin\varphi + \cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}, \\
X_3^4 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\sin\varphi - E_2^1\cos\varphi}{1-\sin^2\varphi\,\sin^2\theta} \\
X_4^4 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{\sin\varphi + 2(C_1^1+E_1^1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

$$
\begin{aligned}
Y_1^1 &= \sin\varphi\,\sin^2\theta\,\frac{2(C_1^1+E_1^1)\sin\varphi - \cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}, \\
Y_2^1 &= \sin\varphi\,\sin^2\theta\,\frac{E_2^1\cos\varphi - C_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}, \\
Y_3^1 &= -\cos\theta\,\sin\theta\,\frac{2(C_1^1+E_1^1)\sin\varphi + \cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)} \\
Y_4^1 &= \cos\theta\,\sin\theta\,\frac{E_2^1\sin\varphi - C_2^1\cos\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
Y_1^2 &= \cos\varphi\,\sin^2\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta},\\
Y_2^2 &= -\cos\varphi\,\sin^2\theta\,\frac{2(C_1^1+E_1^1)\cos\varphi - \sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)},\\
Y_3^2 &= \cos\theta\,\sin\theta\,\frac{E_2^1\cos\varphi - C_2^1\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}\\
Y_4^2 &= -\cos\theta\,\sin\theta\,\frac{\sin\varphi + 2(C_1^1+E_1^1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

$$
\begin{aligned}
Y_1^3 &= \cos\theta\,\sin\theta\,\frac{\cos\varphi - 2(C_1^1+E_1^1)\sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)},\\
Y_2^3 &= -\cos\theta\,\sin\theta\,\frac{E_2^1\cos\varphi - C_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta},\\
Y_3^3 &= -\sin\varphi\,\sin^2\theta\,\frac{\cos\varphi + 2(C_1^1+E_1^1)\sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}\\
Y_4^3 &= \sin\varphi\,\sin^2\theta\,\frac{E_2^1\sin\varphi - C_2^1\cos\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
Y_1^4 &= \cos\theta\,\sin\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta},\\
Y_2^4 &= -\cos\theta\,\sin\theta\,\frac{2(C_1^1+E_1^1)\cos\varphi - \sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)},\\
Y_3^4 &= \cos\varphi\,\sin^2\theta\,\frac{C_2^1\sin\varphi - E_2^1\cos\varphi}{1-\sin^2\varphi\,\sin^2\theta}\\
Y_4^4 &= \cos\varphi\,\sin^2\theta\,\frac{\sin\varphi + 2(C_1^1+E_1^1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

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