On the connection between the Wiener-Hopf method an the theory of ordinary differential equations

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Abstract

The application of the classic Wiener-Hopf method involves the procedure of additive decomposition of a function into two terms, one of which is regular in the upper complex half-plane and another is regular in the lower half-plane. Usually this operation is performed by using the integral operator with Cauchi's kernel. In the current paper we show that this procedure can be performed in many cases using the ordinary differential equations. We obtain the representation of the seeked functions in quadratures.

1 Introduction

Wiener-Hopf method is used in many problems of diffraction and mechanics [1]; as a rule, these problems have mixed boundary conditions. Here we study only the scalar problems.

Let us discuss briefly the scheme of Wiener-Hopf method. Consider the functional equation

$$V_+(k) + K(k)V_-(k) = N(k), \tag{1.1}$$

where k — is a complex independent variable, K and N are some known functions, V_+ and V_- are unknown functions. We require that the function V_+ is regular in the upper half-plane of k, the function V_- is regular in the lower half-plane. Besides, we imply that the unknown functions have some known order of growth at infinity and at singular points.

The equation (1.1) is solved in two steps. on the **first step** the coefficient K is factorized, i.e. the following representation is seeked:

$$K(k) = K_-(k)/K_+(k), \tag{1.2}$$

where K_- ϒK_+ are some functions, regular and having no zeros in the lower and the upper half-planes respectively. Also these functions have some known growth at infinity. Usually, the logarithmic derivatives are used for calculation of K_- and K_+:

$$\frac{K'}{K} = \frac{K'_-}{K} - \frac{K'_+}{K},$$

where prime denotes the differentiation with respect to k. Now factorization is reduced to additive decomposition, which is performed using the integral operator with Cauchi's kernel:

$$\frac{K'_+}{K} = -F_+\left[\frac{K'}{K}\right], \qquad \frac{K'_-}{K} = -F_-\left[\frac{K'}{K}\right], \tag{1.3}$$

where for arbitrary $V(k)$ decaying on the real axis

$$F_+[V(k)] = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{V(\tau)d\tau}{\tau - k} \text{ for } \operatorname{Im} k > 0, \qquad F_-[V(k)] = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{V(\tau)d\tau}{\tau - k} \text{ for } \operatorname{Im} k > 0. \tag{1.4}$$

The values of $F_+[V]$ are defined by the formula (1.4) only for $\mathrm{Im}[k] > 0$. To determine the values of the desired functions for other k one must perform analytic continuation.

When the logarithmic derivatives of K_+ and K_- are calculated, one can reconstruct the functions using the obvious formula

$$K_\pm = \exp\left\{ \int^k \frac{K'_\pm}{K_\pm} d\tau. \right\}$$

Th equation (1.1) now can be rewritten in the form

$$K_+(k)V_+(k) + K_-(k)V_-(k) = K_+(k)N(k). \tag{1.5}$$

Note that the left-hand part is decomposed into two terms, regular in the lower and the upper half-planes respectively.

On the **second step** we decompose the right-hand part of (1.5) into the terms, regular in the upper and lower half-planes:

$$K_+(k)N(k) = M_+(k) + M_-(k),$$

where

$$M_+ = F_+[K_+(k)R(k)], \qquad M_- = -F_-[K_+(k)R(k)].$$

Afterwards, the equation (1.5) can be rewritten in the form

$$K_+(k)V_+(k) - M_+(k) = -K_-(k)V_-(k) + M_-(k). \tag{1.6}$$

This equation is valid for the analytic continuation of the functions, involved in it. It means that the function, which is regular in the upper half-plane (the left-hand side) is equal to the functtion, which is regular in the lower half-plane. The restrictions on the growth guarantee that this function grows no faster than a power of k. Applying the Liouville's theorem, one concludes that

$$K_+(k)V_+(k) - M_+(k) = -K_-(k)V_-(k) + M_-(k) = Q(k),$$

where Q is a polynomial of some known order, whose coefficients are undetermined yet. These coefficients must be found using some additional physical constraints. In the simpliest cases this polynomial is identically equal to zero. Using the last equation, one can find unknown functions.

It is clear that application of Wiener-Hopf method involves the usage of the operators $F_\pm$. This gives the representation of the unknown functions in the form of the integrals with parameters. Such a representation is convenient enough only if the integrals can be calculated analytically. However, in most cases it is not so.

The representation of the functions $K_\pm$ and $M_\pm$ in the form of the integrals with parameters can be inconvenient, for example, if it is necessary to solve a sequence of successive Wiener-hopf problems, i.e. if the field scattered by on obstacle falls on another one e.t.c. This situation occurs when the solution of complicated diffraction problem is seeked in the form of diffraction series. Besides, this represenation gives a lack of information if one needs to find the dependence of the solution on some parameters of the problem (the wavelength, the angle of incidence e.t.c.).

Here we propose another approach to additive decomposition of functions. Namely, for a wide class of functions $V(k)$ we show that the functions $F_\pm[V]$ are the solutions of ordinary differential equations with the coefficients of simple structure. In some cases these equations can be solved in the quadratures, thus providing convenient representations of unknown functions. In other cases this representation provides additional information or can be used for effective numerical calculation of $F_\pm[V]$.

2 Elementary properties of the operators $F_\pm$

Here we study the elementary properties of the operators $F_\pm$. Th properties 1, 2 and 3 are obvious. Th first states thelinearity of the operators, the second states the invariance with respect to translations along the real axis. The fourth property is less obvious. All further calculations are based on it.

We imply below that the functions V, V_1, V_2 are regular in the strip $|\,\mathrm{Im}[k]| < \delta$ for some $\delta > 0$ and decay along the real axis as some negative power of k.

1. For arbitrary constant c and arbitrary $V(k)$, $V_1(k)$, $V_2(k)$

$$F_\pm[cV(k)] = cF_\pm[V(k)], \qquad F_\pm[V_1(k) + V_2(k)] = F_\pm[V_1(k)] + F_\pm[V_2(k)].$$

2. The identity is valid:

$$(F_\pm[V])' = F_\pm[V'] \tag{2.1}$$

This property can be proven using the integration by parts:

$$(F_\pm[V(k)])' = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{V(\tau)d\tau}{(\tau-k)^2} = -\frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} V(\tau) d\left(\frac{1}{\tau - k}\right) = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{V'(\tau)d\tau}{\tau - k}.$$

3. Let $R(k)$ be a rational function growing at infinity no faster than k^{-1}. Then $F_\pm[R]$ are also rational functions. This can be easily proven by decomposing R into the partial fractions

4. For arbitry ξ not lying on the real axis,

$$F_\pm\left[\frac{V}{k-\xi}\right] = \frac{F_\pm[V]}{k-\xi} + \frac{\mathcal{F}(V,\xi)}{k-\xi}, \tag{2.2}$$

where

$$\mathcal{F}(V,\xi) = -\frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{V(\tau)d\tau}{\tau - \xi}. \tag{2.3}$$

The last property follows from the elementary relation

$$\frac{1}{(\tau-\xi)(\tau-k)} = \frac{1}{k-\xi}\left(\frac{1}{\tau-k} - \frac{1}{\tau-\xi}\right).$$

Note that the value $\mathcal{F}(V,\xi)$ does not depend on k, i.e. it is a constant with respect to k.

The equation (2.3) can be interpreted as follows. The operators $F_\pm$ perform the decomposition of V into the terms $V = V_+ - V_-$, regular in the upper and the lower half-planes, respectively. Try to decompose the function $V/(k-\xi)$ the same way. The decomposition of the form

$$\frac{V}{k-\xi} = \frac{V_+}{k-\xi} - \frac{V_-}{k-\xi}$$

is almost valid, but one of the terms has the undesired pole at the point $k=\xi$. However, this pole can be easily subtracted, so

$$\mathcal{F}(V,\xi) = \begin{cases} -V_+(\xi), & \xi \text{ for } \operatorname{Im}[\xi] > 0 \\ -V_-(\xi), & \xi \text{ for } \operatorname{Im}[\xi] < 0 \end{cases} \tag{2.4}$$

The fourth property can be generalized. Applying it several times, we obtain for an integer $n > 0$

$$F_\pm\left[\frac{V(k)}{(k-\xi)^n}\right] = \frac{F_\pm[V]}{(k-\xi)^n} + \sum_{m=1}^{n} \frac{1}{(k-\xi)^m}\mathcal{F}\left(\frac{V}{(k-\xi)^{n-m}}, \xi\right). \tag{2.5}$$

Note that all values $\mathcal{F}(\ldots)$ are constants with respect to k.

Let $R(k)$ be a rational function of k groing at infinity no faster than a constant. Decomposing R into the partial fractions and the equation (2.5), we obtain

$$F_\pm[R(k)V(k)] = R(k)F_\pm[V] + r(k), \tag{2.6}$$

where $r(k)$ is a rational function of k, whose coefficients can be calculated using (2.5) and the definition of the operators $\mathcal{F}$.

3 Differential equations for $F_\pm[V]$

Let the following differential equation is known:

$$\begin{pmatrix} V_1' \\ \ldots \\ V_n' \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} V_1 \\ \ldots \\ V_n \end{pmatrix} + \begin{pmatrix} R_1 \\ \ldots \\ R_n \end{pmatrix}, \tag{3.1}$$

where $a_{ml}(k)$ are rational functions of k, groing at infinity no faster than conctants; $R_1 \ldots R_n$ are rational functions groing no faster than k^{-1}; $V_1(k) = V(k)$. The functions $V_2(k) \ldots V_n(k)$ can be, for example, the derivatives of V if the differential equation of order n is known for $V(k)$. We remind that prime denotes the differentiation with respect to k.

Note that the equations of the form (3.1) are obeyed by many practically important functions, for example all algebraic functions.

Applying the relation (2.6), we obtain:

$$\begin{pmatrix} (F_\pm[V_1])' \\ \dots \\ (F_\pm[V_n])' \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} F_\pm[V_1] \\ \dots \\ F_\pm[V_n] \end{pmatrix} + \begin{pmatrix} r_1 \\ \dots \\ r_n \end{pmatrix}, \tag{3.2}$$

where $r_1(k)\dots r_n(k)$ are new rational functions, groing at infinity no faster than k^{-1}. The coefficients of these functions can be calculated using (2.6) and (2.3). Represent the functions $a_{ml}(k)$ explicitly in the form of the sum of partial fractions:

$$a_{ml}(k) = a^0_{ml} + \sum_{\alpha,\beta} \frac{a^{\alpha,\beta}_{ml}}{(k-k_\alpha)^\beta}.$$

Then

$$r_m = F_\pm[R_m] + \sum_{l=1}^{n} \sum_{\alpha,\beta} \sum_{\gamma=1}^{\beta} \frac{a^{\alpha,\beta}_{ml}}{(k-k_\alpha)^\gamma} \mathcal{F}\left(\frac{V_l}{(k-k_\alpha)^{\beta-\gamma}}, k_\alpha\right) \tag{3.3}$$

Note that the first term is rational function of k according to the property 3.

The relation (3.2) is a system of ordinary differential equations for $F_\pm[V_m]$ with rational coefficients and rational inhomegenuety.

Let the equation (3.1) $R_1(k) = \dots = R_n(k) \equiv 0$, i.e. the equation is homogenuous. This equation has n different solutions: $(V^s_1(k)\dots V^s_n(k))$, where the upper index s corresponds to the number of the solution and runs over the range $1\dots n$. It is not difficult to find these solutions. As the first solution $(V^1_1 \dots V^1_n)$ we take $\Delta V_1 \Gamma \Xi$fl$\exists V_n()$, and other $n-1$ solutions are different analytic continuations of the first solution.

The solution of (3.2) can be obtained by variation of the constants:

$$\mathrm{V}_\pm(k) = \hat{\mathrm{V}}(k)\left(\int_{k_0}^{k} \hat{\mathrm{V}}^{-1}(\tau)\mathrm{r}(\tau)d\tau + \mathrm{c}\right), \tag{3.4}$$

where r is the vector of functions r_m, $\mathrm{V}_\pm$ is the vector of functions $F_\pm[V_m]$, c is the vector of the constants of integration,

$$\hat{\mathrm{V}} = \begin{pmatrix} V^1_1(k) & \dots & V^n_1(k) \\ \dots & \dots & \dots \\ V^1_n(k) & \dots & V^n_n(k) \end{pmatrix},$$

k_0 ia almost arbitry point, such that k_0 is not a singular point of the equation (3.1), and besides, k_0 belongs to the upper half-plane if $F_+[V]$ is seeked and to lower one if $F_-[V]$ is seeked. It is obvious that the following vector equation is valid: $\mathcal{F}(\mathrm{V}, k_0) = -\hat{\mathrm{V}}\mathrm{c}$, where V is the vector of functions V_m, $m = 1\dots n$. Finally,

$$\mathrm{V}_\pm(k) = \hat{\mathrm{V}}(k)\int_{k_0}^{k} \hat{\mathrm{V}}^{-1}(\tau)\mathrm{r}(\tau)d\tau - \hat{\mathrm{V}}(k)\hat{\mathrm{V}}^{-1}(k_0)\mathcal{F}(\mathrm{V}, k_0). \tag{3.5}$$

Note that the infomogeneous system (3.1) for $V_1\dots V_n$ can be easily transformed into a homogeneous sytem by introducing a new function $V_{n+1} \equiv 1/(k-k_0)$ to the set $V_1\dots V_n$.

Finally, we have the representation (3.5) for $F_\pm[V]$ in quadratures instead of the inegral with parameters as in (1.4). The obtained representation has the following advantages. First, the integral in (3.5) can be simplier than the corresponding Cauchi's integral. Second, the representation (3.5) is valid on the whole Riemann surface of $F_\pm[V]$, while the represenation by Cauchi's integral is valid only either in the upper or in the lower half-plane and it requires analytic continuation. Third, numerical calculation of $F_\pm[V]$ using (3.5) is faster than that with Cauchi's integral. Namely, we need the values of $F_\pm[V]$ at N points of the real axis, the Cauchi's integral calculation requires $\sim N^2$ operations and our representation requires only $\sim N$ operations.

4 Examples

Here we considet some typical cases of differential equations for $F_\pm[V]$. The examples are taken from the problems of diffraction theory.

1. Consider the function

$$V(k) = \frac{e^{\mathrm{i}dk}}{\sqrt{(k^2-k_1^2)(k^2-k_2^2)\dots(k^2-k_h^2)}}. \tag{4.1}$$

We imply that $\mathrm{Im}[k_m] > 0$ for all $m = 1\dots h$. Such functions emerge in 2D problems of diffraction by a halfspace, separating different media. If the elastic media are studied the number of different wavenumbers h can be as much as 6. The presence of the exponential factor indicates the presence of the screens of finite dimensions.

The function obeys the homogeneous equation of order 1:

$$V' = KV, \qquad K(k) = \mathrm{i}d - \frac{k}{k^2-k_1^2} - \frac{k}{k^2-k_2^2}\dots - \frac{k}{k^2-k_h^2} \tag{4.2}$$

Let us find the function $V_+ = F_+[V]$. According to the theory developed above, it obeys the equation

$$V_+' = KV + r, \qquad r(k) = -\frac{1}{2}\sum_{m=1}^{h}\left(\frac{\mathcal{F}(V,k_m)}{k-k_m} + \frac{\mathcal{F}(V,-k_m)}{k+k_m}\right). \tag{4.3}$$

The general solution of this equation is given by

$$V_+(k) = V(k)\int^k \frac{r(\tau)}{V(\tau)}d\tau. \tag{4.4}$$

The lower limit of integration must be chosen such that the function V_+ is regular at k_1. For this one can simply choose the lower limit to be equal to k_1. One can prove that the function V_+ defined this way is also regular at all other k_m, $m = 2\dots h$.

Thus, the representation for $V_+(k)$ is obtained, For $d = 0$ and $h = 1$ this representation coincides with known formula [1]:

$$V_+(k) = \frac{\arccos(k/k_1)}{\pi\sqrt{k^2-k_1^2}}.$$

For $d = 0$ and $h = 2$ the function V_+ is represented in ellyptic functions. Such representation is refered in [1] (the author is grateful to Dr.V.D.Lukjanov for this reference).

2. Consider the function

$$V(k) = \frac{1}{(\sqrt{k^2 - k_1^2} - k_2)^{1/2}}, \tag{4.5}$$

where k_1, k_2 are the constants, $\mathrm{Im}[k_1, k_2] > 0$. Such functions emerge in 3D diffraction problems. Let the function $F_+[V]$ is seeked.

Let be $V_1(k) = V(k)$, $V_2(k) = V(k)/\sqrt{k^2 - k_1^2}$. We have the system

$$V_1' = -\frac{k}{2(k^2 - k_1^2 - k_2^2)} V_1 - \frac{k k_2}{2(k^2 - k_1^2 - k_2^2)} V_2, \tag{4.6}$$

$$V_2' = -\frac{k k_2}{2(k^2 - k_1^2)(k^2 - k_1^2 - k_2^2)} V_1 + \frac{k(2k_2^2 - 3k^2 + 3k_1^2)}{2(k^2 - k_1^2)(k^2 - k_1^2 - k_2^2)} V_2. \tag{4.7}$$

The coefficients of the system have simple poles at $k = \pm k_1$ and $k = \pm\sqrt{k_1^2 + k_2^2} = \pm k_3$.

According to the theory developed above, we have the system of equations for $F_+[V_1]$ and $F_+[V_2]$:

$$(F_+[V_1])' = -\frac{k F_+[V_1]}{2(k^2 - k_1^2 - k_2^2)} - \frac{k k_2 F_+[V_2]}{2(k^2 - k_1^2 - k_2^2)} + r_1, \tag{4.8}$$

$$(F_+[V_2])' = -\frac{k k_2 F_+[V_1]}{2(k^2 - k_1^2)(k^2 - k_1^2 - k_2^2)} + \frac{k(2k_2^2 - 3k^2 + 3k_1^2) F_+[V_2]}{2(k^2 - k_1^2)(k^2 - k_1^2 - k_2^2)} + r_2, \tag{4.9}$$

where

$$r_1(k) = -\frac{\mathcal{F}(V_1, k_3) + k_2 \mathcal{F}(V_2, k_3)}{4(k - k_3)} - \frac{\mathcal{F}(V_1, -k_3) + k_2 \mathcal{F}(V_2, -k_3)}{4(k + k_3)} \tag{4.10}$$

$$\begin{aligned} r_2(k) = {} & \frac{\mathcal{F}(V_1, k_1) - 2k_2 \mathcal{F}(V_2, k_1)}{4k_2(k - k_1)} + \frac{\mathcal{F}(V_1, -k_1) - 2k_2 \mathcal{F}(V_2, -k_1)}{4k_2(k + k_1)} - \\ & -\frac{\mathcal{F}(V_1, k_3) + k_2 \mathcal{F}(V_2, k_3)}{4k_2(k - k_3)} - \frac{\mathcal{F}(V_1, -k_3) + k_2 \mathcal{F}(V_2, -k_3)}{4k_2(k + k_3)}. \end{aligned} \tag{4.11}$$

Thus, the equation of the form (3.2) is constructed. We need to find the matrix $\hat{\mathrm{V}}$ consisting of different solutions of (4.6), (4.7) to be able to apply the formula (3.5). In our case this matrix can be found easily:

$$\hat{\mathrm{V}} = \begin{pmatrix} \dfrac{1}{(\sqrt{k^2 - k_1^2} - k_2)^{1/2}}; & \dfrac{1}{(\sqrt{k^2 - k_1^2} + k_2)^{1/2}} \\ \dfrac{1}{\sqrt{k^2 - k_1^2}(\sqrt{k^2 - k_1^2} - k_2)^{1/2}}; & -\dfrac{1}{\sqrt{k^2 - k_1^2}(\sqrt{k^2 - k_1^2} + k_2)^{1/2}} \end{pmatrix}.$$

Using (3.5), we conclude that $F_\pm[V]$ can be expressed in Abelian integrals. This conclusion is valid for any algebraic function V.

3. Consider the function

$$V(k) = \exp\{\mathrm{i}k x_0 + \mathrm{i}\sqrt{k_0^2 - k^2} y_0\}, \tag{4.12}$$

where x_0 and y_0 are real constants, $y_0 > 0$; $\mathrm{Im}[k_0] > 0$, the branch of the square root is chosen such way that the function $V(k)$ decays as $k \to \pm\infty$ along the real axis. Such functions emerge in the problems of diffraction of cylindrical waves.

Let be $V_1(k) = V(k)$, $V_2(k) = \sqrt{k_0^2 - k^2}V(k)$. The functions V_1, V_2 obey the equations

$$V_1' = \mathrm{i}x_0 V_1 + \frac{\mathrm{i}ky_0}{k^2 - k_0^2}V_2, \tag{4.13}$$

$$V_2' = -\mathrm{i}y_0 V_1 + \left(-\mathrm{i}x_0 + \frac{k}{k^2 - k_0^2}\right)V_2. \tag{4.14}$$

The system of infomogeneous equations can be constructed for $F_\pm[V_{1,2}]$. The inhomogeneous part of it is represented by the functions

$$r_1(k) = \frac{\mathrm{i}y_0\mathcal{F}(V_2, k_0)}{2(k - k_0)} + \frac{\mathrm{i}y_0\mathcal{F}(V_2, -k_0)}{2(k + k_0)},$$

$$r_2(k) = \frac{\mathcal{F}(V_2, k_0)}{2(k - k_0)} + \frac{\mathcal{F}(V_2, -k_0)}{2(k + k_0)}.$$

The matrix $\hat{\mathrm{V}}$ has the form

$$\hat{\mathrm{V}} = \begin{pmatrix} \exp\{\mathrm{i}kx_0 + \mathrm{i}\sqrt{k_0^2 - k^2}y_0\}; & \exp\{\mathrm{i}kx_0 - \mathrm{i}\sqrt{k_0^2 - k^2}y_0\} \\ \sqrt{k_0^2 - k^2}\exp\{\mathrm{i}kx_0 + \mathrm{i}\sqrt{k_0^2 - k^2}y_0\}; & -\sqrt{k_0^2 - k^2}\exp\{\mathrm{i}kx_0 - \mathrm{i}\sqrt{k_0^2 - k^2}y_0\} \end{pmatrix}.$$

Now the formula (3.5) can be applied.

Note that in the second coloumn of $\hat{\mathrm{V}}$ there are exponentially groing functions. However, one can still aply the method, because it is required only that the functions in the first coloumn are decaying.

5 Conclusion remarks

In the current paper we constructed the representation for the functions of the form $F_\pm[V]$, where the function $V(k)$ is a solution of a linear homogeneous differential equation with rational coefficients. For example, this class involves the functions $A(k)\exp\{B(k)\}$, where A and B are algebraic functions. We show that if $B(k) \equiv 0$, then the result can be represented in Abelian integrals.

The current paper shows the links between the results obtained in [4], [5] and the classical Wiener-Hopf method. In the first of these works it was shown that the solution of the problem of a plane wave diffraction on an ideal strip is a solution of a differential equation with rational coefficients. Note that Wiener-Hoph method cannot be applied to this problem. In the second paper the diffraction series was constructed for the problem diffraction by a strip, i.e. the process of diffraction was treated as the sequaence of the acts of diffraction by the edges of the strip. Both papers utilised the theorey of ordinary differential equations. Some sophisticated properties of Fuchsian equations were found to be useful when the dependence of the solution on the paprameter was studied. Note that earlier ordinary differential equations were used for solving the problem of diffraction by a strip in [2] and [3]. It was the purpose of the current paper to show that the solution of the diffraction problem obtained by classic Wiener-Hopf method obeys an ordinary differential equation, namely the equation (3.2).

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