Embedding formula for an electromagnetic diffraction problem

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Abstract

Embedding formulae is a powerful tool enabling one to reduce the dimension of the space of variables for a diffraction problem. Let the scatterer be finite, planar and perfectly conducting. The idea of the method is to substitute the initial problem of diffraction of a plane wave by finding an *edge Green's function*, i.e. to solve a problem with a source located near the edge of a scatterer. Embedding formula is an integral relation connecting the solution of the initial plane wave incidence problem with the edge Green's function. Earlier, the embedding formulae have been derived for acoustic and elasticity problems. Here we derive en embedding formula for an electromagnetic problem.

1 Introduction

Embedding formulae belong to a rather new type of the relations in diffraction theory. For the first time (up to our knowledge) the embedding formula was introduced by M. H. Williams [1]. The idea was the following. The 2D scalar problem of scattering of a plane wave by a thin strip has been considered. The main objective of the research was to find the diffraction coefficient $f(\varphi, \varphi_0)$ depending on the angle of incidence and the angle of scattering. M. H. Williams introduced a set of auxiliary functions, who were the diffraction coefficients calculated for the case of a fixed incidence angle φ_0, namely for the grazing incidence $\varphi_0 = \pm\pi/2$. Since the incidence angle was fixed, the auxiliary functions depended only on a *single* variable. By manipulation with the integral equation M. H. Williams managed to express the diffraction coefficient $f(\varphi, \varphi_0)$ in terms of the auxiliary functions $f_{1,2}(\varphi)$ in rather simple manner. Since the diffraction coefficient depends on two variables and each of the auxiliary functions depends on a single variable, we can say that diffraction coefficient was factorized in some sense.

For the next time, the embedding formula appeared in the paper by P. A. Martin and G. R. Wickham [2]. They studied the problem of of a plane wave scattering by a penny-shaped flat crack in a bulk of a solid. As the result of tedious calculations, the authors obtained a formula expressing the solution for an arbitrary incidence angle in terms of the solutions related to the grazing incidence.

After this, the embedding formulae were forgotten for a long time. Recently, the embedding formulae have been revived Biggs et. al. [3, 4, 5]. They obtained and checked the embedding formulae for the problems of scattering by several thin strips, by strips of finite thickness and by the walls of a perforated duct.

In all cases the way to derive the embedding formulae and the form of the formulae themselves remained rather complicated. As the result, such a bright result of diffraction theory remained almost unknown to most of the specialists.

The author of this paper in collaboration with Dr. R. V. Craster developed an easier way to derive the embedding formula for some diffraction problems [6, 7]. Earlier R. V. Craster applied the embedding ideas to solving the ordinary differential equations of Heun's type [8].

In our approach we choose the auxiliary functions as follows. Instead of the grazing incident plane wave we use a point source located close to one of the edges of the scatterer. The solution corresponding to this source is called an *edge Green's function*. Formally, the process of constructing the edge Green's function is described by a limiting procedure, since one cannot place a source directly at the edge. However, the approach based on the edge Green's functions seems to be simple and physically transparent.

The purpose of the current work is to derive the embedding formula for the case of an electromagnetic wave diffraction by a scatterer containing the edges. For this we use our standard approach and modify it according to the vector nature of the electromagnetic field.

2 Problem formulation

We study the problem of diffraction of a plane electromagnetic wave by an ideal (conducting) plane scatterer S of zero thickness located in the plane (x, y). The edge of the scatterer is a curve Γ, which is smooth enough. A coordinate l is defined on Γ; here l is the length along the curve counted from some starting point.

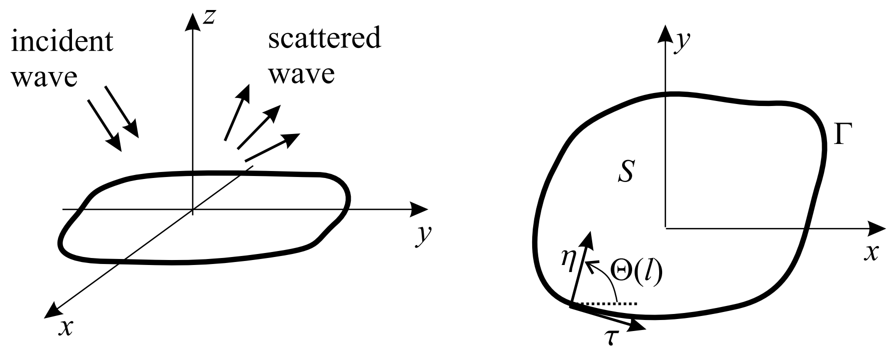

Figure 1: Geometry of the problem

At each point of Γ construct a unit internal normal vector η in the plane (x, y). The angle between this vector and the x-axis will be denoted as $\Theta(l)$ (see Figure 1). Introduce the local cylindrical coordinates near each point of the edge. One of these coordinates is l, two other ones are ρ and α (see Figure 2). For the correctness we can consider these coordinates as curvilinear ones making the edge be described by the relation $\rho = 0$.

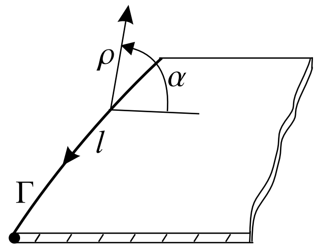

Figure 2: Local cylindrical coordinates near the edge of the scatterer

The Maxwell equations are assumed to be valid in the 3D space for the vectors of electric and magnetic fields:

$$\begin{aligned} \nabla \times \mathbf{E} &= \mathrm{i}k_0 \mathbf{H}, & (1) \\ \nabla \times \mathbf{H} &= -\mathrm{i}k_0 \mathbf{E}, & (2) \\ \nabla \cdot \mathbf{E} &= 0, & (3) \\ \nabla \cdot \mathbf{H} &= 0, & (4) \end{aligned}$$

where $k_0 = \Omega / c_0$, Ω is the circular frequency (the time dependence has the form of $e^{-\mathrm{i}\Omega t}$ and it is omitted henceforth), c_0 is the speed of light.

The boundary conditions on the surfaces of the scatterer can be written using the unit vector $\mathbf{n}$ normal to the surface as

$$\mathbf{E} \times \mathbf{n} = 0, \tag{5}$$

i.e. the tangential component of $\mathbf{E}$ should be equal to zero.

A correct formulation of the diffraction problem includes the Meixner's edge condition. The edge condition denotes the fact that the total energy concentrated near a finite fragment of the edge is finite. This means that the combination $E^2 + H^2$ grows slower than ρ^δ for $\rho \to 0$, where $\delta > -2$.

Formulate the radiation conditions for our problem. The total field should be represented as a sum of incident wave and the outgoing spherical wave. The form of the spherical wave is given below.

Introduce the notation for the directions of propagation and for the polarizations of the plane electromagnetic waves as follows. Denote the directions of incidence and scattering by the points on a unit sphere (or the unit vectors, which is the same) ω_0 and ω (see Figure 3). The points are identified by their spherical coordinates $\omega(\theta, \varphi)$, $\omega_0(\theta_0, \varphi_0)$.

At each point ω of the sphere define a tangential plane. Let $\mathbf{V}(\omega)$ be the 2D vector space in this plane. Indicate the amplitude and polarization of the plane electromagnetic wave by the vector $\mathbf{E}$ belonging to $\mathbf{V}(\omega)$. The vector $\mathbf{H}$ for the outgoing wave can be expressed by the well-known identity

$$\mathbf{H} = \omega \times \mathbf{E}. \tag{6}$$

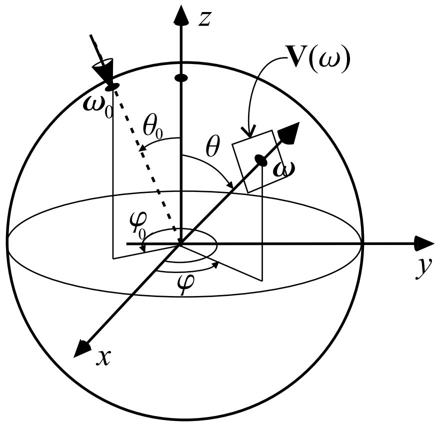

Figure 3: Directions of incidence and scattering

The incident plane wave in these notations have the form:

$$\mathbf{E} = \mathbf{E}_0 \exp\{-\mathrm{i}k_0(x \sin\theta_0 \cos\varphi_0 + y \sin\theta_0 \sin\varphi_0 + z\cos\theta_0)\}, \tag{7}$$

$\mathbf{E}_0$ is the amplitude.

Our purpose here will be to establish the embedding relations for the tensor function $\mathbf{f}(\omega, \omega_0)$ having its values in $\mathbf{V}(\omega) \otimes \mathbf{V}(\omega_0)$ and describing the spherical part of the scattered field:

$$\mathbf{E}_{\mathrm{sc}}(r, \omega) = \frac{2\pi}{k_0 r} e^{\mathrm{i}k_0 r} \mathbf{f}(\omega, \omega_0)\mathbf{E}_0 + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}) \qquad \text{for} \qquad r \to \infty, \tag{8}$$

where r is the distance from the origin.

To make the last formula clear, introduce some orthonormal coordinates in the spaces $\mathbf{V}(\omega)$ and $\mathbf{V}(\omega_0)$. Let E^i_{sc} — be the components of the polarization of the scattered wave projected onto $\mathbf{V}(\omega)$, and let E^j_0 be the components of the vector of polarization of the incident wave. We are looking for such a tensor f^{ij}, that

$$E^i_{\mathrm{sc}}(r, \omega) = \frac{2\pi}{k_0 r} e^{\mathrm{i}k_0 r} \sum_j f^{ij}(\omega, \omega_0) E^j_0 + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}). \tag{9}$$

2.1 Overview of the procedure of deriving the embedding formula

Let us give a sketch of the procedure leading to the embedding formula. On the *first* (auxiliary) step we find the form of the edge asymptotics of our solution obeying the edge (Meixner's) conditions. These asymptotics can be obtained from the solution of the classical Sommerfeld half-plane problem. Also we define *the edge Green's functions* by placing point sources near the edge of the scatterer. Obviously, the edge Green's function violates the edge conditions due to the presence of the sources. However, the degree of growth (the exponent of ρ^{-1}) is only by 1 higher than that of the solution obeying the edge conditions. We shall say that such a function is *slightly oversingular*.

On the *second* step we apply to the total field (both electric and magnetic part of it) the differential operator

$$P_x = \frac{\partial}{\partial x} + \mathrm{i}k_0 \sin\theta_0 \cos\varphi_0. \tag{10}$$

Note that this operator kills the incident wave. The result $P_x[\mathbf{E}, \mathbf{H}]$ will be interpreted as a new electromagnetic field. This field obeys Maxwell equations and the radiation condition. Moreover, a simple analysis shows that this field is slightly oversingular.

On the *third* step we prove the Lemma stating that any slightly oversingular solution of the Maxwell equation obeying the radiation conditions can be represented as a linear combination of the edge Green's functions. The sources are unknown but they are connected with the main term of edge asymptotics of the field. Therefore $P_z[\mathbf{E}, \mathbf{H}]$ can be represented as a convolution-form integral of the edge Green's function over the coordinate of the source with the unknown density of the source. This representation is a *weak form* of the embedding formula.

On the *fourth* step we express the density of the source through the edge Green's function. The reciprocity theorem is used for this.

3 Edge asymptotics of the field

The edge asymptotics of the field can be found from the solution of the Sommerfeld problem. Namely, the singular part of the field can be split into two modes: in one of them the electric component is parallel to the edge, and in another the magnetic component is parallel to the edge:

$$\left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right\} = \left\{ \begin{array}{c} \mathcal{E}_E \\ \mathcal{H}_E \end{array} \right\} + \left\{ \begin{array}{c} \mathcal{E}_H \\ \mathcal{H}_H \end{array} \right\}, \tag{11}$$

where the fields are determined by their asymptotics as $\rho \to 0$

$$\mathcal{E}_E = \tau \frac{2C_E(x)}{\sqrt{\pi}} \rho^{1/2} \sin\frac{\alpha}{2} + O(\rho), \qquad \mathcal{H}_E = \frac{\nabla \times \mathcal{E}_E}{\mathrm{i}k_0}, \tag{12}$$

$$\mathcal{H}_H = \tau \frac{2C_H(x)}{\sqrt{\pi}} \rho^{1/2} \cos\frac{\alpha}{2} + O(\rho), \qquad \mathcal{E}_H = -\frac{\nabla \times \mathcal{H}_H}{\mathrm{i}k_0}. \tag{13}$$

Here τ is the unit vectors tangential to Γ and directed in the positive l direction. The functions $C_E(l)$ and $C_H(l)$ are unknown coefficients playing an important role below.

4 Edge Green's functions

Define a pair of edge Green's functions $\mathbf{G}_E(x, y, z; \xi)$ and $\mathbf{G}_H(x, y, z; \xi)$:

$$\mathbf{G}_E = \left\{ \begin{array}{c} \mathbf{E}_E \\ \mathbf{H}_E \end{array} \right\}, \qquad \mathbf{G}_H = \left\{ \begin{array}{c} \mathbf{E}_H \\ \mathbf{H}_H \end{array} \right\} \tag{14}$$

Define the field $\mathbf{G}_E$ as a result of the following limiting procedure. Denote the approximations to $\mathbf{E}_E$ and $\mathbf{H}_E$ by $\hat{\mathbf{E}}_E$ and $\hat{\mathbf{H}}_E$. Consider the inhomogeneous Maxwell equations for $\hat{\mathbf{E}}_E$ and $\hat{\mathbf{H}}_E$:

$$\nabla \times \hat{\mathbf{E}}_E - \mathrm{i}k_0\hat{\mathbf{H}}_E = 0, \tag{15}$$

$$\nabla \times \hat{\mathbf{H}}_E + \mathrm{i}k_0\hat{\mathbf{E}}_E = \frac{4\pi}{c_0}\hat{\mathbf{j}}, \tag{16}$$

$$\nabla \cdot \hat{\mathbf{E}}_E = 0, \tag{17}$$

$$\nabla \cdot \hat{\mathbf{H}}_E = 0, \tag{18}$$

where

$$\hat{\mathbf{j}} = \tau\frac{\pi^{1/2}}{\epsilon^{3/2}}\delta(l-\xi)\delta(\alpha-\pi)\delta(\rho-\epsilon), \tag{19}$$

i.e. $\hat{\mathbf{j}}$ is the element of current located at the distance ϵ from the point $l = \xi$ of the edge (see Figure 4). The amplitude of the current is chosen to be such that there exists a finite non-zero limit of the field as $\epsilon \to 0$. Equations (15)–(18) are considered with the edge conditions, radiation conditions and the boundary conditions.

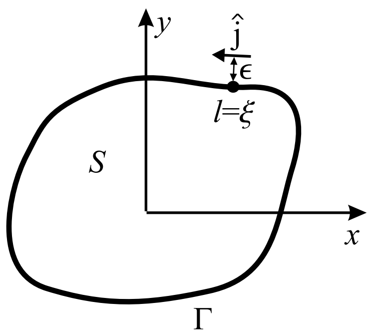

Figure 4: Location of the source for the edge Green's function

We define $\mathbf{E}_E$ and $\mathbf{H}_E$ as follows:

$$\mathbf{E}_E = \lim_{\epsilon\to 0}\hat{\mathbf{E}}_E, \qquad \mathbf{H}_E = \lim_{\epsilon\to 0}\hat{\mathbf{H}}_E. \tag{20}$$

Consider the edge asymptotics of the edge Green function $\mathbf{G}_E$. Due to the source located near the edge, the field has a complicated structure near the edge along the l coordinate. Avoid considering this structure by constructing the convolution-type integral

$$\mathbf{I}(x,y,z) = \int_{-\infty}^{\infty} h(x)\mathbf{G}_E(x,y,z;\xi)d\xi,$$

where $h(\xi)$ is an arbitrary smooth enough density function, which is non-zero on a small segment of the edge.

Consider the vicinity of the point $l = \xi$ of the edge. Return to the limiting procedure and find the asymptotics of the solution for some small but finite ϵ in a small area near this point.

Locally the inhomogeneous Maxwell equations can be approximately reduced to the 2D inhomogeneous Poisson equation

$$\Delta_\perp E_\tau = -h(\xi)\frac{4\pi}{c_0}\sqrt{\frac{\pi}{\epsilon}}\delta(\alpha-\pi)\delta(\rho-\epsilon), \tag{21}$$

where $\Delta_\perp$ is the Laplacian in the plane $l=\xi$.

Equation (21) can be solved using the conformal mappings method. Taking into account the boundary conditions we find:

$$E_\tau \approx -\frac{2h(\xi)}{c_0}\sqrt{\frac{\pi}{\epsilon}}\mathrm{Re}\left[\log(\sqrt{\kappa}-\sqrt{\epsilon})-\log(\sqrt{\kappa}+\sqrt{\epsilon})\right], \tag{22}$$

where $\kappa = \rho(\mathrm{i}\sin\alpha - \cos\alpha)$.

Calculating the outer asymptotics of this solution as $\epsilon \to 0$, obtain the following representation for the electric field in $\mathbf{I}$:

$$\mathbf{E} = \tau\frac{4\sqrt{\pi}h(l)}{c_0}\rho^{-1/2}\sin\frac{\alpha}{2} + O(\rho) \qquad \text{for} \qquad \rho \to 0. \tag{23}$$

The representation for the magnetic field can be obtained by using the equation (15).

Analogously, the components $\mathbf{E}_H$ and $\mathbf{H}_H$ of $\mathbf{G}_H$ are approximated by the functions $\hat{\mathbf{E}}_H$ and $\hat{\mathbf{H}}_H$, for which the inhomogeneous Maxwell equations are written:

$$\begin{aligned}
\nabla\times\hat{\mathbf{E}}_H - \mathrm{i}k_0\hat{\mathbf{H}}_H &= \frac{4\pi}{c_0}\hat{\mathbf{k}}, && (24)\\
\nabla\times\hat{\mathbf{H}}_H + \mathrm{i}k_0\hat{\mathbf{E}}_H &= 0, && (25)\\
\nabla\cdot\mathbf{E}_H &= 0, && (26)\\
\nabla\cdot\mathbf{H}_H &= 0, && (27)
\end{aligned}$$

where

$$\hat{\mathbf{k}} = \tau\frac{\pi^{1/2}}{\epsilon^{3/2}}\delta(l-\xi)\delta(\alpha-\pi)\delta(\rho-\epsilon) \tag{28}$$

is the unphysical "magnetic current". Taking the limit $\epsilon \to 0$ we obtain the components $\mathbf{E}_H$ and $\mathbf{H}_H$. The $\mathbf{H}$-component of the integral of the form

$$\mathbf{I}(x,y,z) = \int_{-\infty}^{\infty} h(\xi)\mathbf{G}_H(x,y,z;\xi)d\xi,$$

has the edge asymptotics as $\rho \to 0$ looking like

$$\mathbf{H} = -\tau\frac{4\sqrt{\pi}h(l)}{c_0}\rho^{-1/2}\cos\frac{\alpha}{2} + O(\rho). \tag{29}$$

The $\mathbf{E}$-component is described by equation (25).

5 Application of the operator P_x to the field

Apply the operator P_x defined according to (10) to the total field. The result will be treated as a new electromagnetic field (obviously, $P_x[\mathbf{E}]$ is its electrical component, and $P_x[\mathbf{H}]$ is the magnetic one). This interpretation is possible because Helmholtz equations are invariant with respect to translations, i.e. they admit differentiations with respect to the coordinates.

Obviously, the new field obeys the same boundary conditions as the old one. Also it obeys the radiation condition, i.e. it contains no components coming from infinity. It could be necessarily equal to zero due to the theorem of uniqueness, however the differentiation with respect to x makes the edge singularity stronger, i.e. the edge (Meixner) conditions become broken. Physically it means that the new field is generated by the sources located near the edge.

The main term of the asymptotics as $\rho \to 0$ can be written as follows:

$$P_x[\mathcal{E}_E] = -\tau \frac{C_E(l)\cos\Theta(l)}{\sqrt{\pi}} \rho^{-1/2} \sin\frac{\alpha}{2} + O(\rho), \qquad P_x[\mathcal{H}_E] = \frac{\nabla \times P_x[\mathcal{E}_E]}{\mathrm{i}k_0}, \tag{30}$$

$$P_x[\mathcal{H}_H] = \tau \frac{C_H(l)\cos\Theta(l)}{\sqrt{\pi}} \rho^{-1/2} \cos\frac{\alpha}{2} + O(\rho), \qquad P_x[\mathcal{E}_H] = -\frac{\nabla \times P_x[\mathcal{H}_H]}{\mathrm{i}k_0}. \tag{31}$$

The main terms of these asymptotics violate the edge conditions. The coeficients at these terms are proportional to the strengths of the linear source of electric and magnetic type located near the edge.

6 Weak form of the embedding formula

Let the theorem of uniqueness is valid for the chosen scatterer, i.e. the following statement is true: *if the field* $(\mathbf{E}, \mathbf{H})$ *obeys homogeneous Maxwell equations, radiation conditions, boundary conditions on the scatterer and Meixner edge conditions, then it is identically equal to zero.* We are not going to prove this theorem, however, we are sure that it is true at least in the simplest cases (S is compact, its boundary is smooth enough). In worse case if Ω belongs to the discrete spectrum of the problem and has finite degeneration, the method can be modified.

Now we are ready to formulate the following Lemma:

Lemma 1 *Let*

$$\mathbf{F} = \left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right\}$$

be a solution of the homogeneous Maxwell equations obeying the radiation conditions and the ideal conducting boundary conditions on the surfaces of the scatterer introduced above, but violating the Meixner edge conditions. Let the edge behaviour of the solution is given by the asymptotics as $\rho \to 0$*:*

$$\left\{ \begin{array}{c} E_\tau \\ H_\tau \end{array} \right\} = \frac{1}{\sqrt{\pi}} \rho^{-1/2} \left\{ \begin{array}{c} D_E(l)\sin(\alpha/2) \\ D_H(l)\cos(\alpha/2) \end{array} \right\} + \text{Meixner terms.} \tag{32}$$

Then
$$\mathbf{F}(x,y,z) = \frac{c_0}{4\pi}\int_\Gamma (D_E(l)\mathbf{G}_E(x,y,z;l) - D_H(l)\mathbf{G}_H(x,y,z;l))\,dl. \tag{33}$$

To prove the Lemma one can subtract the right-hand side of (33) from $\mathbf{F}$. The result obeys all the conditions of the theorem of uniqueness, therefore it is zero.

Note that the asymptotics (30), (31) has the form obeying the conditions of the Lemma. Therefore if $\mathbf{U}$ is the solution of the initial plane-wave incidence problem, then

$$P_z[\mathbf{U}(x,y,z)] = -\frac{c_0}{4\pi}\int_\Gamma \cos\Theta(l)\,\{C_E(l)\mathbf{G}_E(x,y,z;l) + C_H(l)\mathbf{G}_H(x,y,z;l)\}\,dl. \tag{34}$$

where $C_{E,H}$ are the coefficients introduced in (12), (13). We can rewrite this expression in a slightly different way by introducing the directivities of the edge Green's functions. Namely define $\mathbf{f}_{E,H}$ by the following asymptotic relation as $r \to \infty$:

$$\mathbf{E}_{E,H}(r,\omega;l) = \frac{2\pi}{k_0 r}e^{\mathrm{i}k_0 r}\mathbf{f}_{E,H}(\omega;l) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}), \tag{35}$$

where $\mathbf{E}^{1,2}_{E,H}$ is the electrical vector of $\mathbf{G}_{E,H}$. Note that the operator P_x acts on the directivity as follows:

$$\mathbf{f}(\omega,\omega_0) \xrightarrow{P_x} \mathrm{i}k_0(\sin\theta\cos\varphi + \sin\theta_0\cos\varphi_0)\mathbf{f}(\omega,\omega_0). \tag{36}$$

Substituting the fields far from the source in (34) by their directivities we obtain

$$\begin{aligned}&\mathrm{i}k_0(\sin\theta\cos\varphi + \sin\theta_0\cos\varphi_0)\mathbf{f}(\omega,\omega_0)\mathbf{E}_0 = \\ &-\frac{c_0}{4\pi}\int_\Gamma \cos\Theta(l)\,\{C_E(l)\mathbf{f}_E(\omega;l) + C_H(l)\mathbf{f}_H(\omega;l)\}\,dl,\end{aligned} \tag{37}$$

We call the last expression the weak form of the embedding formula because it contains unknown coefficients C_E and C_H. In the next section we shall express these coefficients in terms of the edge Green's functions. Here we just note that these coefficients depend on the direction of incidence ω_0 and the incident field polarization.

7 Application of the reciprocity principle and obtaining the "strong" embedding formula

We formulate the reciprocity principle as follows. Let $(\mathbf{E}_1,\mathbf{H}_1)$ and $(\mathbf{E}_2,\mathbf{H}_2)$ be two fields obeying the inhomogeneous Maxwell equations

$$\nabla\times\mathbf{E}_{1,2} - \mathrm{i}k_0\mathbf{H}_{1,2} = \frac{4\pi}{c_0}\mathbf{k}_{1,2}, \tag{38}$$
$$\nabla\times\mathbf{H}_{1,2} + \mathrm{i}k_0\mathbf{E}_{1,2} = \frac{4\pi}{c_0}\mathbf{j}_{1,2}, \tag{39}$$
$$\nabla\cdot\mathbf{E}_{1,2} = 0, \tag{40}$$
$$\nabla\cdot\mathbf{H}_{1,2} = 0. \tag{41}$$

We remind that $\mathbf{k}$ denotes the unphysical magnetic currents. Both fields satisfy boundary, edge and radiation conditions common for both fields. Then

$$\iiint [(\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1) + (\mathbf{k}_1 \cdot \mathbf{H}_2 - \mathbf{k}_2 \cdot \mathbf{H}_1)]\, dx\, dy\, dz = 0 \tag{42}$$

where the integral is taken over the whole space (or its part containing the sources).

The relation (42) is the reciprocity principle. The procedure of its derivation is rather standard. Equation (38) is multiplied by $\mathbf{H}_{2,1}$, equation (39) is multiplied by $\mathbf{E}_{2,1}$, then the equation are summed and the divergent part is taken out.

Let us apply the relation (42) to transform the embedding formula (37). Consider, say, the directivity $\mathbf{f}_E(\omega_0; l)$ of the edge Green's function $\mathbf{G}_E$.

By definition, we can find this directivity as follows. Take the source

$$\mathbf{j}_1 = \frac{\pi^{1/2}}{\epsilon^{3/2}} \delta(l - \xi)\delta(\rho - \epsilon)\delta(\alpha - \pi)$$

near the edge of the scatterer and calculate the field $\mathbf{E}_1$ at the point (R, ω_0) produced by this source (R is large). The directivity will be equal to the limit

$$\mathbf{f}_E(\omega_0; l) = \lim_{\epsilon \to 0, R \to \infty} \frac{k_0 R}{2\pi} e^{-\mathrm{i} k_0 R} \mathbf{E}_1(R, \omega_0). \tag{43}$$

Now consider the field $\mathbf{E}_2$ produced by scattering of the incident plane wave (7). Note that this plane wave can be approximately replaced by a spherical wave produced by a point source located far enough. Namely, take the localized source with the amplitude

$$\mathbf{j}_2 = \frac{c_0 R}{\mathrm{i} k_0} e^{-\mathrm{i} k_0 R} \mathbf{E}_0 \tag{44}$$

located at the point (R, ω_0) with R large enough. The τ-component of the field $\mathbf{E}_2$ produced by this source at the point $(l = \xi, \rho = \epsilon, \alpha = \pi)$ is equal to

$$(\mathbf{E}_2)_\tau = \frac{2 C_E(\xi)}{\sqrt{\pi}} \epsilon^{1/2} + o(\epsilon^{1/2})$$

Applying formula (42) and taking the limits $\epsilon \to 0$, $R \to \infty$, we obtain the formula

$$C_E(l) = \frac{\pi c_0}{\mathrm{i} k_0^2} \mathbf{f}_E(\omega_0; l) \cdot \mathbf{E}_0.$$

Repeating the same procedure, we obtain a more general formula

$$C_{E,H} = \frac{\pi c_0}{\mathrm{i} k_0^2} \mathbf{f}_{E,H} \cdot \mathbf{E}_0. \tag{45}$$

So, the coefficients $C_{E,H}$ are now expressed in terms of the directivities of the edge Green's functions and the polarization of the incident field.

Substituting (45) into (37) we obtain the *strong form* of the embedding formula:

$$\mathbf{f}(\omega, \omega_0) = \frac{c_0^2}{4 k_0^3 (\sin\theta \cos\varphi + \sin\theta_0 \cos\varphi_0)} \times$$

$$\int_\Gamma \{\mathbf{f}_E(\omega;l)\otimes\mathbf{f}_E(\omega_0;l)+\mathbf{f}_H(\omega;l)\otimes\mathbf{f}_H(\omega_0;l)\}\,dl. \qquad (46)$$

Here we use the symbol $\otimes$ in the sense on Kronecker product, i.e. in the coordinate reprsentation

$$f^{ij}(\omega,\omega_0)=\frac{c_0^2}{4k_0^3(\sin\theta\cos\varphi+\sin\theta_0\cos\varphi_0)}\times$$

$$\int_\Gamma \left\{f_E^i(\omega;l)f_E^j(\omega_0;l)+f_H^i(\omega;l)f_H^j(\omega_0;l)\right\}dl, \qquad (47)$$

where $f^i_{E,H}$ are the components $\mathbf{f}_{E,H}$ in the cooresponding coordinates in $\mathbf{V}(\omega)$ and $\mathbf{V}(\omega_0)$.

The usage of the edge Green's function can be preferable from different point of views. First, the functions through which we expressed the diffraction coefficient, namely $\mathbf{f}_{E,H}(\omega;l)$ depend on three scalar variables θ, φ and l while the initial diffraction coefficient depend on four scalar variables θ, φ, θ_0 and φ_0. It means that numerical tabulation of the new functions gives some gain in memory and time. Second, $\mathbf{f}_{E,H}(\omega;l)$ are physically measured values. Such a measurement gives the information what area is responsible for the maximums of the diffraction coefficient.

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