Excitation of wave field in a triangle with impedance boundary

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Introduction

Here we consider the problem of excitation of wave field in equilateral triangular area with impedance boundary conditions. This area can be the cross-section of waveguider or resonator.

Such a problem with the ideal boundary (Dirichlet's or Newmann's ones) can be easily solved with the method of images [1]. Unfortunately, this method cannot be enhanced for impedance boundary conditions. Variables also cannot be separated in this case. Here we use functional equations of Maljuzhinetz's type [2] to solve this problem.

The paper is organized as follows.

In the *first section* we derive functional equation with the help of the second Green's formula. Fourier transform of the field on the boundary is the unknown function of this equation. Some obvious properties of geometrical symmetry are used to simplify the equation. The result is the equation that combines linearly the values of unknown function $\hat{u}(\varphi)$ at the points φ, $\varphi + 2\pi/3$ and $\varphi - 2\pi/3$ of arbitrary φ. Also some restrictions are posed on the unknown function. They are the conditions of analiticity and decrease. They follow from some well-known theorems concerning Fourier transform.

In the *second section* the method of images is applied to the functional equation. The result is a functional equation containing two values of unknown function instead of three.

In the *third section* the eigenvalue problem of considered area is solved. Eigenfunctions are calculated. We show that the eigenfunctions are combinations of plane waves.

In the *fourth section* the problem with inhomogenous boundary conditions is solved. The solution is presented in the form of an infinite sum over the zeros of a certain entire function.

In the *fifth section* we show the way of reconstruction of the wave field in the area using Fourier transform of the field on the boundary. An integral representation is constructed.

In the *sixth section* we study the case of a very long side of the triangle (in comparison with the wavelength) and a small dissipation in the media. We show that the solution in this case coinsides with known solution for an angle with impedance boundary.

1 Problem formulation and derivation of functional equation

Consider an equilateral triangle with the side L. This triangle and coordinates are shown at Fig.1. The sides of the triangle are numbered with indexes $0, 1, 2$ in positive (anticlockwize) direction beginning with the horisontal side. Introduce local linear coordinates l_0, l_1, l_2 along each side of the triangle as it is shown on the figure. It is obvious that each coordinate takes values from 0 to L. These coordinates can be arguments of functions defined on the sides of the triangle. We will use the notation $u(l_0)$ for the values of function u on the side 0 and notation $u(a)|_{l_0}$ for the function u at the point on side 0 where coordinate l_0 is equal to a

Let n be an internal normal vector to the boundary of triangle. Angle φ that is used for derivation of functional equation is shown at Fig.1.

Let Helmholtz equation be valid for function $u(x, y)$ inside the triangle:

$$\Delta u + k_0^2 u = 0 \tag{1}$$

Note that the stationary problem is considered and time dependence has the form of factor $e^{i\omega t}$. Hence the wave travelling along x axis in positive direction is $e^{-ik_0 x}$.

Boundary conditions have the impedance form

$$\frac{\partial u}{\partial n} - iuk_0 \sin\beta = \Phi, \tag{2}$$

where $\sin\beta$ is a parameter, and Φ are known functions that correspond to forces on the boundary. If the boundary conditions are homogenous ($\Phi = 0$) than the reflection coefficient for incident plane wave coming at the angle ψ with the boundary is

$$K_{ref} = \frac{\sin\psi - \sin\beta}{\sin\psi + \sin\beta}.$$

Therefore the surface is passive when

$$\mathrm{Re}[\sin\beta] \geq 0.$$

It is necessary to formulate Meixner's conditions at vertexes of the triangle. Let r be the distance from the vertex. Function u must have the asymptotic

$$u \sim U_0 + o(r^\delta), \quad \text{where} \quad \delta > 0.$$

Let us derive the functional equation. Apply the second Green's formula:

$$\int [u' \frac{\partial u}{\partial n} - u \frac{\partial u'}{\partial n}] d\Gamma = 0, \tag{3}$$

where u' is an arbitrary solution of equation (1) in the area and the integration is performed along the whole boundary.

Replace u' with a plane wave travelling at an angle φ with respect to x axis:

$$u' = e^{-ik_0(x\cos\varphi + y\sin\varphi)}.$$

Since the integration in (3) is performed along the boundary, it is useful to find the restriction of u' to the sides of triangle. It is easy to show that

$$\begin{aligned}
u'(l_0) &= e^{-ik_0 l_0 \cos\varphi}, \\
u'(l_1) &= e^{-ik_0 L\cos\varphi} e^{-ik_0 l_1 \cos(\varphi - 2\pi/3)}, \\
u'(l_2) &= e^{-ik_0 L\cos(\varphi - \pi/3)} e^{-ik_0 l_2 \cos(\varphi + 2\pi/3)}.
\end{aligned}$$

The normal derivatives of u' can be found in the same way:

$$\begin{aligned}
\frac{\partial u'(l_0)}{\partial n} &= -ik_0 \sin\varphi e^{-ik_0 l_0 \cos\varphi}, \\
\frac{\partial u'(l_1)}{\partial n} &= -ik_0 \sin(\varphi - 2\pi/3) e^{-ik_0 L \cos\varphi} e^{-ik_0 l_1 \cos(\varphi - 2\pi/3)}, \\
\frac{\partial u'(l_2)}{\partial n} &= -ik_0 \sin(\varphi + 2\pi/3) e^{-ik_0 L \cos(\varphi - \pi/3)} e^{-ik_0 l_2 \cos(\varphi + 2\pi/3)}.
\end{aligned}$$

Normal derivative of u can be calculated from

$$\frac{\partial u}{\partial n} = \Phi + iuk_0 \sin\beta.$$

The values obtained can be put into (3). We obtain

$$0 = \int_0^L [-iu(l_0)k_0(\sin\varphi + \sin\beta) - \Phi(l_0)] e^{-ik_0 l_0 \cos\varphi} dl_0 + \tag{4}$$

$$+ \int_0^L e^{-ik_0 L \cos\varphi} [-iu(l_1)k_0(\sin(\varphi - \frac{2\pi}{3}) + \sin\beta) - \Phi(l_1)] e^{-ik_0 l_1 \cos(\varphi - 2\pi/3)} dl_1 +$$

$$+ \int_0^L e^{-ik_0 L \cos(\varphi - \pi/3)} [-iu(l_2)k_0(\sin(\varphi + \frac{2\pi}{3}) + \sin\beta) - \Phi(l_2)] e^{-ik_0 l_2 \cos(\varphi + 2\pi/3)} dl_2.$$

Use the obvious symmetry of the problem with respect to the rotation about geometrical center of the triangle at angle $2\pi/3$. Split the initial problem into three independent problems which have properties

$$u_\nu(l)|_{l_1} = e^{2\pi i\nu/3} u_\nu(l)|_{l_0}, \qquad u_\nu(l)|_{l_2} = e^{-2\pi i\nu/3} u_\nu(l)|_{l_0},$$

$$\Phi_\nu(l)|_{l_1} = e^{2\pi i\nu/3} \Phi_\nu(l)|_{l_0}, \qquad \Phi_\nu(l)|_{l_2} = e^{-2\pi i\nu/3} \Phi_\nu(l)|_{l_0},$$

were ν takes values 0,1,-1. It is evident that each type of symmetry of u corresponds to the same type of symmetry of Φ. Thus, we make discrete Fourier transform with respect to the number of side. Formulas of direct and inverse transforms are

$$u_\nu(l) = \frac{1}{3}[u(l)|_{l_0} + e^{-2\pi i\nu/3} u(l)|_{l_1} + e^{-4\pi i\nu/3} u(l)|_{l_2}],$$

$$u(l)|_{l_j} = u_0(l) + e^{2\pi ij/3} u_1(l) + e^{4\pi ij/3} u_2(l).$$

The same formulas are valid for Φ.

Further we consider the problem that belongs to one of the types of symmetry and we do not write index ν.

Integrals in (4) have the form of Fourier integrals. Define Fourier transform as follows:

$$\begin{aligned}
\hat{u}(\varphi) &= \int_0^L u(l) e^{-ik_0 l \cos\varphi} dl \\
\hat{\Phi}(\varphi) &= \frac{i}{k_0} \int_0^L \Phi(l) e^{-ik_0 l \cos\varphi} dl.
\end{aligned} \tag{5}$$

Note that the combination of variables corresponding to the spatial frequency is $k_0 \cos\varphi$. Hence, the inverse transform has the form

$$u(l) = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} u(\varphi) e^{ik_0 l \cos\varphi} d\cos\varphi.$$

Using the notations introduced above, equation (4) can be rewritten in the form

$$
\begin{aligned}
&(\sin\varphi + \sin\beta)\hat{u}(\varphi) + \\
&+e^{-i(k_0 L\cos\varphi - 2\pi\nu/3)}(\sin(\varphi - \frac{2\pi}{3}) + \sin\beta)\hat{u}(\varphi - \frac{2\pi}{3}) + \\
&+e^{-i(k_0 L\cos(\varphi-\pi/3) + 2\pi\nu/3)}(\sin(\varphi + \frac{2\pi}{3}) + \sin\beta)\hat{u}(\varphi + \frac{2\pi}{3}) = \\
&= \hat{\Phi}(\varphi) + e^{-i(k_0 L\cos\varphi - 2\pi\nu/3)}\hat{\Phi}(\varphi - \frac{2\pi}{3}) + \\
&e^{-i(k_0 L\cos(\varphi-\pi/3) + 2\pi\nu/3)}\hat{\Phi}(\varphi + \frac{2\pi}{3}).
\end{aligned}
\tag{6}
$$

Note that $\hat{u}(\varphi)$ is unknown function of this equation. If we find this function for each ν we can reconstruct the field and its normal derivative at the boundary. After that we can find the field inside the area.

We must take into account some restrictions on the function $\hat{u}(\varphi)$, which follow from definition (5).

It is evident that $\hat{u}(\varphi)$ is an entire function of variable φ and it has the properties of symmetry and periodicity

$$
\hat{u}(-\varphi) = \hat{u}(\varphi), \qquad \hat{u}(\varphi + 2\pi) = \hat{u}(\varphi) \tag{7}
$$

Meixner's conditions at vertexes lead to restrictions on growth of unknown function

$$
\begin{aligned}
\hat{u}(\varphi) < D|\cos\varphi|^{-1}e^{-ik_0 L\cos\varphi}, \qquad \mathrm{Im}[\cos\varphi] > 0 \\
\hat{u}(\varphi) < D|\cos\varphi|^{-1}, \qquad \mathrm{Im}[\cos\varphi] < 0
\end{aligned}
\tag{8}
$$

We name equation (6) a *functional equation* because it contains values of unknown function taken at different values of arguments.

2 Application of the method of images to the functional equation

Equation (6) contains unknown values $\hat{u}(\varphi)$, $\hat{u}(\varphi - 2\pi/3)$, $\hat{u}(\varphi + 2\pi/3)$. Perform a formal substitution in equation (6) $\varphi \to -\varphi$ and use properties (7). Unknown values will be transformed as follows:

$$
\begin{aligned}
\hat{u}(\varphi) &\to \hat{u}(-\varphi) = \hat{u}(\varphi) \\
\hat{u}(\varphi - 2\pi/3) &\to \hat{u}(-\varphi - 2\pi/3) = \hat{u}(\varphi + 2\pi/3), \\
\hat{u}(\varphi + 2\pi/3) &\to \hat{u}(-\varphi + 2\pi/3) = \hat{u}(\varphi - 2\pi/3).
\end{aligned}
$$

Therefore, equation (6) after the substitution contains the same unknown values but with other coefficients. Namely, we obtain equation

$$
\begin{aligned}
&(-\sin\varphi + \sin\beta)\hat{u}(\varphi) + \\
&+e^{-i(k_0 L\cos\varphi - 2\pi\nu/3)}(-\sin(\varphi + \frac{2\pi}{3}) + \sin\beta)\hat{u}(\varphi + \frac{2\pi}{3}) +
\end{aligned}
$$

$$+e^{-i(k_0 L\cos(\varphi+\pi/3)+2\pi\nu/3)}(-\sin(\varphi-\frac{2\pi}{3})+\sin\beta)\hat{u}(\varphi-\frac{2\pi}{3}) =$$
$$= \hat{\Phi}(\varphi)+e^{-i(k_0 L\cos\varphi-2\pi\nu/3)}\hat{\Phi}(\varphi+\frac{2\pi}{3})+$$
$$e^{-i(k_0 L\cos(\varphi+\pi/3)+2\pi\nu/3)}\hat{\Phi}(\varphi-\frac{2\pi}{3}). \quad (9)$$

We can exclude from (6) and (9) one of three unknown values. Let it be $\hat{u}(\varphi)$. We obtain:

$$\frac{\hat{u}(\varphi-2\pi/3)}{f(\varphi-2\pi/3)}-\frac{\hat{u}(\varphi+2\pi/3)}{f(\varphi+2\pi/3)}=\mu(\varphi), \quad (10)$$

where

$$\mu(\varphi)=-\frac{2\sin\varphi\hat{\Phi}(\varphi)\lambda(\varphi+\frac{2\pi}{3})\lambda(\varphi-\frac{2\pi}{3})}{f(\varphi-\frac{2\pi}{3})f(\varphi+\frac{2\pi}{3})}+ \quad (11)$$
$$+\frac{[(\sin\beta-\sin\varphi)-(\sin\beta+\sin\varphi)\lambda(\varphi+\frac{2\pi}{3})]\hat{\Phi}(\varphi-\frac{2\pi}{3})}{f(\varphi-\frac{2\pi}{3})f(\varphi+\frac{2\pi}{3})}-$$
$$-\frac{[(\sin\beta+\sin\varphi)-(\sin\beta-\sin\varphi)\lambda(\varphi-\frac{2\pi}{3})]\hat{\Phi}(\varphi+\frac{2\pi}{3})}{f(\varphi-\frac{2\pi}{3})f(\varphi+\frac{2\pi}{3})},$$

$$f(\varphi)=(\sin\beta+\sin(\varphi+\frac{2\pi}{3}))(\sin\beta-\sin(\varphi-\frac{2\pi}{3}))-$$
$$(\sin\beta-\sin(\varphi+\frac{2\pi}{3}))(\sin\beta+\sin(\varphi-\frac{2\pi}{3}))\lambda(\varphi),$$

$$\lambda(\varphi)=e^{-i(k_0 L\cos\varphi-2\pi\nu/3)}.$$

Thus, we have functional equation containing unknown function at two (instead of three) values of argument. The procedure performed can be interpreted from the physical point of view. Excluding of $\hat{u}(\varphi)$ with the help of $\hat{u}(-\varphi)$ can be considered as reflection of plane wave from the impedance surface. One can come to this conclusion comparing coefficients with $\hat{u}(\varphi)$ in (6) and (9). The ratio of these coefficients is exactly the coefficient of reflection from impedance surface.

We consider the procedure used in this paper to be an extention of the method of images.

3 Eigenvalues and eigenfunctions

Equation (11) enables to determine eigenvalues of the problem formulated. A transcendental equation can be derived to find eigenvalues.

Consider a homogenous equation i.e. let be $\mu(\varphi)=0$. We obtain

$$\frac{\hat{u}(\varphi-2\pi/3)}{f(\varphi-2\pi/3)}=\frac{\hat{u}(\varphi+2\pi/3)}{f(\varphi+2\pi/3)}.$$

The last equation is valid for arbitrary φ, therefore we can strengthen it:

$$\frac{\hat{u}(\varphi)}{f(\varphi)}=\frac{\hat{u}(\varphi+2\pi/3)}{f(\varphi+2\pi/3)}=\frac{\hat{u}(\varphi-2\pi/3)}{f(\varphi-2\pi/3)}. \quad (12)$$

Moreover, it is evident that

$$\frac{\hat{u}(-\varphi)}{f(-\varphi)} = \frac{\hat{u}(\varphi)}{f(\varphi)}. \tag{13}$$

Function $\hat{u}(\varphi)/f(\varphi)$ decreases at infinity and it is not equal to zero, hence it has some poles.

On the other hand, poles of function $\hat{u}(\varphi)/f(\varphi)$ correspond to zeros of function $f(\varphi)$. Let k_0 be one of the eigenvalues of the area. As it follows from (12), a value γ must exist, such that γ, $\gamma + 2\pi/3$ and $\gamma - 2\pi/3$ are zeros of $f(\varphi)$.

Therefore, there are 3 equations with respect to 2 variables (γ and k_0). One can see that the third equation is a consequence of two other. I.e., if γ and $\gamma + 2\pi/3$ are zeros of $f(\varphi)$ then $\gamma - 2\pi/3$ is also a zero of $f(\varphi)$.

First of all we write down an obvious trigonometric identity

$$\cos\varphi + \cos(\varphi + 2\pi/3) + \cos(\varphi - 2\pi/3) = 0.$$

Hence

$$\lambda(\varphi)\lambda(\varphi - 2\pi/3)\lambda(\varphi + 2\pi/3) = 1.$$

Further we consider equations corresponding to conditions $f(\gamma) = 0$ and $f(\gamma + 2\pi/3) = 0$:

$$\begin{aligned} \lambda(\gamma) &= \frac{(\sin\beta + \sin(\gamma + 2\pi/3))(\sin\beta - \sin(\gamma - 2\pi/3))}{(\sin\beta - \sin(\gamma + 2\pi/3)(\sin\beta + \sin(\gamma - 2\pi/3))}, \\ \lambda(\gamma + 2\pi/3) &= \frac{(\sin\beta + \sin(\gamma - 2\pi/3))(\sin\beta - \sin\gamma)}{(\sin\beta - \sin(\gamma - 2\pi/3)(\sin\beta + \sin\gamma)}. \end{aligned} \tag{14}$$

The third equation corresponding to $f(\gamma - 2\pi/3) = 0$ follows from (14):

$$\lambda(\gamma - 2\pi/3) = \frac{(\sin\beta + \sin\gamma)(\sin\beta - \sin(\gamma + 2\pi/3))}{(\sin\beta - \sin\gamma)(\sin\beta + \sin(\gamma + 2\pi/3))}.$$

Function $f(\varphi)$ is periodical with a period 2π and $f(\varphi) = f(-\varphi)$. Hence if α satisfies equations (14) for certain k_0 then $-\alpha$ satisfies (14) for the same k_0. Therefore we must mean 6 values $\pm\gamma, \quad \pm\gamma \pm 2\pi/3$ as a solution of (14).

The set of two equations (14) with respect to two variables γ and k_0 enables to determine eigenvalues.

Let $|\mathrm{Im}[\varphi]|$ be large. Than zeros of $f(\varphi)$ are situated near lines $\mathrm{Re}[\varphi] = \pi n, \quad n \in Z$ and only finite number of zeros lies outside the strips $\pi n - \delta < \mathrm{Re}[\varphi] < \pi n + \delta$, for $\delta > 0$. So one can conclude that for any given k_0 no more than finite number of values γ satisfies system (14).

The equation for eigenvalues is transcendental. Consider a particular case — Newmann's boundary conditions. In this case $f(\varphi)$ has the form

$$f_N(\varphi) = (1 - \lambda(\varphi))\sin(\varphi + 2\pi/3)\sin(\varphi - 2\pi/3). \tag{15}$$

(For Dirichlet's boundary conditions the procedure described above must be changed. One must take the normal derivative of the field on the boundary as the unknown function.)

Zeros of $f_N(\varphi)$ for any k_0 can be easily found. They are the values $\varphi = \pm 2\pi/3$ and all the roots of equation $\cos(\varphi) = 2\pi(n + \nu/3)/(k_0 L), \quad n \in Z$. Eigenvalue problem for k_0 is reduced to quadratic equation with integer parameters a and b:

$$k_0^2 = \frac{4\pi^2}{L^2}\left[(a + b + \frac{2\nu}{3})^2 + \frac{(a-b)^2}{3}\right],$$

$$\cos\gamma = -\frac{2\pi(a+b+2\nu/3)}{k_0 L}.$$

Besides, there is a solution $\gamma = 0$ for $\nu = 0$ and arbitrary k_0. This solution corresponds to the field $u = \text{const}$ in the area.

Now we return back to the problem with impedance boundary conditions and determine eigenfunctions. Let some values γ_j satisfy (14) for certain eigenvalue k_0. The unknown function $\hat{u}(\varphi)$ satisfying (12) and (13) has the form

$$\hat{u}(\varphi) = \sum_j \frac{A_j f(\varphi)}{\cos 3\varphi - \cos 3\gamma_j}, \tag{16}$$

where A_j are arbitrary coefficients.

System (14) can be interpreted in the terms of the method of reflections. The initial plane wave becomes the same wave multipied by $e^{2\pi i j}$ ($j = \pm 1$) after two successive reflections.

Note that each eigenfunction in (16) is a sum of six plane waves, which are successively reflected from the sides of the triangle. The initial wave propagates at the angle $-\gamma_j$ to x axis.

Reconstruction of wave field in the area is discussed below.

4 Problem with inhomogenous boundary conditions in non-resonant case

The solution for inhomogenous equation (10) can be constructed as follows. Consider the poles in the right-hand side and the left-hand side. Note that functions $\hat{u}(\varphi)$ and $\hat{\Phi}(\varphi)$ have no poles. Hence, all the poles are zeros of $f(\varphi)$. Assume that the frequency is not resonant for the area. It means that no one zero of $f(\varphi)$ is also a zero of $f(\varphi \pm 2\pi/3)$.

Let α_i are all zeros of $f(\varphi)$ in the strip $-\pi \le \mathrm{Re}[\varphi] < \pi$. Function $\mu(\varphi)$ has poles at the points $\alpha_i + 2\pi/3 + 2\pi n$ and $\alpha_i - 2\pi/3 + 2\pi n$. The first term in left-hand side has poles at the points $\alpha_i + 2\pi/3 + 2\pi n$ and the second at the points $\alpha_i - 2\pi/3 + 2\pi n$.

We represent $\mu(\varphi)$ in the form

$$\frac{\hat{u}(\varphi)}{f(\varphi)} = \sum_i \sum_{n=-\infty}^{\infty} [\frac{\mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3]}{\varphi - (\alpha_i + 2\pi/3) + 2\pi n} + \frac{\mathrm{Res}[\mu(\varphi), \alpha_i - 2\pi/3]}{\varphi - (\alpha_i - 2\pi/3) + 2\pi n}]$$

Each term in the left-hand side of (10) can be represented as a sum of corresponding poles of right-hand side:

$$\frac{\hat{u}(\varphi)}{f(\varphi)} = \sum_i \sum_{n=-\infty}^{\infty} \frac{\mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3]}{\varphi - \alpha_i + 2\pi n} = \tag{17}$$

$$= -\frac{1}{2} \sum_i \frac{\sin\alpha_i \ \mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3]}{\cos\varphi - \cos\alpha_i}.$$

Note that the second identity follows from

$$\mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3] = -\mathrm{Res}[\mu(\varphi), -\alpha_i + 2\pi/3], \tag{18}$$

which can be checked directly. We must use the fact that α_i are zeros of $f(\varphi)$.

An entire function can be added to the solution (17) but it must be equal to zero because of the decrease of $\hat{u}(\varphi)/f(\varphi)$ in the areas where it has no poles.

Note that for large integer n the following estimations are valid:

$$\cos\alpha_n \sim \frac{2\pi n + 2\pi\nu/3}{k_0 L}, \tag{19}$$

$$\mathrm{Res}[\mu(\varphi), \alpha_n + 2\pi/3] \sim \cos^{-4}\alpha_n,$$

The last estimation confirms the convergence of (17).

5 Reconstruction of wave field in the area

Unknown functions $\hat{u}(\varphi)$ have been obtained in previous sections as solutions of functional equations for homogenous and inhomogenous problems. Here we reconstruct wave field in the area using.

Apply again the second Green's formula to the area and substitute u' with Green's function of (1):

$$u'(\boldsymbol{r}) = H_0^{(1)}(|\boldsymbol{r} - \boldsymbol{r'}|),$$

where $H_0^{(1)}$ is a cylindrical function.

Using a well-known asymptotics of Hankel's function

$$H_0^{(1)}(z) \sim -\frac{2i}{\pi}\ln z,$$

we obtain

$$u(\boldsymbol{r'}) = -\frac{i}{4}\int \left[\frac{\partial u}{\partial n}u' - \frac{\partial u'}{\partial n}u\right] d\Gamma, \tag{20}$$

where integration is performed along the boundary of the area and the vector $\boldsymbol{r'}$ points at the source of Hankel's function.

Use the integral representation of Hankel's function [3]:

$$H_0^{(1)}(z) = \frac{1}{\pi}\int_S e^{iz\cos(t-\alpha)}dt.$$

Contour of integration S is shown at Fig.2. Parameter α can be a real number from 0 to π.

We are calculating wave field at the point with polar coordinates (ρ, ψ) introduced by

$$x = \rho\cos\psi, \qquad y = \rho\sin\psi.$$

We take the integral over the side with number 0 in (20). Define α for each point of this side as it is shown at Fig.3. Note that integration leads again to expression (5). After performing the same procedure for each side we obtain

$$u(\rho, \psi) = \frac{k_0}{4\pi}\int_S [-\hat{\Phi}(t) + (\sin\beta + \sin t)\hat{u}(t)]\times \tag{21}$$

$$\times[e^{ik_0\rho\cos(\psi-t)} + e^{ik_0\rho\cos(\psi-t-2\pi/3)}\lambda(t+2\pi/3) + e^{ik_0\rho\cos(\psi-t+2\pi/3)}\lambda^{-1}(t)]dt.$$

Another representation can be useful

$$u(\rho, \psi) = \frac{k_0}{4\pi}\int_S [-\hat{\Phi}(t) + (\sin\beta + \sin t)\hat{u}(t)]e^{ik_0\rho\cos(\psi-t)}dt+ \tag{22}$$

$$+\frac{k_0}{4\pi}\int_{S+\frac{2\pi}{3}}[-\hat{\Phi}(t-\frac{2\pi}{3})+(\sin\beta+\sin(t-\frac{2\pi}{3}))\hat{u}(t-\frac{2\pi}{3})]e^{ik_0\rho\cos(\psi-t)}\lambda(t)dt+$$

$$+\frac{k_0}{4\pi}\int_{S-\frac{2\pi}{3}}[-\hat{\Phi}(t+\frac{2\pi}{3})+(\sin\beta+\sin(t+\frac{2\pi}{3}))\hat{u}(t+\frac{2\pi}{3})]e^{ik_0\rho\cos(\psi-t)}\lambda^{-1}(t+\frac{2\pi}{3})dt$$

Formulas (21) and (22) enable to calculate wave field in the area.

A good example how to apply the integrals above is the calculation of eigenfunctions. Use solution (16) from the third section. Consider one of the terms in the sum:

$$\hat{u}(\varphi)=\frac{A_j f(\varphi)}{\cos 3\varphi-\cos 3\gamma_j}=\frac{A_j(\sin\beta+\sin(\varphi+\frac{2\pi}{3}))(\sin\beta-\sin(\varphi-\frac{2\pi}{3}))}{\cos 3\varphi-\cos 3\gamma_j}-$$

$$-\frac{A_j(\sin\beta-\sin(\varphi+\frac{2\pi}{3}))(\sin\beta+\sin(\varphi-\frac{2\pi}{3}))\lambda(\varphi)}{\cos 3\varphi-\cos 3\gamma_j}.$$

Apply formula (22). Regroup terms in integrands

$$u(\rho,\psi)=\frac{k_0}{4\pi}\left(\int_S-\int_{S-\frac{2\pi}{3}}\right)e^{ik_0\rho\cos(\psi-t)}\times$$

$$\times\frac{(\sin\beta+\sin t)(\sin\beta+\sin(t+\frac{2\pi}{3}))(\sin\beta-\sin(t-\frac{2\pi}{3}))}{\cos 3t-\cos 3\gamma_j}dt+\dots$$

(two similar terms are implied).

Contours S and $S-2\pi/3$ can be closed at infinity. Integrals become sums of residues at points $\pm\gamma_j$, $\pm\gamma_j+2\pi/3$, $\pm\gamma_j-2\pi/3$.

Using (14) we obtain

$$u(\rho,\psi)=\frac{k_0}{6}\sum_{n=1}^{6}\frac{e^{ik_0\rho\cos(\psi-\gamma_{j,n})}}{\sin 3\gamma_{j,n}}\times \tag{23}$$

$$\times(\sin\beta+\sin\gamma_{j,n})(\sin\beta+\sin(\gamma_{j,n}+\frac{2\pi}{3}))(\sin\beta-\sin(\gamma_{j,n}-\frac{2\pi}{3})),$$

where $\gamma_{j,n}$ runs over $\pm\gamma_j$, $\pm\gamma_j+2\pi/3$, $\pm\gamma_j-2\pi/3$.

Equation (23) confirms that eigenfunction is a sum of six plane waves.

6 Inhomogenous problem with large L

Equations (17), (23) Is the solution of the inhomogenous problem, but it is not clear how this solution corresponds to known solution for impedance angle of $\pi/3$.

Suppose that L is large with respect to the wavelength and k_0 has a small imaginary part $-i\varepsilon$. We suppose that at first L tends to infinity and then ε tends to zero.

Consider zeros of $f(\varphi)$ in the asymptotics described above. It is obvious that $\lambda(\varphi)$ grows rapidly in the area between contours C and $C+\pi$ (this area is marked at Fig.4.) and decreases between contours $C-\pi$ and C. Note that hereafter all the points of the plane φ are defined up to $2\pi n$. Zeros of function $f(\varphi)$ belong to two kinds. Zeros of the first kind lie on the contours C and $C-\pi=-C$. The quantity of zeroes that lie on the segment $\delta\varphi$ is denoted by δn and can be found from (19):

$$\frac{\delta n}{\delta\varphi}=\frac{k_0L|\sin\varphi|}{2\pi}. \tag{24}$$

Zeros of the second kind are the zeros of the expression $(\sin\beta + \sin(\varphi + 2\pi/3))(\sin\beta - \sin(\varphi - 2\pi/3))$ that are situated in the area of decrease of $\lambda(\varphi)$ and the zeros of the expression $(\sin\beta + \sin(\varphi - 2\pi/3))(\sin\beta - \sin(\varphi + 2\pi/3))$ that are situated in the area of growth of $\lambda(\varphi)$. It is evident that it depends on β, whether such zeros exist and where they are situated. One can check directly that there are no zeros of the second kind in the case $0 < \mathrm{Re}[\beta] < \pi/3$ and $\mathrm{Im}[\beta] < 0$. Further we consider this case.

We calculate $\hat{u}(\varphi)$ for $0 < \mathrm{Re}[\varphi] < \pi/3$ and $\mathrm{Im}[\varphi] < 0$.

Besides, we calculate $u(l_0)$ for values l_0 small relatively to L. This fact is used as follows. As it follows from (17), $\hat{u}(\varphi)$ has the form

$$\hat{u}(\varphi) = -f(\varphi)\frac{1}{2}\sum_i \frac{\sin\alpha_i\ \mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3]}{\cos\varphi - \cos\alpha_i}.$$

For the side with nimber 0 for small l_0 we omit the term containing $\lambda(\varphi)$ in $f(\varphi)$:

$$\hat{u}(\varphi) = -\frac{(\sin\beta + \sin(\varphi + \frac{2\pi}{3}))(\sin\beta - \sin(\varphi - \frac{2\pi}{3}))}{2}\times$$

$$\times\sum_i \frac{\sin\alpha_i\ \mathrm{Res}[\mu(\varphi), \alpha_i + 2\pi/3]}{\cos\varphi - \cos\alpha_i}.$$

The sum over the poles that lie along the contour C can be replaced by integration if the dencity of the poles $\delta n/\delta\varphi$ is large, i.e., if $k_0 L$ is large. Using (18) and (24) we obtain:

$$\hat{u}(\varphi) = \frac{(\sin\beta + \sin(\varphi + \frac{2\pi}{3}))(\sin\beta - \sin(\varphi - \frac{2\pi}{3}))k_0 L}{2\pi}\times \tag{25}$$

$$\times\int_C \frac{\sin^2\alpha\ \mathrm{Res}[\mu(\varphi), \alpha + 2\pi/3]\, d\alpha}{\cos\varphi - \cos\alpha}.$$

Suppose that the force is applied to the points situated at the distance d from the point $(0,0)$ on the sides 0 and 2 and having amplitudes $-ik_0A_0$ and $-ik_0A_2$ respectively:

$$\hat{\Phi}(\varphi) = A_0 e^{-ik_0 d\cos(\varphi)} + A_2 e^{ik_0 d\cos(\varphi)}\lambda(\varphi).$$

We calculate the factor $\mathrm{Res}[\mu(\varphi), \alpha + 2\pi/3]$ in the integrand of (25) neglecting $\lambda(\varphi)$ in the areas where the value $\lambda(\varphi)$ decreases for large L:

$$\mathrm{Res}[\mu(\varphi), \alpha + 2\pi/3] = \frac{i}{k_0 L\sin\alpha}\times$$

$$\times[\frac{A_0 e^{-ik_0 d\cos\alpha} + A_2 e^{ik_0 d\cos(\alpha + 2\pi/3)}}{(\sin\beta + \sin\alpha)(\sin\beta + \sin(\alpha + 2\pi/3))(\sin\beta - \sin(\alpha - 2\pi/3))} -$$

$$-\frac{A_0 e^{-ik_0 d\cos(\alpha - 2\pi/3)} + A_2 e^{ik_0 d\cos(\alpha + 2\pi/3)}}{(\sin\beta + \sin\alpha)(\sin\beta - \sin(\alpha + 2\pi/3))(\sin\beta - \sin(\alpha - 2\pi/3))} +$$

$$+\frac{A_0 e^{-ik_0 d\cos(\alpha - 2\pi/3)} + A_2 e^{ik_0 d\cos\alpha}}{(\sin\beta + \sin\alpha)(\sin\beta - \sin(\alpha + 2\pi/3))(\sin\beta + \sin(\alpha - 2\pi/3))}].$$

The solution obtained can be transformed to the representation from [4] after regrouping the terms and deforming the contours of integration.

7 Discussion

Functional equations similar to used in this paper can be easily derived for any closed 2D area with piecewize-linear boundary. The procedure seemes to be more difficult for open areas but functional equations can be derived for open areas as well.

Functional equation for the equilateral triangle can be solved explicitly because of the geometry of the area. We are not aware of the solutions for such areas as the right pentagon or an arbitrary triangle. Probably some effective numerical methods can be found for such problems.

8 Acknoledgements

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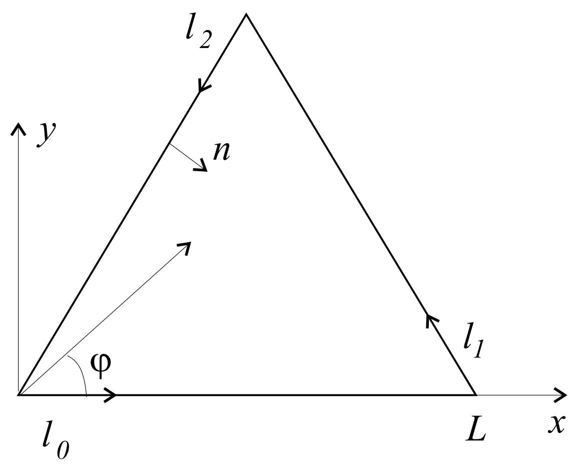

Figure 1:

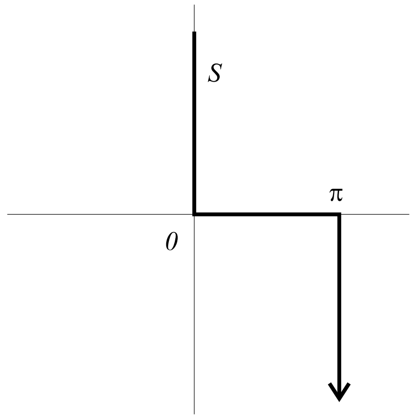

Figure 2:

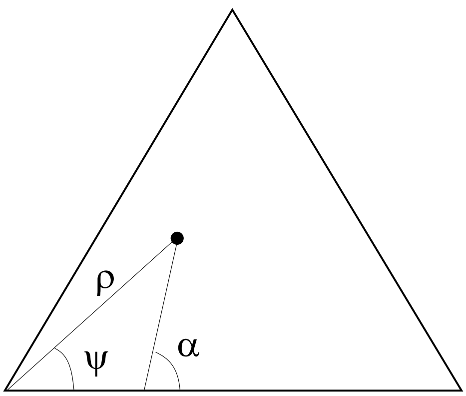

Figure 3:

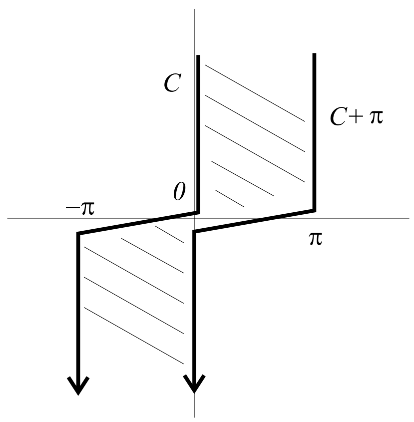

Figure 4: