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Wave Reflection from a Diffraction Grating Consisting of Absorbing Screens: Description in Terms of the Wiener–Hopf–Fock Method

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Abstract—A two-dimensional problem of wave reflection from a diffraction grating is considered for the case of grazing incidence. The wavelength is assumed to be small. The diffraction grating has a periodicity cell consisting of two semi-infinite absorbing screens perpendicular to the edge of the grating. It is known that, for a grating with a cell consisting of one screen, the reflection coefficient tends to –1 as the angle of incidence tends to zero. It is shown that this his result is valid for the case of a period containing two screens. The study is carried out in terms of the Wiener–Hopf–Fock method. A matrix factorization problem is formulated. The solution to the problem is unknown and remains so within the present study. The limiting reflection coefficient is investigated without constructing the solution by using the approach proposed by L.A. Weinstein.

Keywords: diffraction by gratings, Wiener–Hopf–Fock method, parabolic equation of diffraction theory, Weinstein problem, billiard modes

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1. FORMULATION OF THE PROBLEM

We consider a two-dimensional problem of acoustic wave scattering by a periodic structure, shown in Fig. 1. The structure is a diffraction grating with a periodicity cell consisting of two perfectly absorbing screens positioned along the y axis. The period of the grating (along the x axis) is $a + b$.. The screens occupy the half-lines $\{x = (a + b)n, y < 0\}$ and $\{x = a + (a + b)n, y < c\}$, $n \in Z$.

We consider a wave process with a narrow angular spectrum concentrated near the direction of the x axis. Such a process is described by the parabolic equation of the diffraction theory [1],

$$\left(2ik\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right)u = 0, \tag{1}$$

which implies description of the diffraction process in the Fresnel approximation. Here, k is a parameter (the wavenumber) and u is the field variable of the parabolic equation. The latter is related to the physical field variable (e.g., the sound pressure p) by the conventional formula [1]

$$p(x, y) = e^{ikx}u(x, y). \tag{2}$$

It is assumed that $k(a + b) \gg 1$.

It is well known that, for the Helmholtz equation, no perfectly absorbing boundary conditions can be set. However, in the parabolic approximation, such boundary conditions can easily be formulated for the screens perpendicular to the x axis: it is sufficient to require that the field be zero to the right of each of the screens. The reflected waves are not described by the parabolic equation.

The formal statement of the problem for the parabolic equation should include the boundary conditions set at the vertices of the screens, which consist in the absence of any sources at the vertices. It is sufficient to require that the field u near the vertices be limited.

From the upper half-plane ($y > 0$), the following plane wave is incident on the diffraction grating:

$$u^{\text{in}} = \exp\{-ik(x\theta_{\text{in}}^2/2 + y\theta_{\text{in}})\}, \tag{3}$$

where θ_{in} is the (small) angle of incidence measured with respect to the x axis. Note that transformation (2)

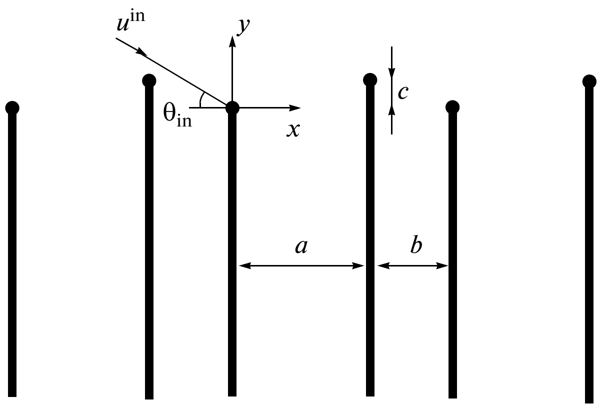

Fig. 1. Geometry of problem.

together with the approximations $\sin\theta_{in} \approx \theta_{in}$, $\cos\theta_{in} \approx 1 - \theta_{in}^2/2$ reduce Eq. (3) to the expression for a common plane wave.

It is necessary to determine the scattered wave u^{sc} decreasing or propagating in the direction of increasing $|y|$. Note that the incident wave is characterized by the property

$$u^{in}(x + a + b, y) = \exp\{-ik(a + b)\theta_{in}^2/2\}u^{in}(x, y).$$

According to the Floquet principle, for periodic structures, a similar property should be characteristic of the scattered field u^{sc}. Evidently, in the upper half-plane (more precisely, for $y > c$)), the scattered field can be represented as the sum over the diffraction orders:

$$u^{sc} = \sum_{n=-\infty}^{\infty} R_n \exp\{-ikx\theta_n^2/2 + iky\theta_n\}, \quad (4)$$

$$\theta_n = \sqrt{\theta_{in}^2 + \frac{4\pi n}{k(a + b)}}. \quad (5)$$

The branch of the square root is chosen so that the values are positive real or positive imaginary ones, which correspond to outgoing or decreasing waves. Evidently, $\theta_0 = \theta_{in}$. The coefficients R_n characterizing the scattering into diffraction orders are of the main interest for our study.

A solution to the problem is known for $a = b$, $c = 0$, i.e., for the case of the period consisting of one screen. This is the classical Weinstein problem [2, 3]. The problem has been solved by the (scalar) Wiener–Hopf–Fock method. The property of the solution that is of main interest to us is as follows: for small values of θ_{in}, all the coefficients except for the zero-order one are small, while the zero-order coefficient behaves as

$$R_0 = -1 - \frac{1 - i}{\sqrt{\pi}}\varsigma(1/2)\sqrt{ka}\theta_{in}, \quad (6)$$

i.e., for $\theta_{in} \to 0$, the mirror reflection coefficient tends to -1 (here, ς is the Riemann function). This seems surprising, because the screens are perfectly absorbing, while a reflection coefficient of -1 corresponds to a perfectly reflecting (acoustically soft) boundary. In more exact terms, dependence (6) makes it possible to ascribe a certain effective impedance to the boundary $y = 0$, as viewed from the upper half-space (see below).

The aim of our paper is to study the coefficient R_0 for the problem stated above with a period consisting of two screens. The problem is reduced to a matrix factorization problem whose solution is unknown. In [4, 5], the Wiener–Hopf method was used to consider a problem that was similar to our formulated one for $a = b$, $c > 0$, with the Helmholtz equation replacing Eq. (1). In the cited papers, the matrix problem was reduced to a scalar Fredholm equation of the second kind. In this paper, we make no attempt to construct a solution to the factorization problem. Instead, we generalize the method developed by L.A. Weinstein [2] to the case of a matrix problem. This method allows us to analyze the coefficient R_0 for $\theta_{in} \to 0$ without constructing a solution.

The latest studies [6] show that the result $R_0 \to -1$ is a general one for a wide class of problems concerned with the reflection from the end of a waveguide for the case where the frequency of the incident mode tends to the cutoff frequency (the classical Weinstein problem is a reformulated version of precisely this problem).

Note that we proposed an alternative to the Wiener–Hopf–Fock method for Weinstein-type problems. The method was reduced to construction of the so-called OE equation and its numerical or asymptotic solution [7–9]. In the Appendix, we discuss the relation between the factorization problem and the OE equation.

Let us consider the impetus of this study from the physical viewpoint . According to Weinstein, application of the reflection method to a planar waveguide with perfectly reflecting walls reduces the problem of diffraction by the end of the waveguide to the problem of diffraction by a grating with perfectly absorbing screens. In our case, this corresponds to gratings with $a = b$. Here, the screens correspond not to the waveguide walls but to transitions from sheet to sheet of a certain multisheeted surface to which the reflection method leads. The case $c > 0$ corresponds to a planar waveguide with an asymmetric end. Such a problem was considered in [3] in relation to simulation of a Fabry–Perot resonator whose mirrors are shifted with respect to each other (the effect of the shift of mirrors on the Q factor of the resonator was investigated). The same problem arises in analyzing the operational efficiency of a planar source near a rigid wall with finite dimensions. Lastly, in recent publications on aviation acoustics, the problem of radiation from a planar waveguide with shifted walls was used to model engine noise amplification at the edge of an airplane wing.

A more general family of diffraction problems (Weinstein-type problems) was introduced in [10]. The authors considered the high-frequency modes of two-dimensional open resonators in the form of rectangular “rooms” with open “windows.” It was demonstrated (in particular, by direct numerical simulation) that the highest-Q modes have a billiard nature; i.e., these modes have the form of beams with narrow angular spectra, which propagate from wall to wall along closed trajectories. For these modes, the main mechanism of energy loss is diffraction by window edges. Experimental data testifying to the existence of such modes can be found in [11].

In [10], billiard modes were studied by the reflection method. It was assumed that the walls of the resonator were perfectly rigid, while the windows had the form of perfectly absorbing surfaces. In this case, a billiard mode is transformed to a wave beam propagating between two diffraction gratings, which consist of

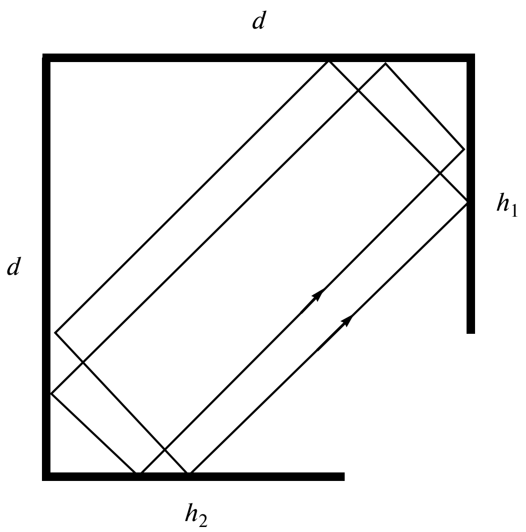

Fig. 2. Open resonator and billiard mode.

absorbing screens. The Q factor of the resonator can be determined if the coefficients of reflection from the gratings are known. According to [10], different geometries of resonators lead to different types of diffraction gratings. We study one of the types here. Figure 2 shows the resonator that gives rise to the diffraction grating under study. It is a square resonator with a side d and a corner window. The sides adjacent to the window have lengths h_1 and h_2.We consider a billiard mode traveling along the diagonal. After applying the reflection method, we find that the beam corresponding to this mode is bounded by the grating shown in Fig. 1 with the following parameters: $a = d\sqrt{2} - (h_1 + h_2)/\sqrt{2}$, $b = d\sqrt{2} + (h_1 + h_2)/\sqrt{2}$, $c = (h_1 - h_2)/\sqrt{2}$.

Now, let us make two additional comments. First, the representation of the resonator windows as perfectly absorbing screens may seem rough or even incorrect from the viewpoint of the description by the Helmholtz equation. A correct but much less illustrative description is obtained by applying the reflection method to an acoustically stiff boundary, shown in Fig. 2 (an infinitely thin boundary in the form of a four-piece broken line). In this case, the waves go to outer space (to infinity). As a result of applying the reflection method, we find that the wave beam corresponding to the billiard mode propagates over a rather complex branching surface. The role of the edges of perfectly absorbing screens is played by branch points.

Second, it is necessary to comment on the use of the parabolic equation to describe the wave process. It is known that the parabolic equation adequately describes the Fresnel diffraction and inadequately describes scattering by the screen edges at large angles. In our study, the parabolic equation is used because the Fresnel processes are of main interest in describing the diffraction by the grating under consideration. In the case of grazing incidence, the problem is complicated by the fact that the half-shadow zone of one of the screens contains a considerable number of other screen edges. Thus, a high-order half-shadow zone is formed. Such half-shadow zones are adequately described by the parabolic equation. Note that Weinstein solved the problem formulated for the Helmholtz equation and also for the parabolic equation. From these solutions, it follows that, for small angles of incidence and small-angle scattering, the parabolic description is quite adequate. The accuracy of the Fresnel approximation can be estimated as the ratio of the wavelength to the size of the first Fresnel zone corresponding to a single run between the screens, i.e., the ratio of $1/k$ to $\sqrt{a/k}$ (a can be replaced by b). The zone lying near the screen edge and characterized by a size on the order of the wavelength is responsible for scattering at large angles, whereas the Fresnel zone is responsible for the formation of the half-shadow region.

The main result of this paper is that, for large values of k and $\theta_{\text{in}} \to 0$, the coefficient of reflection into the fundamental mode R_0 tends to -1. Let us specify for which parameter values this result is valid. Our consideration requires that $ka \gg 1$ and $kb \gg 1$ to ensure applicability of the parabolic equation. Values of these parameters on the order of 10 are sufficient for a fair accuracy of our approximation. Then, by analogy with the classical Weinstein problem, we expect that the deviation of R_0 from -1 is about $\sqrt{ka}\theta_{\text{in}}$ (we assume that a and b are of the same order of magnitude). Hence, the reflection coefficient is close to -1 under the condition that $\sqrt{ka}\theta_{\text{in}} \ll 1$. In our formulation, the screens are assumed to be infinitely thin. The effect of the wall thickness on the result was not studied.

There is a hypothesis based on the results reported by S.A. Nazarov [6] and the comments made in [2] that the conditions $ka \gg 1$ and $kb \gg 1$ are not necessary for the validity of the limit $R_0 \to -1$ at $\theta_{\text{in}} \to 0$,i.e., the result holds for the case of a grating whose dimensions are comparable with the wavelength (evidently, in this case, it is necessary to formulate the problem on the basis of the Helmholtz equation). In addition, the same considerations suggest that the structure of the grating is not really important.

2. DERIVATION OF THE WIENER–HOPF–FOCK EQUATIONS

Let us introduce the functions

$$u_0(y) = \begin{cases} u^{\text{sc}}(0, y), & y > 0 \\ 0, & y < 0 \end{cases}, \tag{7}$$

$$u_1(y) = \begin{cases} u^{\text{sc}}(a, y), & y > c \\ 0, & y < c \end{cases}, \tag{8}$$

$$\psi_0(y) = \begin{cases} 0, & y > 0 \\ u^{\mathrm{sc}}(-0, y), & y < 0 \end{cases}, \tag{9}$$

$$\psi_1(y) = \begin{cases} 0, & y > c \\ u^{\mathrm{sc}}(a-0, y), & y < c \end{cases}. \tag{10}$$

In Eqs. (9) and (10), the arguments $x = -0$ and $x = a - 0$ mean that the values are taken to the left of the screens.

Note that, since the total field $u = u^{\mathrm{in}} + u^{\mathrm{sc}}$ to the right of the screens is zero (the screens are absorbing ones), the following identities are satisfied:

$$u^{\mathrm{sc}}(+0, y) = -u^{\mathrm{in}}(0, y), \quad y < 0, \tag{11}$$

$$u^{\mathrm{sc}}(a+0, y) = -u^{\mathrm{in}}(a, y), \quad y < c. \tag{12}$$

Now, we use an important formula for the parabolic equation [1]. If $u(x, y)$ is a solution to Eq. (1) within the region $x_1 < x < x_2$, and this solution is continuous at the boundaries of the given region, we have

$$u(x_2, y) = \int_{-\infty}^{\infty} G(x_2 - x_1, y - y')u(x_1, y')dy', \tag{13}$$

where G is the Green's function of the parabolic equation

$$G(x, y) = \sqrt{\frac{k}{2\pi x}} \exp\left\{i\frac{ky^2}{2x} - i\frac{\pi}{4}\right\}. \tag{14}$$

Within the region $0 < 0 < x < a$, Eq. (13) yields

$$u_1(y) + \psi_1(y) = \int_0^{\infty} G(a, y - y')u_0(y')dy' - \int_{-\infty}^{0} G(a, y - y')u^{\mathrm{in}}(0, y')dy'. \tag{15}$$

Formula (13) can also be applied within the region $a < x < a + b$. According to the Floquet principle, the scattered field values at the boundary $x = a + b - 0$ can be represented as the scattered field values taken at $x = -0$, and multiplied by $\exp\{-ik(a+b)\theta_{\mathrm{in}}^2/2\}$:

$$\exp\{-ik(a+b)\theta_{\mathrm{in}}^2/2\}(u_0(y) + \psi_0(y)) = \int_c^{\infty} G(a, y - y')u_1(y')dy' - \int_{-\infty}^{c} G(a, y - y')u^{\mathrm{in}}(a, y')dy'. \tag{16}$$

System of equations (15), (16) can be transformed to a system of integral equations closed with respect to the unknown functions u_0 and u_1. For this purpose, it is sufficient to consider the first equation for $y > c$, and the second equation for $y > 0$. For these values, the (unknown) functions ψ_1 and ψ_0 appearing on the right-hand sides of the equations are identically equal to zero. However, to derive the Wiener–Hopf–Fock equations, it is worthwhile to leave Eqs. (15) and (16) in their original form, containing redundant unknowns, but being valid throughout the entire y axis.

We introduce a one-sided Fourier transformation:

$$U_0(\xi) = \int_0^{\infty} u_0(y)e^{i\xi y}dy, \tag{17}$$

$$U_1(\xi) = \exp\left\{-ic\xi + ika\theta_{\mathrm{in}}^2/2\right\} \int_c^{\infty} u_1(y)e^{i\xi y}dy, \tag{18}$$

$$\Psi_0(\xi) = \int_{-\infty}^{0} \psi_0(y)e^{i\xi y}dy, \tag{19}$$

$$\Psi_1(\xi) = \exp\left\{-ic\xi + ika\theta_{\mathrm{in}}^2/2\right\} \int_{-\infty}^{c} \psi_1(y)e^{i\xi y}dy. \tag{20}$$

Note that integral operators (15) and (16) are of convolution nature (a difference kernel). This allows us to apply the convolution theorem and to represent Eqs. (15) and (16) in matrix form:

$$\mathbf{\Psi}(\xi) + \mathbf{K}(\xi)\mathbf{U}(\xi) = \frac{1}{\xi - \xi_{\mathrm{in}}}\mathbf{D}(\xi), \tag{21}$$

where

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{22}$$

$$\mathbf{K}(\xi) = \begin{pmatrix} 1 & -\exp\left\{-\dfrac{ib(\xi^2 - \xi_{\mathrm{in}}^2)}{2k} + ic\xi\right\} \\ -\exp\left\{-\dfrac{ia(\xi^2 - \xi_{\mathrm{in}}^2)}{2k} - ic\xi\right\} & 1 \end{pmatrix}, \tag{23}$$

$$\mathbf{D} = i\begin{pmatrix} \exp\left\{-i\dfrac{b(\xi^2 - \xi_{\mathrm{in}}^2)}{2k} + ic(\xi - \xi_{\mathrm{in}})\right\} \\ \exp\left\{-i\dfrac{a(\xi^2 - \xi_{\mathrm{in}}^2)}{2k} - ic\xi\right\} \end{pmatrix}, \tag{24}$$

$$\xi_{\mathrm{in}} = k\theta_{\mathrm{in}}.$$

Equation (21) should be complemented with a priori satisfied limitations on the unknown vector functions $\mathbf{\Psi}(\xi)$, $\mathbf{U}(\xi)$,. The limitations follow from the general properties of the one-sided Fourier transformation. Namely, the Function $\mathbf{\Psi}(\xi)$ should be analytic in the lower half-plane of the argument while the function $\mathbf{U}(\xi)$ should be analytic in the upper half-plane. In addition, a simple analysis shows that the functions u_0, ψ_0 have discontinuities at $y = 0$, while the functions u_1 and ψ_1 are discontinuous at $y = c$. According to the Watson lemma, this means that the vector $\mathbf{\Psi}(\xi)$ decreases as ξ^{-1} for large values of $|\xi|$ in the lower half-plane while the function $\mathbf{U}(\xi)$ decreases as ξ^{-1} in the upper half-plane. The aforementioned limitations together with equation (21) form a Wiener–Hopf–Fock functional problem [2, 12].

The right-hand side of Eq. (21) can be represented in the following equivalent form:

$$\mathbf{\Psi}(\xi) + \mathbf{K}(\xi)\mathbf{U}(\xi) = \frac{1}{\xi - \xi_{\text{in}}}(\mathbf{K}(\xi) - \mathbf{I})\mathbf{r}, \qquad \mathbf{r} = -i\begin{pmatrix} 1 \\ \exp\{-ic\xi_{\text{in}}\} \end{pmatrix}. \tag{25}$$

For the subsequent consideration, it is important that the vector $\mathbf{r}$ does not depend on ξ. Here and below, $\mathbf{I}$ is a 2×2 unit matrix.

The determinant of the matrix $\mathbf{K}$ is

$$\det(\mathbf{K}(\xi)) = 1 - \exp\left\{-i\frac{(a+b)}{2k}(\xi^2 - \xi_{\text{in}}^2)\right\}.$$

The zero points of the determinant are $\xi = \pm k\theta_n$.. According to the principle of ultimate absorption, we assume that the points $k\theta_n$ belong to the upper half-plane (i.e., the real axis bypasses them from below). Accordingly, the points $-k\theta_n$ belong to the lower half-plane. In particular, the point $\xi_{\text{in}} = k\theta_{\text{in}}$ belongs to the upper half-plane.

3. FORMAL SOLUTION TO THE WIENER–HOPF–FOCK FUNCTIONAL PROBLEM

The general solution to the Wiener–Hopf–Fock matrix problem is unknown and is not constructed here. Instead, by the method proposed in [2], we analyze the mirror reflection coefficient R_0 for $\theta_{\text{in}} \to 0$.

Let us formally represent the solution to the Wiener–Hopf–Fock functional problem [12]. Let the following factorization be known:

$$\mathbf{K}(\xi) = \mathbf{K}_-(\xi)\mathbf{K}_+(\xi), \tag{26}$$

where $\mathbf{K}_-$ and $\mathbf{K}_+$ are nonsingular regular matrices in the lower and upper half-planes, respectively. In addition, we assume that $\mathbf{K}_- \to \mathbf{I}$ as $|\xi| \to \infty$ in the lower half-plane and $\mathbf{K}_+ \to \mathbf{I}$ as $|\xi| \to \infty$ in the upper half-plane. The construction of such a factorization is the main task in solving the Wiener–Hopf–Fock matrix problem.

We multiply Eq. (25) by $\mathbf{K}_-^{-1}$:

$$\mathbf{K}_-^{-1}(\xi)\mathbf{\Psi}(\xi) + \mathbf{K}_+(\xi)\mathbf{U}(\xi) = \frac{1}{\xi - \xi_{\text{in}}}(\mathbf{K}_+(\xi) - \mathbf{K}_-^{-1}(\xi))\mathbf{r}. \tag{27}$$

We expand the right-hand side in the regular decreasing vector functions $\mathbf{F}_+(\xi)$ and $\mathbf{F}_-(\xi)$, in the upper and lower half-planes, respectively:

$$\frac{1}{\xi - \xi_{\text{in}}}(\mathbf{K}_+(\xi) - \mathbf{K}_-^{-1}(\xi))\mathbf{r} = \mathbf{F}_+(\xi) + \mathbf{F}_-(\xi). \tag{28}$$

In the case under consideration, this expansion can be represented in explicit form:

$$\mathbf{F}_+(\xi) = \frac{1}{\xi - \xi_{\text{in}}}(\mathbf{K}_+(\xi) - \mathbf{K}_+(\xi_{\text{in}}))\mathbf{r}, \tag{29}$$

$$\mathbf{F}_-(\xi) = \frac{1}{\xi - \xi_{\text{in}}}(\mathbf{K}_+(\xi_{\text{in}}) - \mathbf{K}_-^{-1}(\xi))\mathbf{r}. \tag{30}$$

We rearrange Eq. (27) as follows:

$$\mathbf{K}_-^{-1}(\xi)\mathbf{\Psi}(\xi) - \mathbf{F}_-(\xi) = \mathbf{F}_+(\xi) - \mathbf{K}_+(\xi)\mathbf{U}(\xi). \tag{31}$$

According to the logic of the Wiener–Hopf–Fock method, note that the left-hand side is regular and decreases in the lower half-plane, whereas the right-hand side is regular and decreases in the upper half-plane. Since the left-hand and right-hand sides represent a single function, by applying the Liouville theorem we conclude that this function is identically equal to zero. As a consequence, we obtain

$$\mathbf{K}_+(\xi)\mathbf{U}(\xi) = \mathbf{F}_+(\xi) \tag{32}$$

and

$$\mathbf{U}(\xi) = \mathbf{K}_+^{-1}(\xi)\mathbf{F}_+(\xi). \tag{33}$$

We invert the Fourier transformation and construct the function u_0:

$$u_0(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1,0)\mathbf{K}_+^{-1}(\xi)\mathbf{F}_+(\xi)e^{-i\xi y}d\xi. \tag{34}$$

Here, (1,0) is the row vector consisting of two elements.

The matrix K(ξ) has zero points of its determinant, which, in particular, occur at $\xi = -k\theta_n$ for all the integral numbers n. These zero points lie in the lower half-plane and, hence, refer to the factor $\mathbf{K}_+$. Let us consider the point $\xi = -\xi_0 = -\xi_{\text{in}}$. At this point, the (matrix) function $\mathbf{K}_+^{-1}(\xi)$ has a simple pole. We denote the residue of the integrand at this point as

$$\text{res}[(1,0)\mathbf{K}_+^{-1}(\xi)\mathbf{F}_+(\xi),\ \ \xi = -\xi_{\text{in}}] = h. \tag{35}$$

Note that

$$\mathbf{K}_+^{-1}(\xi)\mathbf{F}_+(\xi) = \frac{\mathbf{r} - \mathbf{K}_+^{-1}(\xi)\mathbf{K}_+(\xi_{\text{in}})\mathbf{r}}{\xi - \xi_{\text{in}}}, \tag{36}$$

and, hence,

$$h = \frac{1}{2\xi_{\text{in}}}(1,0)\text{res}[\mathbf{K}_+^{-1}(\xi),\ \xi = -\xi_{\text{in}}]\mathbf{K}_+(\xi_{\text{in}})\mathbf{r}. \tag{37}$$

We close the integration contour involved in Eq. (34) by a large arc lying in the lower half-plane (because the exponential factor decreases in the lower half-plane) and calculate the integral by the Cauchy method (the integrand has poles and does not have any branch points). As a result, the function u_0 takes the form of a series expansion in the poles of the integrand:

$$u_0(y) = -i\sum_n \exp\{ik\theta_n y\} \\ \times \operatorname{res}[(1,0)\mathbf{K}_+^{-1}(\xi)\mathbf{F}_+(\xi), \xi = -k\theta_n]. \tag{38}$$

The contribution to the integral corresponding to the residue at the point $\xi = -\xi_{\text{in}}$, is $-ih\exp\{ik\theta_0 y\}$. Comparing Eq. (38) with Eq. (4), we obtain

$$R_0 = -ih. \tag{39}$$

4. STUDY OF THE REFLECTION COEFFICIENT IN THE LIMITING CASE

We apply Weinstein's technique to study the coefficient R_0 for $\xi_{\text{in}} \to 0$. The study is complicated by the zero points of the determinant of matrix $\mathbf{K}(\xi)$, that occur at $\xi = \pm\xi_{\text{in}}$.. We represent matrix $\mathbf{K}$ in the form

$$\mathbf{K}(\xi) = \mathbf{L}_-(\xi)\bar{\mathbf{K}}(\xi)\mathbf{L}_+(\xi),$$

where matrix $\bar{\mathbf{K}}(\xi)$ is regular as a function of two variables for small values of ξ and ξ_{in}, in particular, it has no poles at $\xi = \pm\xi_{\text{in}}$. At the same time, let matrix $\mathbf{L}_-$ together with its reciprocal matrix be regular in the lower half-plane and let it tend to $\mathbf{I}$ within this half-plane for high values of $|\xi|$. Correspondingly, let matrix $\mathbf{L}_+$ together with its reciprocal matrix be regular in the upper half-plane and tend to $\mathbf{I}$ in the corresponding half-plane for high values of $|\xi|$. Such matrices can easily be constructed, which is demonstrated below.

Let matrices $\bar{\mathbf{K}}_-$ and $\bar{\mathbf{K}}_+$ factorize matrix $\bar{\mathbf{K}}$,; i.e.,

$$\bar{\mathbf{K}}(\xi) = \bar{\mathbf{K}}_-(\xi)\bar{\mathbf{K}}_+(\xi)$$

with all the limitations. Then, evidently, we have

$$\mathbf{K}_- = \mathbf{L}_-\bar{\mathbf{K}}_-, \quad \mathbf{K}_+ = \bar{\mathbf{K}}_+\mathbf{L}_+. \tag{40}$$

Matrices $\mathbf{L}_-$ and $\mathbf{L}_+$ are constructed as follows:

$$\mathbf{L}_- = \begin{pmatrix} 1 & 0 \\ 0 & \exp\{-ic\xi_{\text{in}}\} \end{pmatrix} \\ \times \begin{pmatrix} 1 & i/(\xi - \xi_{\text{in}} - i) \\ i/(\xi - \xi_{\text{in}} - i) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp\{ic\xi_{\text{in}}\} \end{pmatrix}, \tag{41}$$

$$\mathbf{L}_+ = \begin{pmatrix} 1 & 0 \\ 0 & \exp\{ic\xi_{\text{in}}\} \end{pmatrix} \\ \times \begin{pmatrix} 1 & i/(\xi + \xi_{\text{in}} - i) \\ i/(\xi + \xi_{\text{in}} - i) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp\{-ic\xi_{\text{in}}\} \end{pmatrix}. \tag{42}$$

Let us explain the structure of matrix $\mathbf{L}_-$. The central factor is chosen to satisfy the following requirements:

(i) the matrix tends to $\mathbf{I}$ for large values of $|\xi|$,, while the determinant tends to 1;

(ii) the determinant of the matrix becomes zero at points $\xi = \xi_{\text{in}}$ and $\xi = 2i + \xi_{\text{in}}$ (both zero points lie in the upper half-plane); and

(iii) at point $\xi = \xi_{\text{in}}$, the value of the matrix is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

As a result, the left multiplication of matrix $\mathbf{K}$ by $(\mathbf{L}_-)^{-1}$ not only leads to the disappearance of the zero point of the determinant at $\xi = \xi_{\text{in}}$, but also ensures the regularity of all the elements of the product at these values.

The right-hand factor is necessary to make the entire product tend to $\mathbf{I}$ for large values of $|\xi|$.

We can easily verify that matrix $\bar{\mathbf{K}} = (\mathbf{L}_-)^{-1}\mathbf{K}(\mathbf{L}_+)^{-1}$ has no singularities in the region of small (compared to $(a+b)^{-1}$) values of ξ. For small values of variables ξ and ξ_{in}, matrices $\bar{\mathbf{K}}_+$ and $\bar{\mathbf{K}}_-$ are nonsingular and, for small values of these variables, they are described by linear increments, for example:

$$\bar{\mathbf{K}}_+(\xi) = \mathbf{K}_0 + \xi\mathbf{K}_1 + \xi_{\text{in}}\mathbf{K}_2 + \ldots. \tag{43}$$

Matrix $\mathbf{K}_0$ is nonsingular by its structure.

Let us return to estimating the coefficient R_0. We substitute Eqs. (40), (41), and (43) in Eq. (37). Note that residue $\operatorname{res}[\mathbf{K}_+^{-1}(\xi), \xi = -\xi_{\text{in}}]$ has the form following from Eqs. (40), (41), and (43),

$$\operatorname{res}[\mathbf{K}_+^{-1}(\xi), \xi = -\xi_{\text{in}}] = \frac{i}{2}\begin{pmatrix} 1 & \exp\{-ic\xi_n\} \\ \exp\{ic\xi_{\text{in}}\} & 1 \end{pmatrix} \\ \times (\mathbf{K}_0 + \xi_{\text{in}}(\mathbf{K}_2 - \mathbf{K}_1) + \ldots)^{-1},$$

and the value of $\mathbf{K}_+(\xi_{\text{in}})$ is determined as

$$\mathbf{K}_+(\xi_{\text{in}}) = (\mathbf{K}_0 + \xi_{\text{in}}(\mathbf{K}_2 + \mathbf{K}_1) + \ldots) \\ \times \begin{pmatrix} 1 & -1 + \xi_{\text{in}}(-2i + ic) + \ldots \\ -1 + \xi_{\text{in}}(-2i - ic) + \ldots & 1 \end{pmatrix}.$$

Within the first approximation in ξ_{in}, Eq. (37) yields

$$h = -i + O(\xi_{\text{in}}), \tag{44}$$

which corresponds to

$$R_0 = -1 + O(\xi_{\text{in}}). \tag{45}$$

Thus, the Weinstein method yields the result $R_0 \to -1$ for $\theta_{\text{in}} \to 0$ in the matrix case as well. This is the main result of our study. It guarantees the presence of high-Q modes in resonators similar to the resonator shown in Fig. 2.

5. CONCLUSIONS

Let us assume that we somehow succeeded in calculating the coefficient involved in Eq. (45), i.e., representing the reflection coefficient in the form

$$R_0 = -1 + \alpha\theta_{\text{in}} + o(\theta_{\text{in}}). \tag{46}$$

The method of such a calculation for the case of small c and $a = b$, was described in [9]. This method can be generalized to the case of $a \neq b$ with a small difference between a and b. Let us derive approximate impedance boundary conditions for the boundary $y = 0$ of the upper half-space. Namely, we determine the boundary conditions so that, in the case of the reflection shown in Fig. 3, the reflection coefficient is determined by Eq. (46) to a first approximation in θ_{in}.

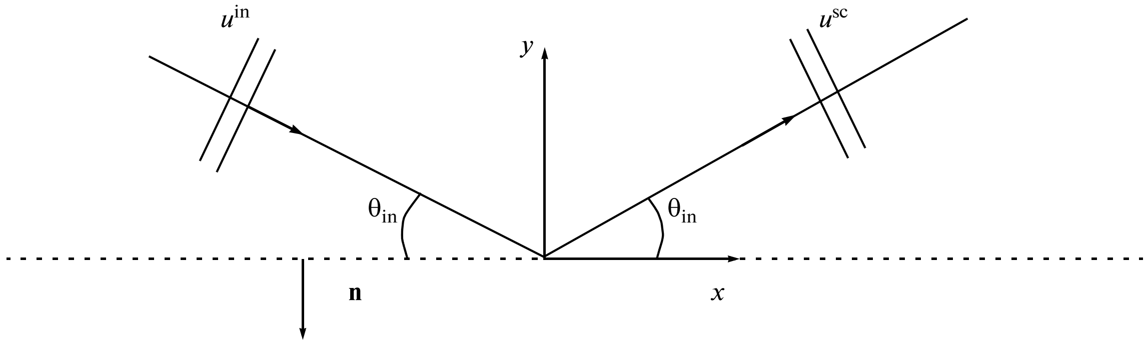

Fig. 3. Construction of impedance boundary conditions.

The impedance boundary conditions have the form

$$\frac{\partial u}{\partial n} = \eta u, \tag{47}$$

where $\partial_n = -\partial_y$, η is the impedance. Under these boundary conditions, the reflection coefficient is

$$R = -\left(1 - \frac{ik}{\eta}\sin\theta_{\text{in}}\right)\left(1 + \frac{ik}{\eta}\sin\theta_{\text{in}}\right)^{-1}. \tag{48}$$

For small values of θ_{in}, the reflection coefficient has the form of Eq. (46) at

$$\alpha = 2\frac{ik}{\eta}. \tag{49}$$

Thus, when θ_{in} is small, the behavior of the diffraction grating consisting of absorbing screens is similar to that of an impedance boundary. This approximation is more accurate, as compared to the approximation of a perfectly soft boundary. It allows for the reflection loss.

Let us summarize the results of our study. According to [10], to estimate the Q factor of the resonator shown in Fig. 2, it is necessary to calculate the scattering coefficient $R_0(\theta_{\text{in}})$ for the diffraction grating consisting of perfectly absorbing screens (Fig. 1). The grating is considered within the parabolic equation approximation of diffraction theory. We derived integral equations (15), (16) describing the system. These equations were in a standard way transformed to Wiener–Hopf–Fock matrix problem (25). We constructed a formal solution to this problem (formal, because the solution to matrix factorization problem is unknown) and derived expression (37), (39) for the coefficient R_0. We used the Weinstein method, consisting in explicit separation of singular factors, to analyze this expression and to obtain the result in the form of Eq. (45). We showed that such a reflection coefficient approximately corresponds to the impedance boundary conditions that are set at the line $y = 0$ with the impedance satisfying Eq. (49).

APPENDIX: DERIVATION OF THE OE EQUATION

Earlier, we developed another approach [8, 9] to Weinstein-type problems. It was based on the spectral equation and the so-called OE equation. The spectral equation is an ordinary differential equation for the directional pattern of the field. The solution of the OE equation consists in determining the coefficient of the ordinary differential equation (the spectral equation) from the known boundary values. The spectral and OE equations were derived in [8, 9] by using a technique that was specially developed for this purpose on the basis of the uniqueness theorem. Here, we demonstrate how the spectral and OE equations can be derived immediately from the Wiener–Hopf–Fock functional problem. We consider the case $c = 0$, which is studied in detail in [8].

We represent functional equation (26) in the form

$$\mathbf{K}'(\xi)\mathbf{K}_+^{-1}(\xi) = \mathbf{K}_-(\xi), \tag{50}$$

$$\mathbf{K}'(\xi) \equiv$$

$$\equiv \begin{pmatrix} 1 & -\exp\left\{-\dfrac{ib(\xi^2 - \xi_{\text{in}}^2)}{2k}\right\} \\ -\exp\left\{-\dfrac{ia(\xi^2 - \xi_{\text{in}}^2)}{2k}\right\} & 1 \end{pmatrix}. \tag{51}$$

All the matrices involved in Eq. (50) depend on two variables: ξ and ξ_{in}. All the functional limitations imposed on the regularity and growth concern only variable ξ. Variable ξ_{in} plays the role of a fixed parameter.

A simple analysis based on the ideas described in [8] shows that, for $|\xi| \to \infty$, the behavior of matrix $\mathbf{K}$ in the lower half-plane and at the negative real semiaxis is as follows:

$$\mathbf{K}_- = \mathbf{I} + \sum_{n=1}^{\infty} \xi^{-n}\mathbf{C}_n(\xi_{\text{in}}) \tag{52}$$

where quantities $\mathbf{C}_n$. are unknown. A similar representation is valid for $\mathbf{K}_+^{-1}$ in the upper half-plane on at the positive real semiaxis. The representation obtained for

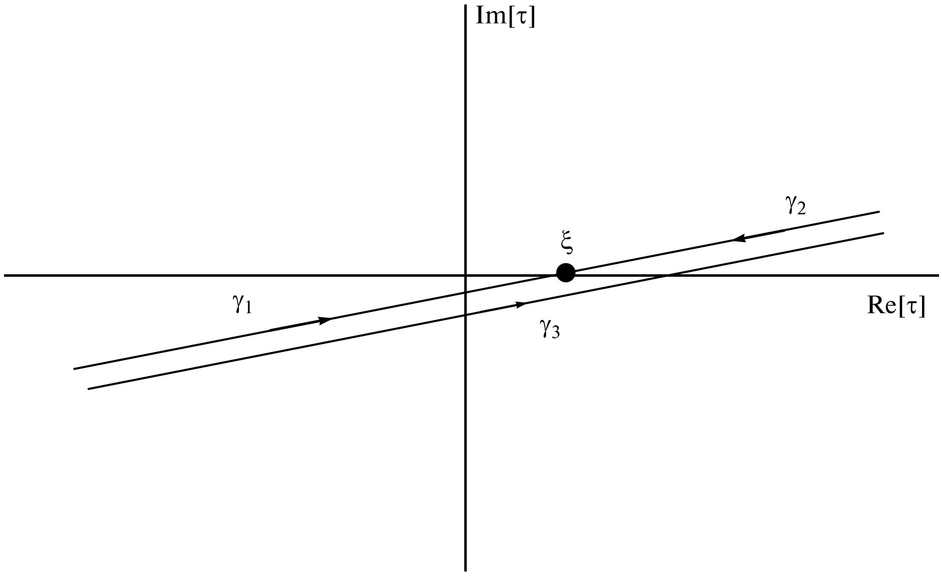

Fig. 4. Contours for derivation of OE equation.

$\mathbf{C}_1$, in [8] shows that this function rapidly decreases with increasing $\xi_{\rm in}$ within the intervals $0 < \mathrm{Arg}[\xi_{\rm in}] < \pi/2$ and $\pi < \mathrm{Arg}[\xi_{\rm in}] < 3\pi/2$.

We introduce the differential operator

$$D = \frac{\partial}{\partial\xi} + \frac{\xi}{\xi_{\rm in}}\frac{\partial}{\partial\xi_{\rm in}}. \tag{53}$$

Note that $D[\mathbf{K}'] = 0$, and, therefore,

$$\mathbf{K}'(\xi)D[\mathbf{K}_+^{-1}(\xi)] = D[\mathbf{K}_-(\xi)]. \tag{54}$$

From Eqs. (53) and (54), we obtain

$$\mathbf{K}_+D[\mathbf{K}_+^{-1}] = \mathbf{K}_-^{-1}D[\mathbf{K}_-]. \tag{55}$$

From the properties of the matrices $\mathbf{K}_+^{-1}$ and $\mathbf{K}_-$ it follows that the right- and left-hand sides of Eq. (55) involve matrix functions that are regular in variable ξ and increase at infinity no faster than a constant does. From Eq. (52), we find that this constant is

$$\mathbf{P}(\xi_{\rm in}) \equiv \frac{1}{\xi_{\rm in}}\frac{d\mathbf{C}_1}{d\xi_{\rm in}}. \tag{56}$$

Hence, matrices $\mathbf{K}_+^{-1}$ and $\mathbf{K}_-$ satisfy the differential equation

$$\left(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi_{\rm in}}\frac{\partial}{\partial\xi_{\rm in}}\right)\mathbf{A}(\xi;\xi_{\rm in}) = \mathbf{A}(\xi;\xi_{\rm in})\mathbf{P}(\xi_{\rm in}), \tag{57}$$

where $\mathbf{A}$ should be replaced by $\mathbf{K}_+^1$ or $\mathbf{K}_-$. Equation (57) is the spectral equation for the given problem.

Now, let us derive the OE equation. We change from variables $(\xi, \xi_{\rm in})$to variables $(\xi, p = \xi_{\rm in}^2 - \xi^2)$. We introduce a new dependent variable

$$\mathbf{B}(\xi, p) \equiv \mathbf{A}(\xi, \sqrt{\xi^2 + p}). \tag{58}$$

For this variable, we represent Eq. (57) as

$$\frac{\partial}{\partial\xi}\mathbf{B}(\xi, p) = \mathbf{B}(\xi, p)\mathbf{P}(\sqrt{\xi^2 + p}). \tag{59}$$

We introduce the OE notations. Let a certain differential equation

$$\frac{d}{d\xi}\mathbf{X}(\xi) = \mathbf{X}(\xi)\mathbf{F}(\xi) \tag{60}$$

with matrix coefficient $\mathbf{F}$ and unknown matrix function $\mathbf{X}$ be solved along the contour γ beginning at point ξ_1 and ending at point ξ_2. Let the initial condition be

$$\mathbf{X}(\xi_1) = \mathbf{I}.$$

We introduce the notation

$$\mathrm{OE}_\gamma[\mathbf{F}(\xi)d\xi] \equiv \mathbf{X}(\xi_2), \tag{61}$$

In fact, we introduce a notation similar to the integral notation for solving an ordinary differential equation.

Using this notation and taking into account the analyticity region of the functions $\mathbf{K}_+^{-1}$ and $\mathbf{K}_-$, we represent the latter functions in the form

$$\mathbf{K}_-(\xi;\xi_{\rm in}) = \mathrm{OE}_{\gamma_1}[\mathbf{P}(\sqrt{\tau^2 + \xi_{\rm in}^2 - \xi^2})d\tau], \tag{62}$$

$$\mathbf{K}_+^{-1}(\xi;\xi_{\rm in}) = \mathrm{OE}_{\gamma_2}[\mathbf{P}(\sqrt{\tau^2 + \xi_{\rm in}^2 - \xi^2})d\tau], \tag{63}$$

where the contours γ_1 and γ_2 are shown in Fig. 4. Both of these contours go from infinity to point ξ, but the contour γ_1 comes from the third quadrant and the contour γ_2 from the first quadrant. Precisely this choice of contours guarantees the analyticity and growth conditions for the functions $\mathbf{K}_-$ and $\mathbf{K}_+$.

Let us consider the evident properties of the OE symbol, namely: let $-\gamma$ denote the contour traveled along the contour γ but in the opposite direction. Then,

$$\mathrm{OE}_{-\gamma}[\mathbf{F}(\xi)d\xi] = (\mathrm{OE}_\gamma[\mathbf{F}(\xi)d\xi])^{-1}. \tag{64}$$

Let $\gamma_1 + \gamma_2$ be the concatenation of the contours γ_1 and γ_2 (the contour γ_1 be traveled first). Then,

$$\mathrm{OE}_{\gamma_1+\gamma_2}[\mathbf{F}(\xi)d\xi] = \mathrm{OE}_{\gamma_1}[\mathbf{F}(\xi)d\xi]\mathrm{OE}_{\gamma_2}[\mathbf{F}(\xi)d\xi]. \qquad (65)$$

Combining Eqs. (64) and (65) and taking into account the functional equation $\mathbf{K}_-\mathbf{K}_+ = \mathbf{K}$,, we obtain

$$\mathrm{OE}_{\gamma_3}[\mathbf{P}(\sqrt{\tau^2 + p})d\tau] = \begin{pmatrix} 1 & -\exp\{ibp/(2k)\} \\ -\exp\{iap/(2k)\} & 1 \end{pmatrix}, \qquad (66)$$

where the contour $\gamma_3 = \gamma_1 + (-\gamma_2)$ is shown in Fig. 4. Equation (66) is the OE equation for the given system. It represents the problem of determining the unknown coefficient $\mathbf{P}(\xi)$ according to the boundary values given by Eq. (66) for different values of the parameter p.

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