To commutative factorization of algebraic matrices

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Abstract

The problem of matrix factorization motivated by diffraction or elasticity is studied. A powerful tool for analyzing its solutions is introduced, namely analytical continuation formulae are derived. Necessary conditions for commutative factorization are found; a link with previous works is established.

Keywords: matrix factorization, Wiener-Hopf method

1 Introduction

A matrix factorization problem is usually motivated by elasticity or wave diffraction and typically its formulation does not contain a requirement of *commutative* factorization. However, the possibility to perform a commutative factorization is usually studied very carefully. The reason for this is intuitively clear. Commutating matrices (under some other conditions) have a common set of eigenvectors, thus matrix logarithms can be computed by taking logarithms of the eigenvalues. So the matrix problem becomes reduced to several scalar ones.

The authors here had in mind the following targets:

— to introduce a powerful tool for studying properties of solutions of matrix factorization problems even in cases when solutions themselves are not known;

— to introduce new formulations of matrix factorization problems;

— to formulate a condition, under which a commutative factorization is possible.

A condition, under which a commutative factorization is possible, has been studied in details for matrices 2×2 [4]. A necessary condition for commutative factorization has been found in an "Ansatz" form, i.e. a matrix should have a specific representation including some entire (polynomial) matrices and some arbitrary functions as coefficients. An explicit form of factorization for such matrices has been found by Khrapkov [8]. Some cases of matrices with dimension more than 2×2 have been investigated by Lukyanov [10], Lewis et al. [9] and some other authors using modifications of Khrapkov's approach.

A weak point of Khrapkov's solution is a behaviour of factors at infinity. For some cases factors have polynomial growth, but in general the elements of the decomposition grow exponentially. Various techniques have been proposed to overcome this difficulty. Most known methods have been suggested by Daniele [5] and Abrahams [1]. Alternative approach based on Jacobi's inversion problem was described by Zverovich [11], Antipov and Silvestrov [2, 3].

More general $(N \times N)$ case has been studied by Jones [7]. He also obtained a necessary condition in the "Ansatz" form, but his theory is valid for the matrices having distinct eigenvalues everywhere. In fact, practically interesting matrices have distinct eigenvalues *almost* everywhere (everywhere except the branch points of characteristic equation), so the formulation of Jones' theorem is excessively strict, as well as his Ansatz for the entire matrix is. Below we propose a refined formulation of Jones' theorem. However, we should note that the changes of Jones' result happen to be not too big.

In the current paper we are focused mainly on a necessary condition of commutative matrix factorization. Constructive methods (leading to sufficient conditions) are beyond the scope of this work. Also we study only algebraic matrices, and even sometimes restrict the class of matrices to the so called "diffraction" ones, who are algebraic matrices of a special form peculiar to diffraction theory.

For an algebraic matrix of a big dimension it is not always easy to find a good "Ansatz" representation. That is why we pay some attention to "check-up" conditions of commutative factorization. I.e. we explicitly describe the manipulations giving the answer, whether the matrix is factorizable or not.

Certainly, a work on such a subject cannot appear without being motivated by previous research of other authors. Where it is possible we mention the parallels to (and sources of) our ideas.

The paper is organized as follows. In Section 2 we formulate the problem of matrix factorization. A definition for diffraction matrices is given there.

In Section 3 we describe Riemann surface of a solution of a matrix factorization problem (commutative or not). Explicit notation is introduced for bypasses about branch points. Formulae of analytical continuation are derived. These formulae connect values of unknown functions on the physical sheet with values on all other sheets. We introduce a set of *bypass matrices* describing the transformations of unknown functions happening when the argument is carried about branch points.

In Section 4 we formulate a necessary condition for commutative factorization in the check-up form, and then reformulate the same conditions in the Ansatz form. Connections with Jones' theory are discussed there.

In Section 5 we study the group structure existing in the set of bypass matrices. We reformulate the factorization problem taking into account this structure and discuss Hurd's ideas [6], which get a clear interpretation in our terms.

2 Problem formulation and important notations

2.1 Problem under consideration

The initial problem of matrix factorization is as follows:

Problem 1 (boundary factorization problem) *For a matrix $G(k)$ defined in a nar-*

row strip along the real axis ($-\epsilon < \mathrm{Im}\,[k] < \epsilon$) *find matrices* $Q^+(k)$, $Q^-(k)$ *analytical (maybe except some isolated poles), continuous, and having algebraic growth in the upper* ($\mathrm{Im}\,[k] > -\epsilon$) *and lower* ($\mathrm{Im}\,[k] < \epsilon$) *half-planes, respectively, and satisfying the equation*

$$G(k) = Q^+(k)Q^-(k). \tag{1}$$

Algebraic growth hereafter means that there exists a number l, such that all elements of corresponding matrices grow at the corresponding half-plane no faster than $|k|^l$. We cannot expect that the elements will grow exactly as some powers of k, since the elements of the solutions can have logarithmic behaviour.

Assume everywhere that the determinant of G is not equal to zero identically.

In some places below we specify the form of G in more details. Namely, we introduce a class of *diffraction* matrices, which are typically coefficients of the Wiener-Hopf problems appearing in diffraction and elasticity:

Definition 1 (diffraction matrix) *If elements of matrix* $G(k)$ *are rational functions of* k *and of square roots* $\sqrt{\tau_m^2 - k^2}$, $m = 1 \dots s$, $\mathrm{Re}\,[\tau_m] > 0$ *then it is called a diffraction matrix.*

3 Formulae of analytical continuation. Bypass matrices

3.1 Notations for bypasses

Let $\mathcal{R}_G$ be the Riemann surface of matrix $G(k)$, i.e. $\mathcal{R}_G$ is such surface that every element of the matrix $G(k)$ is a single-valued function on this surface. Below we shall call k an *affix* of a point $(k, G(k)) \in \mathcal{R}_G$.

Let branch points of $G(k)$ have affixes τ_m^+ and τ_m^-, where $\mathrm{Im}\,[\tau_m^+] > 0$ and $\mathrm{Im}\,[\tau_m^-] < 0$. Each affix has its order $n_m^\pm$, which is the least common multiple of all orders of branch points with corresponding affix.

Make G single-valued on $\mathbb{C}$ by performing cuts going from branch points to infinity. The cuts can be chosen as $\gamma_m^+ = (\tau_m^+, +\mathrm{i}\infty)$ and $\gamma_m^- = (\tau_m^-, -\mathrm{i}\infty)$. It is important that the cuts should not cross the real axis and each other. As a result, the surface $\mathcal{R}_G$ becomes split into several sheets. There is a special sheet of $\mathcal{R}_G$, on which equation (1) is assumed to be valid. Name this sheet a *physical sheet*.

Introduce a notation for the sheets of $\mathcal{R}_G$. Note that later the same notation will be used for the sheets of the Riemann surfaces of $Q^\pm$. Each point of the surface will be denoted by $(k)\{w\}$, where k is the affix, and $\{w\}$ is a *word* describing the path, along which the argument k should be carried from physical sheet to a selected sheet. The structure of this word is explained below.

Denote bypasses about points τ_i^+ in positive direction by letters a_i and bypasses about points τ_i^- in positive direction by b_i (fig. 1).

A series of consecutive bypasses will be denoted by a word of letters a_i and b_i. The word must be read from left to right, i.e. the first performed bypass corresponds to the

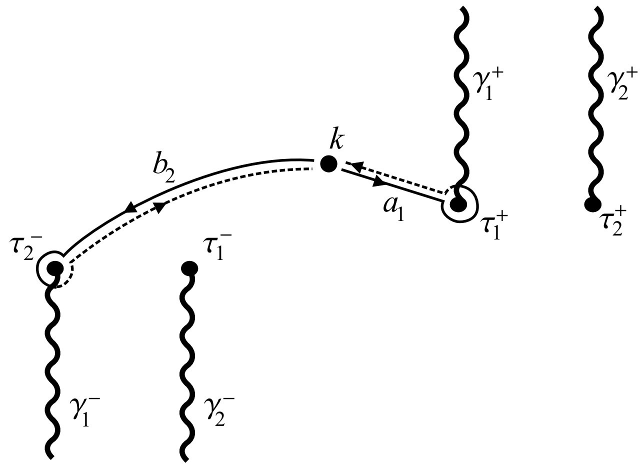

Figure 1: Notation for bypasses

left end of the word and the last bypass corresponds to the right end. By default a series of bypasses begins from the "physical sheet". A trivial bypass will be denoted by letter e.

Define the composition of words w and v as the bypass performed along the way composed of w and v. The bypass w is performed first. Denote this composition by wv. Let $\mathcal{W}$ be the set of all words, and let $\mathcal{W}_a$, $\mathcal{W}_b$ be the sets of words composed only of the letters a_i, and only of letters b_i, respectively.

Let $G(k)\{e\}$ be the value of function $G(k)$ on the "physical sheet". Denote by $G(k)\{w\}$ the value of $G(k)$ on the sheet that can be reached by performing the bypass w starting from the point $(k, G(k)\{e\})$.

The set $\mathcal{W}$ can be considered as a *group* of words, a subject of combinatorial group theory. Its generators are the letters a_i, b_i, and the relations have the form

$$a_i^{n_i^+} = e, \qquad b_i^{n_i^-} = e. \tag{2}$$

As we shall see below, the same relations are valid for the words describing the Riemann surfaces of $Q^{\pm}$.

Relations (2) enable one to determine an inverse element for each $w \in \mathcal{W}$ without introducing new letters for bypasses in negative direction (or without using the symbols a_i^{-1} and b_i^{-1}). In an important particular case of a diffraction matrix G, obviously we have $a_i^{-1} = a_i$; $b_i^{-1} = b_i$. Using (2), below we assume that for each word w there exists a word w^{-1}, such that $ww^{-1} = w^{-1}w = e$.

Let us demonstrate an example of Riemann surfaces for G and Q^+. Take matrix G from Daniele's paper [5]:

$$G(k) = \begin{pmatrix} 1 & \frac{k_1 - s(k)}{k_2 + s(k)} \\ \frac{k_2 - s(k)}{k_1 + s(k)} & 1 \end{pmatrix}, \tag{3}$$

where $s(k) = \sqrt{k_0^2 - k^2}$; k_0, k_1 and k_2 are some complex constants.

In this case the Riemann surface of $G(k)$ has two sheets and two quadratic branch points, namely $k = \pm k_0$. Let be $\mathrm{Re}\,[k_0] > 0$. Let letter a denote a bypass about k_0, and letter b denote a bypass about $-k_0$.

Traditionally, the structure of Riemann surface is displayed by graphical diagrams. Horizontal lines correspond to sheets (cut along $\gamma_j^{\pm}$), nodes correspond to branch points, and vertical lines link sheets, which are connected at a branch point.

The scheme for the Riemann surface of G is shown in Fig. 2 a. The upper sheet is physical (i.e. it contains the "physical" real axis).

The scheme of Q^+ corresponding to this problem is shown in Fig. 2 b. The number of sheets is infinite, but all branch points are of second order, and positive physical half-plane contains no branch points. This structure can be revealed, e.g. from [5].

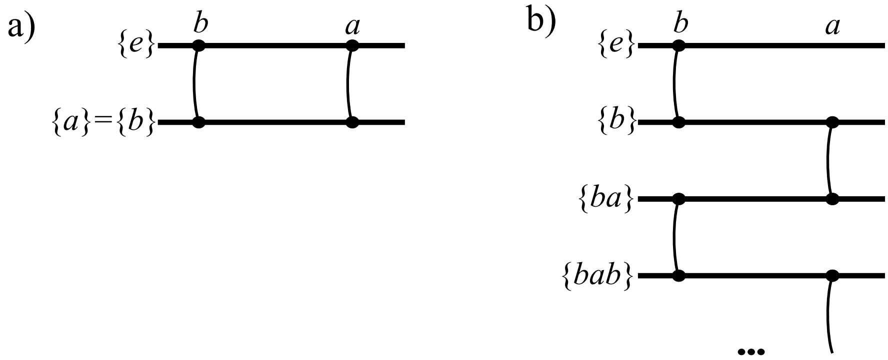

Figure 2: Diagrams of Riemann surfaces for $G(k)$ and $Q^+(k)$

3.2 Truncation operators

Let be $w = \alpha_1\alpha_2\dots\alpha_n$ where α_i substitutes an arbitrary single letter. Denote by p the maximal number, such that the word $\alpha_1\alpha_2\dots\alpha_p \in \mathcal{W}_a$. Analogically let m be the maximal number, such that $\alpha_1\alpha_2\dots\alpha_m \in \mathcal{W}_b$. Obviously, one of this integers is zero, since the first letter of the word is either a_j or b_j.

Define the truncation operators $^+$ and $^-$ by

$$w^+ = \alpha_{p+1}\alpha_{p+2}\dots\alpha_n,$$
$$w^- = \alpha_{m+1}\alpha_{p+2}\dots\alpha_n.$$

For example, applying operators $^+$ and $^-$ to the words $w = a_1a_2b_1b_2$, $v = b_1b_2a_1a_2$ we obtain

$$w^+ = b_1b_2, \qquad w^- = w = a_1a_2b_1b_2, \qquad w^{+-} \equiv (w^+)^- = e,$$
$$v^- = a_1a_2, \qquad v^+ = v = b_1b_2a_1a_2, \qquad v^{-+} = e.$$

3.3 Derivation of the formulae of analytical continuation

Consider boundary functional equation (1). Both right and left sides of this equation are analytic functions in some neighbourhood of the real axis of the physical sheet. Continue Q^+ and Q^- analytically to this domain and, further, onto some Riemann surfaces. Continue also the relation (1) onto the Riemann surfaces of G, Q^+ and Q^-. Obviously, the continuation of the relation (1) can be written in the form:

$$Q^{+}(k)\{w\}\,Q^{-}(k)\{w\} = G(k)\{w\}. \tag{4}$$

At this moment we have not proved yet that functions Q^+ and Q^- have no branch points except $\tau_i^{\pm}$, so formula (4) has sense only for geometrically fixed bypasses.

Here we are going to find the *formulae of analytical continuation* for $Q^{\pm}$, i.e. algebraic relations connecting $Q^{\pm}(k)\{w\}$ with $Q^{\pm}(k)\{e\}$.

First, as an example, obtain a formula of analytical continuation of Q^+ into the lower physical half-plane. Since the determinant of matrix G is not equal to zero identically, the determinants of matrices Q^+ and Q^- are not equal to zero identically too. Using the expression

$$Q^{+}(k)\{e\} = G(k)\{e\}\,(Q^{-}(k)\{e\})^{-1}, \tag{5}$$

we obtain the formula of analytical continuation of Q^+ onto the lower half-plane of the physical sheet. Since the function $Q^-(k)$ has no branch points in the lower half-plane we conclude that the branch points of matrix Q^+ in the lower half-plane should coincide with the branch points of matrix G. Moreover, behaviour of unknown function Q^+ at the branch points is defined by the behaviour of function G at the same points, e.g. if G has branch points of second order then Q^+ should have branch points of second order also.

General formulae of analytical continuation can be written in a recursive form as follows:

Theorem 1 *Let $Q^+(k)$ and $Q^-(k)$ form a solution of Problem 1. Then the following relations are valid*

$$Q^{+}\{w\} = G\{w^{+}\}\,G^{-1}\{w^{+-}\}\,Q^{+}\{w^{+-}\}, \tag{6}$$

$$Q^{-}\{w\} = Q^{-}\{w^{-+}\}\,G^{-1}\{w^{-+}\}\,G\{w^{-}\}. \tag{7}$$

(A dependence on k is implied for all functions in (6), (7)).

Note that for any word w their exists some constant c, such that $w^{(+-)^c} = e$, therefore formula (6) being repeated several times connects $Q^+\{w\}$ with $Q^+\{e\}$. Analogously, $Q^-\{w\}$ is connected with $Q^-\{e\}$. The coefficients are always products of known matrices.
Proof: We are starting from regularity conditions

$$Q^{+}\{w_a\} = Q^{+}\{e\} \quad \text{for} \quad w_a \in \mathcal{W}_a; \qquad Q^{-}\{w_b\} = Q^{-}\{e\} \quad \text{for} \quad w_b \in \mathcal{W}_b$$

and following from them relations with truncation operators

$$Q^{+}\{w\} = Q^{+}\{w^{+}\}, \qquad Q^{-}\{w\} = Q^{-}\{w^{-}\}, \tag{8}$$

which are obvious.

Denote by $|w|$ the length of word w (note that the word $\{e\}$ has zero length). Consider two following cases:

1. Let be $|w^+| < |w|$. Using (8) we represent the function $Q^+\{w\}$ through the function $Q^+\{w^+\}$, which depends on the word of smaller length.

2. Let be $|w^+| = |w|$ and therefore $|w^-| < |w|$. Applying equation (4) to the sheets labelled by words w and w^- obtain

$$G\{w\} = Q^+\{w\}\, Q^-\{w^-\}, \tag{9}$$

$$G\{w^-\} = Q^+\{w^-\}\, Q^-\{w^-\}. \tag{10}$$

Using equations (8) obtain

$$Q^+\{w\} = G\{w\}(G\{w^-\})^{-1}Q^+\{w^-\}.$$

Combining both cases, we obtain equation (6). Relation (7) can be derived similarly. Theorem 1 is proved.

Analytical continuation in the form (6) has been obtained by Hurd [6] for a particular case of a single bypass. Hurd's ideas are discussed later in details.

Using analytical continuation we can investigate the structure of Riemann surface of unknown function Q^+. For example, the following proposition can be easily proved:

Proposition 1 *Let $G(k)$ be a diffraction matrix, and let the functions $Q^+(k)$ and $Q^-(k)$ form a solution of Problem 1. Then both functions $Q^+(k)$ and $Q^-(k)$ can be analytically continued onto some Riemann surfaces; both functions have branch points only at affixes $\tau_i^\pm$. The order of each branch point is a divisor of corresponding $n_i^\pm$.*

A formal proof can be conducted by induction with respect to the length of the word w, which is the argument of $Q^\pm(k)\{w\}$.

Generally, solution of Problem 1 is not unique: for example the behaviour of different solutions at infinity can be different. However, all solutions are similar up to a meromorphic factor:

Proposition 2 *Let Problem 1 have two different solutions, namely $(Q_1^+(k), Q_1^-(k))$ and $(Q_2^+(k), Q_2^-(k))$. Then there exists a matrix $K(k)$ meromorphic at all finite points of $\mathbb{C}$, such that*

$$Q_2^+(k) = Q_1^+(k)K(k), \qquad Q_2^-(k) = K^{-1}(k)Q_1^-(k). \tag{11}$$

Proof: Consider the matrix

$$K(k) = (Q_1^+(k))^{-1}Q_2^+. \tag{12}$$

It is clear that matrix K can have branch point on the physical sheet only in the lower half-plane. Apply equation (1) continued onto the whole physical sheet. As the result the same matrix becomes expressed in another form:

$$K = Q_1^-(k)(Q_2^-(k))^{-1}. \tag{13}$$

According to this representation, matrix K can have branch points only in the upper half-plane. Therefore, this matrix has no branch points. Proposition is proved.

The idea to use such matrix K to compensate the growth of the solution belongs to Daniele [5].

3.4 Bypass matrices and reformulation of factorization problem

Let G be an algebraic matrix, and let $\mathcal{W}$ be the set of words associated with this matrix.

Definition 2 *A bypass matrix $P_w(k)$ for a word w is defined by the relation:*

$$Q^+(k)\{w\} = P_w(k)\, Q^+(k)\{e\}. \tag{14}$$

According to the formulae of analytical continuation (6),

$$P_w(k) = G\{w^+\}G^{-1}\{w^{+-}\}G\{w^{+-+}\}G^{-1}\{w^{+-+-}\}\dots \tag{15}$$

Denote by $P_w(k)\{v\}$ the value of the matrix $P_w(k)$ continued along the path $v \in \mathcal{W}$ (formally, $P_w(k) = P_w(k)\{e\}$).

Proposition 3 *Let $\mathcal{W}$ be a group of words; let the alphabet consist of the letters a_i and b_i associated with the affixes $\tau_i^\pm$. Let $P_w(k)$ be a set of algebraic matrices having branch points only at the affixes $\tau_i^\pm$. Let be*

$$P_e \equiv I,$$

(I is the identity matrix), and for any two words w and v

$$P_{wv}(k) = P_w(k)\{v\}\ P_v(k). \tag{16}$$

Let $T(k)$ be a matrix function with a selected physical sheet, having branch points on the physical sheet only at $\tau_i^\pm$, and

$$T(k)\{\alpha_i\} = P_{\alpha_i}(k)\, T(k)\{e\}$$

for any single letter α_i.

Then $T(k)$ has branch points only at τ_i, and the formula of analytical continuation for $T(k)$ has the form (15).

A formal proof of Proposition 3 can be easily performed by induction with respect to the length of w.

Proposition 4 *Let $G(k)$ be an algebraic matrix with a selected physical sheet, and P_w be a set of matrices defined by the relations (15). Then the set P_w obeys the condition (16).*

The proof of this proposition is rather straightforward, but quite lengthy, so we do not put it here.

Matrices P_w generally do not form a representation of the group $\mathcal{W}$, since one of the factors in relation (16) depends on the second one. Later (in Section 5) we shall introduce a proper group structure on $\mathcal{W}$.

Let us demonstrate here two immediate consequences of Propositions 3 and 4. These formulae will be used below.

1. If v is an arbitrary word and $w_a \in \mathcal{W}_a$ then

$$P_{w_a v}(k) = P_v(k). \tag{17}$$

2. For any word w and its inverse w^{-1}

$$P_{ww^{-1}} = I. \tag{18}$$

Problem 2 (bypass problem) *Let $G(k)$ be an algebraical matrix, let $\tau_i^{\pm}$ be affixes of its branch points, and let $\mathcal{W}$ be a corresponding group of words. Let $P_w(k)$ be a set of bypass matrices, associated with the matrix $G(k)$ and constructed by relation (15). Find a function $Q^+(k)$ having branch points only with affixes $\tau_m^{\pm}$, obeying relation (14), and having algebraic growth at infinity.*

The equivalence of the problems Problem 1 and Problem 2 can be proved by construction of the set P.

4 Necessary condition for commutative matrix factorization

4.1 Necessary condition in "check-up" form

The formulation of Problem 1 does not contain a requirement of commutative factorization. Here, however, we try to find an answer for the following question: Under which condition a commutative factorization is possible, i.e. when there exist matrices $Q^{\pm}(k)$, such that

$$Q^+(k)\, Q^-(k) = Q^-(k)\, Q^+(k) = G(k) \tag{19}$$

on the real axis of the physical sheet?

We shall call the relation $Q^+(k)Q^-(k) = G(k)$ right factorization, and $Q^-(k)Q^+(k) = G(k)$ left factorization.

First introduce a definition:

Definition 3 *Algebraical matrix $G(k)$ is called commutative, if for any k the values of G on different sheets of its Riemann surface commute:*

$$G_1(k)G_2(k) = G_2(k)G_1(k).$$

In a word notation this condition has form

$$[G(k)\{w_1\}, G(k)\{w_2\}] \equiv G(k)\{w_1\}G(k)\{w_2\} - G(k)\{w_2\}G(k)\{w_1\} = 0. \tag{20}$$

for any different words w_1 and w_2.

Note that our definition of commutativeness is weaker than Chebotarev's definition of functional commutativeness [4].

The necessary condition of commutative factorization is given by the following theorem:

Theorem 2 *If a diffraction matrix G admits commutative factorization, then it is a commutative matrix.*

Proof: The formula of analytical continuation (6) has been derived for right factorization. One can obtain a similar formula for left factorization:

$$Q^+\{w\} = Q^+\{w^{+-+-}\}\, G^{-1}\{w^{+-}\}G\{w^+\}. \tag{21}$$

Perform the rest of the proof step by step. Here we mark the statements and make some comments if the statements are not obvious:

1. For any word w $\quad Q^+\{w\}Q^-\{w\} = Q^-\{w\}Q^+\{w\} = G\{w\}$. It is an analytical continuation of (19).

2. For any word w $\quad [G\{w\},(Q^+\{w\})^{-1}] = 0$. This can be obtained by multiplying the previous relation by $(Q^+\{w\})^{-1}$.

3. For any word w $\quad [G\{w\},Q^+\{w\}] = 0$. This can be obtained from the previous point by multiplication by G^{-1} at left and right.

4. For any $v \in \mathcal{W}_a$ $\quad [G\{v\},Q^+\{e\}] = 0$. This follows from the previous point and (8).

5. For any word w $\quad [G\{w\},Q^+\{e\}] = 0$. Note that for a diffraction matrix for any word w there exists a word $v \in \mathcal{W}_a$, such that $G\{v\} = G\{w\}$.

6. For any $v_1 \in \mathcal{W}_b$, $v_2 \in \mathcal{W}_a$ $\quad [G\{v_1v_2\},G^{-1}\{v_2\}] = 0$. This statement can be obtained by applying left and right analytical continuation formulae to the word v_1v_2 and by using the previous point.

7. For any $v_1 \in \mathcal{W}_b$, $v_2 \in \mathcal{W}_a$ $\quad [G\{v_1v_2\},G\{v_2\}] = 0$.

8. The statement of the theorem, by noting that for any w_1 and w_2 one can find the words $v_b \in \mathcal{W}_b$ and $v_a \in \mathcal{W}_a$, such that $G\{v_bv_a\} = G\{w_1\}$, $G\{v_a\} = G\{w_2\}$.

Theorem 2 is the main result of the paper. Note that since the number of sheets of G is finite, the necessary condition can be checked by checking a finite number of matrix identities.

4.2 Diagonalization and properties of eigenvectors

Let a *diffraction* matrix $G(k)$ have distinct eigenvalues almost everywhere (i.e. on the whole complex plane excluding several points). Represent this matrix in the form

$$G(k) = M(k)\,\mathrm{diag}\{\lambda_1, \dots \lambda_N\}M(k)^{-1} \tag{22}$$

Here matrix $M(k)$ consists of vector-columns, which are right eigenvectors of G; $\lambda_1 \dots \lambda_N$ are corresponding eigenvalues; N is dimension of G. Normalize the columns of M by making all elements of the first raw of M equal to 1.

Obviously, for obtaining representation (22) one should first solve the characteristic equation for G, and then find a solution of an inhomogeneous linear system for each eigenvector.

Denote Riemann surface of matrix $M(k)$ by $\mathcal{R}_M$. Now we have associated with a matrix G two Riemann surfaces: $\mathcal{R}_G$ and $\mathcal{R}_M$. Typically, say for Khrapkov matrices, $\mathcal{R}_M$ has a structure very different from $\mathcal{R}_G$.

Different authors studied matrix factorization problems by formulating a functional problem on a Riemann surface. It is important to mention that most of them had in mind the surface $\mathcal{R}_M$, not $\mathcal{R}_G$.

Obviously, Riemann surface for the eigenvalues $\lambda_j(k)$ should contain branch points of both structures, i.e. of G and of M.

Let $G(k)$ be a *commutative* matrix. In this case the set of normalized eigenvectors must be the same on all sheets of $\mathcal{R}_M$. Therefore, matrix $M(k)$ possesses an important property: any bypass about branch points leads to a permutation of the columns, i.e. an analytical continuation of each column along a closed contour c on $\mathbb{C}$ is some other column of M.

As an example, consider matrix (3), which is commutative. As it was mentioned, it has only two branch points, namely $\pm k_0$. The scheme of Riemann surface for this matrix is shown is Fig. 2 a. It is easy to find that matrix M for this G is as follows:

$$M(k) = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{k_0^2-k^2-k_2^2}}{\sqrt{k_0^2-k^2-k_1^2}} & -\frac{\sqrt{k_0^2-k^2-k_2^2}}{\sqrt{k_0^2-k^2-k_1^2}} \end{pmatrix}. \tag{23}$$

Matrix M has four branch points, namely $\pm\sqrt{k_0^2 - k_1^2}$ and $\pm\sqrt{k_0^2 - k_2^2}$. Generally (i.e. if $k_1 \neq 0$ and $k_2 \neq 0$) the branch points of $\mathcal{R}_M$ are different from the branch points of $\mathcal{R}_G$. The scheme of $\mathcal{R}_M$ is shown in Fig. 3.

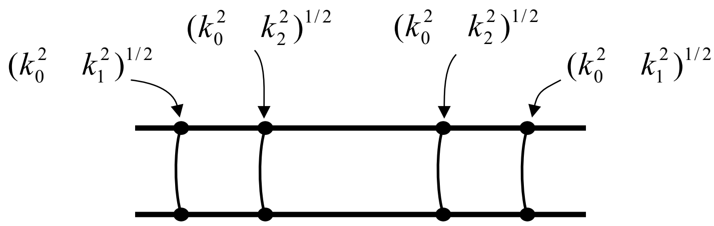

Figure 3: Diagram of $\mathcal{R}_M$

A transition from one sheet of $\mathcal{R}_M$ to another leads to a permutation of the columns of M.

4.3 "Ansatz" form of necessary condition

Theorem 3 *Let G be a commutative matrix $N \times N$, whose eigenvalues are distinct almost everywhere. Then it can be represented in the form*

$$G = \sum_{m=0}^{N-1} g_m(k) A^{m-1}(k), \tag{24}$$

where $A(k)$ is a rational matrix, $g_m(k)$ are algebraic functions. Vice versa, any matrix admitting a decomposition of the form (24) is commutative.

Proof: The second part of the theorem is obvious, so we are concentrating our efforts on the first one. Consider matrix $M(k)$. Let π_c be a permutation of columns of M occuring when the argument is carried along a contour c on $\mathbb{C}$ starting and terminating at k. Let Π_c be a matrix containing only numbers 0 and 1, describing permutation π_c in matrix language, i.e.

$$(\Pi_c)_n^m = \delta_{m,\pi_c(n)}, \tag{25}$$

and the permutation of columns of M looks like $M \to M\Pi_c$.

Construct N functions $f_m(k)$, $m = 1 \dots N$ as follows. Take N constants $\beta_1 \dots \beta_N$ such that the combinations

$$f_m(k) = \sum_{n=1}^{N} \beta_n (M)^n_m, \tag{26}$$

almost everywhere obey the relation $f_{m_1}(k) \neq f_{m_2}(k)$ as $m_1 \neq m_2$. (Here $(M)^n_m$ are the elements of M.) Obviously, $f_m \to f_{\pi_c(m)}$ when the argument is carried along c.

Construct a combination

$$A(k) = M(k)\,\mathrm{diag}\{f_1(k), \dots, f_N(k)\} M^{-1}(k). \tag{27}$$

Note that the diagonal matrix obeys the relation

$$\mathrm{diag}\{f_{\pi_c(1)}, \dots, f_{\pi_c(N)}\} = \Pi_c^{-1}\,\mathrm{diag}\{f_1, \dots, f_N\}\Pi_c. \tag{28}$$

Substituting (25) and (28) into (27), conclude that A remains unchanged after any bypass c. Since A is an algebraic matrix by construction, it should be a rational matrix.

Finally, let us show that G can be expressed in the form (24) with matrix A constructed above. The matrix composed of the elements $(F)^m_n = f_n^{m-1}$ (here $m-1 = 0 \dots N-1$ is a power) has a non-zero determinant almost everywhere. In the opposite case it would happen that N distinct numbers are roots of a polynomial of order smaller than N. Therefore any set of N numbers, for example the eigenvalues of G, can be represented as

$$\lambda_n(k) = \sum_{m=1}^{N} g_m(k) f_n^{m-1}(k) \tag{29}$$

for almost all k. By construction, g_n are algebraic functions.

The theorem is proved.

The form (24) is close to that of [7], however on one hand we impose no restrictions on the behaviour of the matrices $Q^\pm$, and on the other hand, we do not specify the form of equation, which matrix A should obey. Theorem 3 together with Proposition 2 provide a modified Jones' result.

Theorem 3 states that there are two alternative ways to check, whether a diffraction matrix G can be factorized commutatively: 1) by checking whether a matrix can be represented in a certain form, and 2) by checking conditions (20) between different sheets. The second variant seems more easy.

4.4 Formulation of scalar bypass problem for eigenvalues

Let c be a bypass on $\mathbb{C}$, starting and terminating at k. Define two objects depending on c. The first object is the word w_c, describing the bypass, which is the rising of c onto R_G. The second object is the permutation π_c of the columns of matrix M explained in the previous subsection.

Write the bypass matrices $P_w(k)$ in a diagonal form:

$$P_w = M\,\mathrm{diag}\{p_{w,1}, \dots, p_{w,N}\} M^{-1}, \tag{30}$$

where $p_{w,m}(k)$ are certain algebraic functions of k.

Write the solution Q^+ of the matrix factorization problem (1) also in a diagonal form

$$Q^+ = M \operatorname{diag}\{q_1^+, \dots, q_N^+\} M^{-1}, \tag{31}$$

where $q_m^+(k)$ are unknown eigenvalues.

Reformulate Problem 2 for eigenvalues. Let $(q_m^+)_c$ be the value of q_m^+ continued along the contour c. Taking into account the permutations of the eigenvectors, the condition (14) can be written in the form

$$(q_m^+)_c = p_{w_c,m}\, q^+_{\pi_c(m)}. \tag{32}$$

Now we can formulate the problem for the eigenvalues:

Problem 3 *Find N functions $q_1^+, \dots, q_N^+$, growing algebraically at infinity, whose analytical continuations obey relation (32) for any contour c.*

The problem formulated here is a set of scalar problems. After some algebraic manipulations it can be reduced to Jacobi's inversion problem. The way to do this is close to the methods described in details in [12, 2] and some other papers. Here, however, we are not focused on solving this problem.

5 Group structure on bypass matrices

5.1 Closed words

Previously we introduced word notation for bypasses by their projections onto $\mathbb{C}$. Each bypass was closed on $\mathbb{C}$, but not necessarily closed when risen to $\mathcal{R}_G$. Here we study subset of $\mathcal{W}$ consisting of all bypasses closed on $\mathcal{R}_G$. We shall name these bypasses *closed bypasses* and corresponding words *closed words*. For example, if G is a scalar function $G(k) = \sqrt{\tau^2 - k^2}$, letters a and b denote bypasses about τ and $-\tau$, then the words ab, $abab$, ba are closed, and the words a, b, aba are not closed.

A formal definition is as follows:

Definition 4 *Word w is closed if*

$$G(k)\{w\} = G(k)\{e\}.$$

The set of all closed words will be called $\mathcal{W}_c$. Obviously, $\mathcal{W}_c$ is a subgroup of $\mathcal{W}$.

For each word w define its *class* $S(w)$ by the following property:

$$v \in S(w) \qquad \text{iff} \qquad G\{w\} = G\{v\}.$$

It is clear that the number of classes is equal to the number of sheets of G. All closed words form the class $S(e)$.

In this section we restrict our consideration to diffraction matrices. Let us formulate several obvious properties of such matrices. First, for any two words v and w

$$G\{wv\} = G\{vw\}. \tag{33}$$

This follows from the fact that each square root in G has this property.

Second, if v is an arbitrary word, and w is a closed word, then

$$P_v\{w\} = P_v. \tag{34}$$

This relation can be checked by combining (15) and (33).

5.2 Further reformulations of factorization problem

Consider a restriction of Problem 2 to the set $\mathcal{W}_c$:

Problem 4 (bypass problem on closed words) *Let $G(k)$ be a diffraction matrix, and $P_w(k)$ be the set of corresponding bypass matrices. Find a function $H^+(k)$ having algebraic growth at infinity, having only branch points of order 2 at $k = \tau_m^\pm$, and obeying analytical continuation formulae:*

$$H^+\{w\} = P_w(k)\ H^+\{e\} \tag{35}$$

for all words $w \in \mathcal{W}_c$.

Note that function H^+ is not necessarily analytical on the positive physical half-plane.

Although the condition of Problem 4 seems to be weaker than the condition of Problem 2, the following theorem states that a solution of Problem 2 can be obtained from almost every solution of Problem 4 by the procedure of symmetrization:

Theorem 4 *Let $H^+(k)$ be a solution of Problem 4. Let $\mathcal{W}_{ar}$ be a (finite) set of words $w \in \mathcal{W}_a$, such that there is exactly one word in $\mathcal{W}_{ar}$ belonging to each class S.*

Construct function $\bar{Q}^+(k)$ by the relation

$$\bar{Q}^+(k)\{w\} = \sum_{w_j \in \mathcal{W}_{ar}} H^+(k)\{w_j w\}. \tag{36}$$

Function $\bar{Q}^+$ is either identically equal to zero or it is a solution of Problem 2 for Q^+.

Proof: First, prove that the sum (36) obeys the condition

$$\bar{Q}^+\{w\} = P_w\,\bar{Q}^+\{e\} \tag{37}$$

for any closed word w. Note that the values on the physical sheet of $\bar{Q}^+$ correspond to

$$\bar{Q}^+(k)\{e\} = \sum_j H^+(k)\{w_j\}.$$

Consider the term $H^+\{w_j w\}$ in (36). Construct a word $w_j w w_j^{-1}$. Since G is a diffraction matrix, $w_j w w_j^{-1} \in S(e)$ (see (33)).

Consider the value $H^+\{w_j w w_j^{-1}\}$. By the condition of the theorem

$$H^+\{w_j w w_j^{-1}\} = P_{w_j w w_j^{-1}} H\{e\}. \tag{38}$$

Continue (38) along the contour w_j:

$$H^+\{w_j w\} = P_{w_j w w_j^{-1}}\{w_j\} H\{w_j\}. \tag{39}$$

Using (16) and taking into account that $P_{w_j} = I$ since $w \in \mathcal{W}_a$, obtain that

$$P_{w_j w w_j^{-1}}\{w_j\} = P_{w_j w} P_{w_j}^{-1} = P_{w_j w}. \tag{40}$$

Again, by (16)

$$P_{w_j w} = P_{w_j}\{w\} P_w = P_w. \tag{41}$$

Substituting (41) into (39), we obtain

$$H^+\{w_j w\} = P_w H\{w_j\}. \tag{42}$$

Performing summation over w_j, obtain (37).

Next, note that function $\bar{Q}^+$ has no branch points in the positive physical half-plane, since any bypass there leads to a permutation of the terms of r.-h.s. of (36). This means that if $\bar{Q}^+$ is not identically equal to zero, it should be meromorphic on the positive physical half-plane.

Finally, consider an arbitrary (not necessarily closed) word w. Find a word $v \in \mathcal{W}_a$, such that vw is a closed word. Since $\bar{Q}^+$ has no branch points on the positive physical half-plane,

$$\bar{Q}^+\{w\} = \bar{Q}^+\{vw\}. \tag{43}$$

According to (37),

$$\bar{Q}^+\{vw\} = P_{vw}\, \bar{Q}^+\{e\}, \tag{44}$$

and due to (16),

$$P_{vw} = P_v\{w\} P_w = P_w. \tag{45}$$

Combining (43), (44) and (45), we obtain

$$\bar{Q}^+\{w\} = P_w\, \bar{Q}^+\{e\}, \tag{46}$$

which is the condition of the Problem 2. Theorem 4 is proved.

The following proposition states that bypass matrices of closed words form a representation of $\mathcal{W}_c$ considered as a group.

Proposition 5 *If a closed word w can be represented as a product of several closed words v_i*

$$w = v_1 v_2 \dots v_n$$

then

$$P_w(k) = P_{v_1}(k)\, P_{v_2}(k) \dots P_{v_n}(k).$$

Proof: Use (16) and note that $P_{v_j}(k)\{v_{j+1} v_{j+2} \dots v_n\} = P_{v_j}(k)$ due to (34).

Proposition 6 *There exists a finite set (basis) of closed words $B = \{\bar{w}\}$ such that for any closed word w there can be constructed a closed word v, such that*

$$P_w = P_v,$$

and the word v can be represented as a product of several words of basis B:

$$v = \bar{w}_1 \bar{w}_2 \dots \bar{w}_m.$$

This proposition can pe proved formally, however here we are not going to do this. Instead, we can list all paths on $\mathcal{R}_G$, whose letter notations are the basis words.

Let the number of sheets of $\mathcal{R}_G$ be equal to n. It is well known that a Riemann surface cut along the elements of the canonical cross-section is a 1-connected area. There are n points on this area corresponding to the infinities at each sheet. The basis paths are the elements of the canonical cross-section and $n-1$ bypasses about all infinities, except one. A bypass about the nth infinity can be obtained as the combination of other basis bypasses.

According to the Propositions 5 and 6, we can rewrite the bypass problem for closed words as follows:

Problem 5 *For a diffraction matrix $G(k)$, a set of bypass matrices P_w corresponding to this matrix, and the basis of closed words $\bar{w}_1 \dots \bar{w}_m$ find a function $H^+(k)$ having algebraic growth at infinity and obeying relation (35) for all basis words.*

Note that now the relation (35) should be checked for a finite set of words only.

5.3 Hurd's case

Definition 5 *Let $G(k)$ be a diffraction matrix, and $P_w(k)$ be the set of bypass matrices. Let for any k and for any different $w_1, w_2 \in \mathcal{W}_c$ be*

$$[P_{w_1}(k), P_{w_2}(k)] = 0. \tag{47}$$

Then we shall say that G possesses Hurd's property.

Obviously, Hurd's property can be checked only for basis words. Note also that if relation (47) is fulfilled only closed words, then it is also fulfilled for all words.

If matrix G is commutative then Hurd's property is fulfilled, but not necessarily vice versa. However, as we shall show here, Hurd's property enables one to apply some commutative technique even if G is not commutative. The idea to study the analytical continuation of the solution and to investigate, whether the bypass matrices have structure simpler than G, belongs to Hurd [6].

Hurd's property is quite common, for example, if $\mathcal{R}_G$ is a hyperelliptic surface, then all bypass matrices are just powers of $P = G_2^{-1}(k) G_1(k)$, where G_1 and G_2 are values of G on two different sheets of $\mathcal{R}_G$. It means, that bypass matrices (for the same affix k) are commutating. Thus, all G with hyperelliptic $\mathcal{R}_G$ belong to Hurd's case.

Let the eigenvectors of all $P_w(k) \neq I$ be distinct almost everywhere. All matrices P_w can be diagonalized simultaneously. Let all bypass matrices can be represented in the form (30), where $M(k)$ is now an algebraic matrix consisting of normalized eigenvectors of P. Note that now the set of the columns of M does not necessarily remain unchanged when the argument is carried over an arbitrary closed contour on $\mathbb{C}$. However, the set of columns remains unchanged when the rising of the contour to $\mathcal{R}_G$ is closed.

Seek the solution H^+ in a diagonalized form

$$H^+(k) = M \operatorname{diag}\{h_1^+, \dots, h_N^+\} M^{-1}. \tag{48}$$

For the functions h_n^+ one can derive a problem similar to Problem 3:

Problem 6 *Find N functions $h_1^+, \dots, h_N^+$, growing algebraically at infinity, whose analytical continuations obey relation*

$$(h_m^+)_c = p_{w_c,m}\, h^+_{\pi_c(m)}. \tag{49}$$

for any contour c, whose rising onto $\mathcal{R}_G$ is closed.

This problem also can be treated by the methods of the theory of functions. A solution of Problem 6 produces a solution of Problem 4, and can be transformed into a solution of the initial factorization problem by symmetrization (36).

Although the matrix H^+ provide a commutative solution of Problem 5, the matrices $Q^\pm$ constructed by the relation (36) are not commutating.

6 Conclusion remarks

6.1 A short summary

The main results of this paper are as follows:

1. Formulae of analytical continuation are derived (Theorem 1).
2. Necessary condition of commutative factorization in the “check-up” form are proved (Theorem 2).
3. Connection with the “Ansatz” (Jones’) form of the necessary condition is established (Theorem 3); conditions of Jones’ theorem are refined.
4. Factorization problem is reformulated in the form of a bypass problem for closed words (Theorem 4). It is shown that sometimes (in Hurd’s case) the commutative methods can be applied to the bypass problem, even if the initial problem does not admit commutative factorization.

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