Three Theorems Concerning Diffraction by a Strip or a Slit

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Abstract

The problem of diffraction on a strip or a slit is under consideration. A functional equation of Wiener-Hopf type is derived; analytical restrictions are imposed on unknown functions. The solution of functional problem (the spectrum of the scattered field) is proved to be a solution of an ordinary differential equation (ODE) with rational coefficients known up to several numerical constants. A nonlinear differential equation describing the dependence of unknown constants on the width of the strip (slit) is derived. Connection between the methods developed here and the solution in the form of Schwarzschild series is established.

Introduction

The problem of diffraction of a plane wave on a strip (slit) with ideal boundary conditions has been studied excessively since the 1950s. A number of analytical, semi-analytical, numerical and approximate theories have been developed. We do not have the intention to present here a review of these results. However, one can mention three main ways to treat the problem analytically:

1. Separation of variables can be performed in elliptic coordinates. The solution is expressed as a series of Mathieu functions (e.g. [1]). An interesting generalization of this method has been proposed by Shinbrot in [2], where Mathieu-type equations are obtained for the problem of diffraction on a set of several strips.

2. Physically clear results can be obtained using the series of successive diffractions of the wave on the edges of the obstacle. For the slit problem this series has been studied by Schwarzschild about 100 years ago [3]. Approximate results can be obtained using Keller's GTD (Geometrical Theory of Diffraction) approach [4, 5]. Results provided by GTD are valid for large width of the strip (comparatively to the wavelength), but they found to be in a good agreement with experimental data even if the width and the wavelength are of the same order. Numerous iterative and approximate procedures have been developed to solve the related integral equation (see for example [6, 7, 8]). Some approximate methods were used for solving corresponding Wiener-Hopf problem [9, 10].

3. Very interesting results have been obtained by Latta [11] and later by Williams [12]. They studied integral equation related to diffraction problems and found that due to some specific properties of the kernel the integral equation can be reduced to an ordinary differential equation. The independent variable for this equation is a spatial coordinate of the problem. Some particular cases of our results can be compared with [12], but here we develop a different tecnique (also connected with ordinary differential equations).

Some other attempts to solve the problems of diffraction on a strip or a slit are discussed in a detailed and emotional review by Luneburg [13]. A remarkable list of references in this work must be mentioned.

In spite of a great number of papers, dedicated to the problem of diffraction on a strip, we believe that we can present here some new analytical results. In brief, the idea of the paper is following. Sommerfeld's solution for diffraction on a half-plane is a simple algebraic function of the spatial frequency k (or $\varphi = \arccos k/k_0$, if the angular spectrum is studied). Namely, if the screen with Dirichlet condition is situated along the negative x half-axis, the incident field is a plane wave with wave vector k_0 normal to the screen and k is Fourier transform variable associated with x, then Fourier transform of the solution can be written (up to a constant) in the form [9]:

$$F(k) = \frac{1}{k\sqrt{k_0 - k}}.$$

This function obeys a simple ordinary differential equation (ODE) of order 1:

$$F'(k) + \left(\frac{1}{k} + \frac{1}{2(k - k_0)} \right) F(k) = 0.$$

In this paper we show, that the solution of a slit or a strip problem obeys such equation of order 2. Coefficients of the equation are rational functions.

Direct numerical or analytical calculation of the coefficients of the ODE is not simple because of the nature of restrictions imposed on these coefficients. That is why we develop two indirect techniques to study the coefficients. In Section 2 we construct the "evolution" equation (the term by Williams), i.e., a nonlinear differential equation describing the dependence of unknown parameters as functions of the width of the strip or a slit. The second method is the analysis of successive diffractions on the edges of the strip or a slit (Schwarzschild series). This method enables one to find approximate values of the coefficients of the ODE. Since the two methods are independent, it is possible to compare numerical results provided by them.

1 Formulation of functional problem for diffraction on a strip or a slit

1.1 Functional equation for diffraction on a strip

Consider a problem with mixed boundary conditions. Let the equation

$$\Delta u + k_0^2 u = 0 \tag{1.1}$$

be satisfied by the function $u(x, y)$ in the half-plane $y > 0$, $-\infty < x < \infty$. Time dependence of all values is chosen as $e^{-\mathrm{i}\omega t}$, i.e. a plane wave propagating in the positive direction of x axis has the form $e^{\mathrm{i}k_0 x}$.

Boundary conditions are:

$$u(x, 0) = -e^{-\mathrm{i}k_* x} \qquad \text{for} \qquad -a < x < a. \tag{1.2}$$

and

$$\frac{\partial u(x,0)}{\partial y} = 0 \qquad \text{for} \quad (-\infty < x < -a) \cup (a < x < \infty). \tag{1.3}$$

We suppose that there are no waves coming from infinity and no growing terms (the field satisfies Sommerfeld radiation conditions at infinity). We also expect that the field and its derivative possess known asymptotics near the edges points $(-a, 0)$ and $(a, 0)$; the asymptotics are taken from the exact edge point solution arising from the problem of diffraction on a semi-infinite screen.

The problem (1.1), (1.2), (1.3) is equivalent to the problem of diffraction on an infinite strip (a segment in 2D cross-section). The incident field is a plane wave coming at the angle ψ to y-axis, such that $k_* = k_0 \sin\psi$. Points $(-a, 0)$ and $(a, 0)$ are the edges of the strip. One can note, that (1.1), (1.2), (1.3) corresponds to the symmetrical part of diffractional problem (antisymmetric part is trivial).

The field can be presented in the form of Fourier integral

$$u(x,y) = -\frac{1}{2\pi}\int_C \hat{W}(k) e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2-k^2}y} dk, \tag{1.4}$$

where the contour C of integration coincides with the real axis everywhere except in the neighborhood of the points $\pm k_0$. It passes below the branch point k_0 and above the point $-k_0$ (we shall show below that $\hat{W}$ has no other singularities). The value of the square root is chosen to be positive on the segment $(-k_0, k_0)$.

Representation (1.4) can be used for calculation of $y-$derivative of the field on $x-$axis.

$$\frac{\partial u(x,0)}{\partial y} = -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \sqrt{k_0^2-k^2}\hat{W}(k) e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2-k^2}y} dk. \tag{1.5}$$

Fourier transforming to (1.5) and taking into account (1.3), we obtain

$$\hat{W}(k) = \frac{\mathrm{i}}{\sqrt{k_0^2-k^2}}\int_{-a}^{a} \frac{\partial u(x,0)}{\partial y} e^{\mathrm{i}kx} dx. \tag{1.6}$$

From another point of view, function $\hat{W}(k)$ can be presented in the form of Fourier integral of $u(x, 0)$

$$\hat{W}(k) = -\int_{-\infty}^{\infty} u(x,0) e^{\mathrm{i}kx} dx.$$

The path of integration can be split into 3 parts: $(-\infty, -a)$, $(-a, a)$ and (a, ∞). Taking into account (1.2), we obtain the *functional equation*

$$\hat{U}_+(k) + \hat{U}_-(k) + \hat{W}(k) = 0, \tag{1.7}$$

where

$$\hat{U}_+(k) = \int_a^{\infty} u(x,0) e^{\mathrm{i}kx} dx + \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)a}}{k-k_*}, \tag{1.8}$$

$$\hat{U}_-(k) \;=\; \int\limits_{-\infty}^{-a} u(x,0)e^{\mathrm{i}kx}dx - \frac{\mathrm{i}e^{-\mathrm{i}(k-k_*)a}}{k-k_*}. \tag{1.9}$$

A similar equation has been obtained in [9].

1.2 Functional equation for diffraction on a slit

The problem of diffraction on a slit is a bit more difficult. In addition to the incident and the diffracted field, one must take into account the reflected field. For the diffracted field we have boundary conditions

$$\frac{\partial u(x,0)}{\partial y} = -e^{-\mathrm{i}k_*x} \qquad \text{for} \qquad -a < x < a. \tag{1.10}$$

and

$$u(x,0) = 0 \qquad \text{for} \qquad (-\infty < x < -a) \cup (a < x < \infty). \tag{1.11}$$

Applying Fourier transformation (see previous subsection), we again obtain the functional equation

$$\bar{U}_+(k) + \bar{U}_-(k) + \bar{W}(k) = 0, \tag{1.12}$$

where now

$$\bar{W}(k) \;=\; \int\limits_{-a}^{a} u(x,0)e^{\mathrm{i}kx}dx, \tag{1.13}$$

$$\bar{U}_+(k) \;=\; \frac{\mathrm{i}}{\sqrt{k_0^2-k^2}} \int\limits_{a}^{\infty} \frac{\partial u(x,0)}{\partial y} e^{\mathrm{i}kx}dx - \frac{1}{\sqrt{k_0^2-k^2}} \frac{e^{\mathrm{i}a(k-k_*)}}{k-k_*}, \tag{1.14}$$

$$\bar{U}_-(k) \;=\; \frac{\mathrm{i}}{\sqrt{k_0^2-k^2}} \int\limits_{-\infty}^{-a} \frac{\partial u(x,0)}{\partial y} e^{\mathrm{i}kx}dx + \frac{1}{\sqrt{k_0^2-k^2}} \frac{e^{-\mathrm{i}a(k-k_*)}}{k-k_*}. \tag{1.15}$$

1.3 Analytic properties of unknown functions

Consider the problem of diffraction on a strip. Let k be a complex variable. Equation (1.7) contains three unknown functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$. In this subsection we shall find the properties of these functions, such that no other information will be necessary to solve the problem. Namely, we shall find some asymptotics at infinity and at singular points k_* and $\pm k_0$.

Functions

$$\hat{U}_+(k) - \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)a}}{k-k_*}, \qquad \hat{U}_-(k) + \frac{\mathrm{i}e^{-\mathrm{i}(k-k_*)a}}{k-k_*}, \quad \text{and} \quad \sqrt{k_0^2-k^2}\hat{W}(k)$$

are Fourier integrals; integration is performed along half-lines or a segment (see (1.8), (1.9), (1.6)). The following properties are known for such functions [9]:

a) $\hat{U}_+(k) - \mathrm{i}e^{\mathrm{i}(k-k_*)a}/(k-k_*)$ is defined and has no singularities in the upper half-plane k;

b) $\hat{U}_-(k) + \mathrm{i}e^{-\mathrm{i}(k-k_*)a}/(k-k_*)$ is defined and has no singularities in the lower half-plane;
c) $\sqrt{k_0^2 - k^2}\hat{W}(k)$ is an entire function of k.

The upper half-plane is the part of complex plane lying above the contour C and the lower half-plane is the part below C. Note that k_0 belongs to the upper half-plane and $-k_0$ belongs to the lower-half plane. The behavior of unknown functions at $\pm k_0$ is very important, because as we will see below, unknown functions have no other branch points. Thus, $\hat{U}_+(k)$ is a regular function at $k = k_0$ and $\hat{U}_-(k)$ is a regular function at $k = -k_0$.

Note that similar conclusions concerning the location of the points $\pm k_0$ can be made if we suppose that k_0 has a small positive imaginary part, corresponding to the dissipation in the medium. Although now we don't need the artificial dissipation, below it will be used for establishing the asymptotic properties of Schwarzchild's series.

We will say that function $F(k)$ *changes its sign* at some point $k = k_1$, if $F(k)$ can be represented in the vicinity of this point as a series convergent for some $0 < |k - k_1| < \delta$:

$$F(k) = (k - k_1)^{1/2} \sum_{n=m}^{\infty} a_n (k - k_1)^n,$$

where m is an integer (in this paper m is equal to 0 or -1). In other words, function $(k - k_1)^{1/2} F(k)$ is either regular at $k = k_1$ or has a pole at this point. One can see, that $\hat{W}(k)$ changes its sign at $\pm k_0$.

Note also that there are no waves coming from the infinity, therefore $\hat{U}_+(k)$ has no singularities on the positive real half-axis and $\hat{U}_-(k)$ has no singularities on the negative real half-axis. The only exception is the point $k = k_*$, where both functions have a simple pole with known residues.

1.4 Growth of unknown functions at infinity

According to the exact solution of the Sommerfeld problem (diffraction on a semi-infinite screen [9]), we are looking for the functions $\hat{U}_+(k)$, $\hat{U}_-(k)$, $\hat{W}(k)$ with the following asymptotics as $|k| \to \infty$ in the upper and lower half-planes:

$$\begin{aligned} \hat{U}_+(k) &\sim \frac{e^{ika}}{k^{3/2}} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ \hat{U}_-(k) &\sim \frac{e^{-ika}}{k^{3/2}} \quad \text{for} \quad \mathrm{Im}[k] < 0, \\ \hat{W}(k) &\sim \frac{e^{-ika}}{k^{3/2}} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ \hat{W}(k) &\sim \frac{e^{ika}}{k^{3/2}} \quad \text{for} \quad \mathrm{Im}[k] < 0, \end{aligned} \tag{1.16}$$

These asymptotics can be illustrated as follows. Let r and ϕ be local polar coordinates in the proximity of the point $x = a$ and let the direction $\phi = 0$ correspond to the negative x direction. Then the total field (the incident plane wave plus the scattered field u) can be represented as a series of Bessel functions

$$u_{\mathrm{tot}}(r, \phi) = \sum_{n=1}^{\infty} \alpha_n J_{n/2}(k_0 r) \sin(n\phi/2).$$

This representation leads to (1.16).

Another (Meixner's) argument takes into account the energy flow from the edges of the strip and also yields (1.16).

1.5 Analytic continuation of functions $\hat{U}_+$ and $\hat{U}_-$

Functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$ can be analytically continued onto their Riemann surfaces. Let us introduce a new independent variable

$$\beta = \arccos(k/k_0) \tag{1.17}$$

and study the properties of $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$ as functions of β

The definitions (1.8) and (1.9) can be directly applied for k such that $\mathrm{Im}[k] \geq 0$ and $\mathrm{Im}[k] \leq 0$ respectively. This corresponds to the regions I and II shown in Fig. 1a. The particular shape of their boundaries depends on the imaginary part of k_0. The definition (1.6) is valid on the whole β-plane.

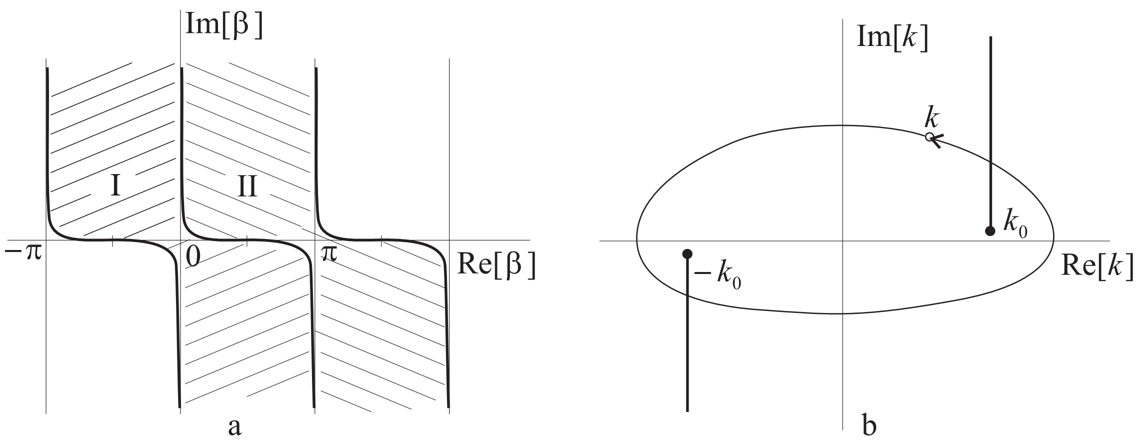

Figure 1: a) regions in β-plane, corresponding to $\mathrm{Im}[k] \geq 0$ and $\mathrm{Im}[k] \leq 0$; b) transformation in k-plane, corresponding to $\beta \to \beta + 2\pi$.

Using the functional equation (1.7), one can continue $\hat{U}_+(\beta)$ into the region II and $\hat{U}_-(\beta)$ into the region I, setting

$$\hat{U}_- = -\hat{U}_+ - \hat{W} \text{ for } \beta \in \mathrm{I}, \qquad \hat{U}_+ = -\hat{U}_- - \hat{W} \text{ for } \beta \in \mathrm{II}.$$

The regularity of $\hat{U}_+(k)$ at k_0 and of $\hat{U}_-(k)$ at $-k_0$ can be expressed as

$$\hat{U}_+(-\beta) = \hat{U}_+(\beta). \tag{1.18}$$

$$\hat{U}_-(2\pi - \beta) = \hat{U}_-(\beta). \tag{1.19}$$

Function $\hat{W}(k)$ changes its sign at $\pm k_0$, so

$$\hat{W}(\beta) = -\hat{W}(-\beta) = \hat{W}(2\pi + \beta). \tag{1.20}$$

Continuing the identities (1.17)–(1.19) and functional equation (1.7), we obtain that

$$\hat{U}_+(2\pi - \beta) = -\hat{U}_-(2\pi - \beta) - \hat{W}(2\pi - \beta) = -\hat{U}_-(\beta) + \hat{W}(\beta) = \hat{U}_+(\beta) + 2\hat{W}(\beta). \tag{1.21}$$

Combining the last identity with (1.18) and (1.20), we obtain

$$\hat{U}_+(\beta + 2\pi) = \hat{U}_+(\beta) - 2\hat{W}(\beta). \tag{1.22}$$

Similar reasoning gives

$$\hat{U}_-(\beta + 2\pi) = \hat{U}_-(\beta) + 2\hat{W}(\beta). \tag{1.23}$$

Equations (1.22), (1.23) enable one to continue functions $\hat{U}_+$ and $\hat{U}_-$ to the whole β-plane.

Let us return to the variable k. The transformation $\beta \to \beta + 2\pi$ corresponds to the bypass of infinity in k plane $\arg k \to \arg k + 2\pi$, shown in Fig. 1b. Bold lines correspond to branch cuts.

It is useful to rewrite (1.22), (1.23) and (1.20) in the form

$$\begin{aligned} \hat{U}_+(ke^{2\pi \mathrm{i}}) &= \hat{U}_+(k) - 2\hat{W}(k), \\ \hat{U}_-(ke^{2\pi \mathrm{i}}) &= \hat{U}_-(k) + 2\hat{W}(k), \\ \hat{W}(ke^{2\pi \mathrm{i}}) &= \hat{W}(k). \end{aligned} \tag{1.24}$$

As it follows from (1.24), the values of $\hat{U}_+(k)$ on different sheets of its Riemann surface differ by the value $2n\hat{W}$. Therefore, there are no other singularities except k_* and $\pm k_0$ on the whole Riemann surface.

Now we can formulate a functional problem that will be considered below.

We seek functions $\hat{U}_+(k)$, $\hat{U}_-(k)$, $\hat{W}(k)$, such that

1. *equation (1.7) is valid;*

2. *$\hat{U}_+(k)$, $\hat{U}_-(k)$, $\hat{W}(k)$ have no singularities except $\pm k_0$, k_* and infinity;*

3. *$\hat{U}_+$ is regular at k_0 on the physical sheet; $\hat{U}_-$ is regular at $-k_0$ on the physical sheet; $\hat{W}$ changes its sign at $\pm k_0$;*

4. *$\hat{U}_+$ has the residue equal to* i *at k_*, $\hat{U}_-$ has the residue equal to* $-$i *at k_*;*

5. *Asymptotics (1.16) are valid.*

Similar properties can be easily established for the functions $\bar{U}_-$, $\bar{U}_+$, $\bar{W}$, related to the slit problem. One must substitute everywhere "changes sign" by "regular" and vice versa.

2 Ordinary differential equation associated with the functional equation

2.1 An analogy with the cylindrical functions

To make clear the subsequent analysis, in this subsection we will draw an analogy of the functional problem formulated above with the theory of cylindrical functions. Consider as an example a set of Bessel and Hankel functions $J_0(ka)$, $H_0^{(1)}(ka)$ and $H_0^{(2)}(ka)$. These functions are considered as the functions of k (a is a constant). The listed functions are the analogues of

functions $\hat{W}(k)$, $\hat{U}_+(k)$ and $\hat{U}_-(k)$ respectively. The following properties should be mentioned:
1. Three cylindrical functions obey the relation

$$H_0^{(1)}(ka) + H_0^{(2)}(ka) - 2J_0(ka) = 0. \tag{2.1}$$

This equation is the analog of (1.7).
2. The following asymptotics are known as $|k| \to \infty$:

$$\begin{aligned} H_0^{(1)}(ka) &\sim \frac{e^{\mathrm{i}ka}}{k^{1/2}} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ H_0^{(2)}(ka) &\sim \frac{e^{-\mathrm{i}ka}}{k^{1/2}} \quad \text{for} \quad \mathrm{Im}[k] < 0, \\ J_0(ka) &\sim \frac{e^{-\mathrm{i}ka}}{k^{1/2}} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\ J_0(ka) &\sim \frac{e^{\mathrm{i}ka}}{k^{1/2}} \quad \text{for} \quad \mathrm{Im}[k] < 0. \end{aligned} \tag{2.2}$$

These estimations are analogous to (1.16).
3. Cylindrical functions have branch points at $k = 0$ and ∞. The following properties are valid

$$\begin{aligned} H_0^{(1)}(kae^{2\pi\mathrm{i}}) &= H_0^{(1)}(ka) - 2J_0(ka), \\ H_0^{(2)}(kae^{2\pi\mathrm{i}}) &= H_0^{(2)}(ka) + 2J_0(ka), \\ J_0(kae^{2\pi\mathrm{i}}) &= J_0(ka). \end{aligned} \tag{2.3}$$

These properties are analogous to (1.24).

Cylindrical functions are the solutions of Bessel equation, which is a confluent hypergeometric equation with one regular singularity $k = 0$ and one irregular singularity $k = \infty$. The features listed above are not specific properties of cylindrical functions, but they are peculiar to the solutions of an ordinary differential equation of rather general kind (see [14]). Below we look for such equation for the set $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$.

2.2 Ordinary differential equation for $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$

Theorem 1 *Functions $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$ obey the equation (later it will be named the* **ODE***):*

$$V''(k) - f(k)V'(k) - g(k)V(k) = 0, \tag{2.4}$$

where prime corresponds to the differentiation with respect to k. Functions $f(k)$ and $g(k)$ are the ratios of the polynomials of prescribed order.

It is obvious that if two functions obey the equation, then the third obeys it due to (1.7).

One can find a differential equation, such that two arbitrary functions obey it. For example let $\hat{U}_+(k)$ and $\hat{W}(k)$ be such functions. Then the pair of equations is valid:

$$\begin{aligned} f(k)\hat{U}_+'(k) + g(k)\hat{U}_+(k) &= \hat{U}_+''(k), \\ f(k)\hat{W}'(k) + g(k)\hat{W}(k) &= \hat{W}''(k). \end{aligned} \tag{2.5}$$

This set can be interpreted as a system of linear algebraic equations with respect to $f(k)$ and $g(k)$. Its solution is following

$$f(k) = \frac{D'(k)}{D(k)}, \tag{2.6}$$

$$g(k) = \frac{E(k)}{D(k)}, \tag{2.7}$$

where $D(k)$ and $E(k)$ are the determinants

$$D(k) = \begin{vmatrix} \hat{U}'_+(k), & \hat{U}_+(k) \\ \hat{W}'(k), & \hat{W}(k) \end{vmatrix}, \tag{2.8}$$

$$E(k) = \begin{vmatrix} \hat{U}'_+(k), & \hat{U}''_+(k) \\ \hat{W}'(k), & \hat{W}''(k) \end{vmatrix}. \tag{2.9}$$

Note that

$$D'(k) = \begin{vmatrix} \hat{U}''_+(k), & \hat{U}_+(k) \\ \hat{W}''(k), & \hat{W}(k) \end{vmatrix}.$$

Study the properties of the determinants $D(k)$ and $E(k)$. First note, that each determinant can be written in two different forms. Using equation (1.7) and the properties of determinants, we obtain

$$D = \begin{vmatrix} \hat{U}'_+, & \hat{U}_+ \\ \hat{W}', & \hat{W} \end{vmatrix} = \begin{vmatrix} -\hat{W}' - \hat{U}'_-, & -\hat{W} - \hat{U}_- \\ \hat{W}', & \hat{W} \end{vmatrix} = - \begin{vmatrix} \hat{U}'_-, & \hat{U}_- \\ \hat{W}', & \hat{W} \end{vmatrix}, \tag{2.10}$$

$$E = \begin{vmatrix} \hat{U}'_+, & \hat{U}''_+ \\ \hat{W}', & \hat{W}'' \end{vmatrix} = \begin{vmatrix} -\hat{W}' - \hat{U}'_+, & -\hat{W}'' - \hat{U}''_+ \\ \hat{W}', & \hat{W}'' \end{vmatrix} = - \begin{vmatrix} \hat{U}'_-, & \hat{U}''_- \\ \hat{W}', & \hat{W}'' \end{vmatrix}, \tag{2.11}$$

i.e. each determinant can be expressed in terms of either the pair $(\hat{U}_+, \hat{W})$ (the first representation) or the pair $(\hat{U}_-, \hat{W})$ (the second representation). We shall use the first pair to study the properties of D and E in the upper half-plane of k, and the second one to study the determinants in the lower half-plane.

Consider the determinant $D(k)$. In the upper half plane of the argument it grows as $\sim O(k^{-3})$ due to the first representation and asymptotics (1.16). It has the same order of growth in the lower half plane due to the second representation. Besides, $D(k)$ changes its sign at the point k_0 ($\hat{U}_+$ is regular and $\hat{W}$ changes its sign). Also, $D(k)$ changes its sign at $-k_0$ due to the second representation. Since there are no other branch points, the function $\sqrt{k_0^2 - k^2} D(k)$ *is single-valued on the whole complex plane.*

As it follows from the definition of functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$, function $\sqrt{k_0^2 - k^2} D(k)$ has some singularities on the complex plane. It has the pole of second order at $k = k_*$ and simple poles at the points k_0 and $-k_0$. No other singularities can appear. Taking into account the order of growth of the function $D(k)$ and applying Liouville theorem, we conclude that

$$D(k) = \frac{P(k)}{(k^2 - k_0^2)^{3/2} (k - k_*)^2}, \tag{2.12}$$

where $P(k)$ is a polynomial of degree 2, whose coefficients depend on k_0, a and k_*.

Similar reasoning yields to the following representation for $E(k)$:

$$E(k) = \frac{Q(k)}{(k^2 - k_0^2)^{5/2}(k - k_*)^3}, \tag{2.13}$$

where $Q(k)$ is a polynomial of degree 5. Note that the ratios (2.6) and (2.7) do not contain radicals. Theorem 1 is proven.

Note that we implicitly used relations (1.22) and (1.23) in the argument performed above.

It is convenient to express the numerator of (2.12) in the form

$$P(k) = M(k - \lambda_1)(k - \lambda_2), \tag{2.14}$$

where λ_1 and λ_2 are the roots of the polynomial $P(k)$. In general case $\lambda_1 \neq \lambda_2$. As it follows from (2.6) and (2.12), one can write $f(k)$ in the form

$$f(k) = \frac{1}{k - \lambda_1} + \frac{1}{k - \lambda_2} - \frac{2}{k - k_*} - \frac{3k}{k^2 - k_0^2}. \tag{2.15}$$

Using (2.7) and (2.13), we obtain

$$g(k) = \frac{Q(k)/M}{(k - \lambda_1)(k - \lambda_2)(k^2 - k_0^2)(k - k_*)}. \tag{2.16}$$

Theorem 1 can be compared with the results of Williams [12]. Using a quite different method he obtained a differential equation of the type (2.4) for the particular case of tangential incidence (i.e. $k_* = \pm k_0$). However, the result proven above for arbitrary k_* cannot be obtained directly from [12] and the links between two methods are unclear, as yet.

2.3 Restrictions on unknown parameters

Note that $f(k)$ and $g(k)$ are the rational functions of k, which are known up to 8 constant parameters. There are 9 unknown coefficients of the polynomials $P(k)$ and $Q(k)$, but the leading coefficient of P can be chosen arbitrarily, since $f(k)$ and $g(k)$ involve the ratios of the polynomials.

We must chose functions $f(k)$ and $g(k)$, such that functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$ possessing known properties can be chosen among the solutions of the ODE (2.4). The ODE has 6 singular points: infinity, k_*, $\pm k_0$, λ_1 and λ_2. We should demand that the ODE has two solutions with prescribed asymptotics at each singular point and, besides, these asymptotics match, i.e. for example $\hat{W}(k)$ has one prescribed asymptotic at ∞, another at k_0 etc. Therefore, two types of the restrictions can be posed: "local" and "global" ones. For local restriction it is enough to study the coefficients near the singular point. The procedure of investigation of each point is standard [14]. The coefficients $f(k)$ and $g(k)$ are expanded as power series and two linearly independent solutions of known anzats are substituted in the equation. A recurrent set of relations can be obtained for each solution, and for some specific term the relation degenerates to the restriction on the coefficients. Local restrictions are discussed below.

a.1. Consider the behavior of the functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$ at infinity (1.16). Follow the standard procedure to investigate the singular points of ODE [14]. Let the coefficients of the equation be expanded as power series at infinity:

$$f(k) = \sum_{n=1}^{\infty} \frac{f_n^{\infty}}{k^n} = -\frac{3}{k} + \ldots, \qquad g(k) = \sum_{n=0}^{\infty} \frac{g_n^{\infty}}{k^n} \tag{2.17}$$

Infinity is the irregular point of the ODE, so two linearly independent solutions can be found in the form

$$V_{1,2}(k) \sim k^{\mu_{1,2}} e^{k\eta_{1,2}}.$$

Substituting this anzats into (2.17) and taking into account that according to (1.16), $\mu_{1,2} = -3/2$, $\eta_{1,2} = \pm \mathrm{i}a$, we find that

$$g_0^{\infty} = -a^2, \tag{2.18}$$

$$g_1^{\infty} = 0. \tag{2.19}$$

Note that g_0 and g_1 depend on the coefficients of the polynomials $P(k)$ and $Q(k)$. Therefore, (2.18) and (2.19) are the restrictions on unknown constants.

a.2. Consider the behavior of the solutions near the point $k = k_*$. The exponents of this point are 0 and -1. Their difference is integer. As it follows from the elementary theory of differential equations [14], in general case the fundamental solution with exponent 0 can contain logarithmic terms. However, we are sure that function $\hat{W}$ has no logarithmic terms. So we must impose special restrictions on the coefficients of the equation to exclude logarithmic terms.

Let the following expansions be valid near k_*:

$$\begin{aligned} f(k) &= \frac{1}{k-k_*} \sum_{n=0}^{\infty} f_n^*(k-k_*)^n, \\ g(k) &= \frac{1}{(k-k_*)^2} \sum_{n=1}^{\infty} g_n^*(k-k_*)^n. \end{aligned} \tag{2.20}$$

There are two linearly independent solutions near the point $k = k_*$. They are, for example,

$$\hat{W}(k) \sim \sum_{n=0}^{\infty} w_n^*(k-k_*)^n, \qquad \hat{U}_+(k) \sim \sum_{n=-1}^{\infty} u_n^*(k-k_*)^n. \tag{2.21}$$

Substitute (2.21) for $\hat{U}_+$ and expand the ODE as power series in $k - k_*$. Coefficicent at $(k-k_*)^{-3}$ is identically equal to zero. At $(k-k_*)^{-2}$ we obtain the restriction

$$g_1^* = f_1^*. \tag{2.22}$$

a.3. Points $k = \lambda_{1,2}$ are the poles of the function $f(k)$, so they are the singular points of the differential equation. Consider the point λ_1. Its exponents are 0 and 2. Let the following expansions be valid near $k = \lambda_1$:

$$f(k) = \frac{1}{k-\lambda_1} \sum_{n=0}^{\infty} f_n(k-\lambda_1)^n, \qquad g(k) = \frac{1}{(k-\lambda_1)^2} \sum_{n=1}^{\infty} g_n(k-\lambda_1)^n. \tag{2.23}$$

Substituting power series for linearly independent solutions and expanding the equation as power series in $k - \lambda_1$ we find that

$$g_2 - g_1(f_1 + g_1) = 0, \tag{2.24}$$

Equation (2.24) is a restriction on the coefficients of $P(k)$ and $Q(k)$. Another restriction of the same kind can be obtained for the point λ_2.

If $\lambda_1 = \lambda_2$, then only one restriction can be obtained, but there is one less unknown parameter.

So there are 5 local restrictions for 8 parameters.

Local restrictions can be easily taken into account. For example, consider the case of normal incidence ($k_* = 0$). Due to the symmetry of the problem, $\lambda_1 = -\lambda_2 = \lambda$. Due to (2.22), $g_1^* = 0$ and the ODE can be written in the form

$$V''(k) + \left[\frac{2}{k} - \frac{2k}{k^2 - \lambda^2} + \frac{3k}{k^2 - k_0^2}\right] V'(k) + \left[a^2 - \frac{X}{k^2 - \lambda^2} - \frac{Y}{k^2 - k_0^2}\right] V(k) = 0, \tag{2.25}$$

where λ, X and Y are unknown numerical parameters depending on a.

Restriction (2.24) leads to the relation between these three parameters:

$$4\lambda^2 Y + 2(4\lambda^2 - k_0^2)X + (X^2 + 4a^2\lambda^2)(k_0^2 - \lambda^2) = 0. \tag{2.26}$$

Thus, if we find λ and X, then we can calculate Y from (2.26).

Equation (2.25) was obtained for the strip problem. It can be shown, however, that the slit problem leads to the same equation. Relation (2.26) also remains unchanged. Only the parameters X and λ are different from those of the strip problem.

If the parameters obey the restrictions formulated above, then at each singular point there is a pair of solutions with required expansion. One must belong to $\hat{W}(k)$, another — to $\hat{U}_+(k)$ or $\hat{U}_-(k)$. But there is no warranty that all asymptotics (at different singular points), that must belong for example to $\hat{W}(k)$ actually belong to one solution of ODE. So one must build solutions of ODE along the lines connecting singular points and check whether asymptotic expansions match at the ends of the lines.

Take the polynomials $P(k)$ and $Q(k)$ satisfying only the local restrictions Consider the point $k = k_0$. Near this point one can choose 2 linearly independent solutions $V_1(k)$ and $V_2(k)$ of ODE (2.4), such that

$$V_1(k) = \varphi_1(k - k_0),$$
$$V_2(k) = \frac{\varphi_2(k - k_0)}{\sqrt{k - k_0}},$$

where φ_1 and φ_2 are regular at zero.

If the polynomials $P(k)$ and $Q(k)$ are chosen correctly, then $V_1(k)$ is proportional to $\hat{U}_+(k)$ and $V_2(k)$ is proportional to $\hat{W}(k)$. The following properties must be valid for V_1 and V_2

b.1. Function $V_1(k)$ can be presented at infinity in the upper half plane as a sum of exponentially decaying and growing functions:

$$V_1(k) = A_0 e^{\mathrm{i}ak} k^{-3/2}(1 + a_1 k^{-1} + a_2 k^{-2} + \ldots) + B_0 e^{-\mathrm{i}ak} k^{-3/2}(1 + b_1 k^{-1} + b_2 k^{-2} + \ldots).$$

Taking into account the asymptotics (1.16), we insist that

$$B_0 = 0. \tag{2.27}$$

This condition can be checked only numerically.

Note that it is not necessary to formulate a similar condition for $k = -k_0$. A detailed analysis (see [14]) shows that it follows from the condition for $k = k_0$ formulated above.

b.2. Consider the second solution V_2. Near the point $k = -k_0$ function V_2 can be presented in the form

$$V_2(k) = \varphi_3(k + k_0) + \frac{\varphi_4(k + k_0)}{\sqrt{k + k_0}},$$

where φ_3 and φ_4 are regular at zero. One must demand that

$$\varphi_3 \equiv 0. \tag{2.28}$$

b.3. Consider the solution V_2. In the vicinity of the point $k = k_*$ it can be presented in the form

$$V_2(k) = \frac{A_1}{k - k_*} + \varphi_5(k - k_*).$$

We must demand that

$$A_1 = 0. \tag{2.29}$$

Thus, there are 3 global restrictions for obtaining unknown parameters. Unfortunately, there is no effective direct numerical or analytical procedure known to find the parameters satisfying these global restrictions. In two following sections we develop some indirect methods to study the coefficients of the ODE.

3 Evolution equation for the coefficients of the ODE

3.1 Evolution equation in matrix form

Let the width of the strip a be a variable. Coefficients of the ODE (2.4) are now the functions of two variables: $f(k, a)$ and $g(k, a)$. In the previous section we discussed the properties of f and g as functions of k. Here we shall discuss the behaviour of f and g in a. For the simplicity, we perform the detailed calculations only for the case of normal incidence (2.25).

Note, that the restrictions (both local and global) on the coefficients of the ODE, listed in the previous section, are named *monodromy data* of the equation. Whilst parameter a varies, monodromy data must remain unchanged. Such deformations of the coefficients of ODEs are well-studied. So-called Slezinger equation describes the monodromy preserving deformations of Fuchsian equation. For our case a similar theory was developed in [15]. However, the theory in [15] is described in a very formal manner, so here we are going to translate it into the language of our particular applied problem.

Below we derive a general form of evolution equation and then consider a particular case of normal incidence. The result of this section is a system of nonlinear ordinary differential equations for $X(a)$ and $\lambda(a)$, where X and λ are the parameters of equation (2.25).

First of all, let us rewrite (2.4) in matrix form. Denote by $\hat{\mathbf{U}}_+$, $\hat{\mathbf{U}}_-$ and $\hat{\mathbf{W}}$ the vectors

$$\hat{\mathbf{U}}_+ = \begin{pmatrix} \hat{U}_+ \\ \hat{U}'_+ \end{pmatrix}, \qquad \hat{\mathbf{U}}_- = \begin{pmatrix} \hat{U}_- \\ \hat{U}'_- \end{pmatrix}, \qquad \hat{\mathbf{W}} = \begin{pmatrix} \hat{W} \\ \hat{W}' \end{pmatrix}, \tag{3.1}$$

where prime corresponds to differentiation with respect to k.

Equation (2.4) can be rewritten in the form

$$\frac{\partial}{\partial k}\mathbf{V} = \mathbf{K}\mathbf{V}, \tag{3.2}$$

where $\mathbf{V}$ stands for $\hat{\mathbf{U}}_+$, $\hat{\mathbf{U}}_-$ or $\hat{\mathbf{W}}$ and

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ g & f \end{pmatrix}.$$

Theorem 2 *There exists matrix $\mathbf{A}$ depending on k and a, such that*

$$\frac{\partial}{\partial a}\mathbf{V} = \mathbf{A}\mathbf{V} \tag{3.3}$$

for $\mathbf{V}$ equal to $\hat{\mathbf{U}}_+$, $\hat{\mathbf{U}}_-$ or $\hat{\mathbf{W}}$. The elements of $\mathbf{A}$ are rational functions of k. The following evolution equation is valid

$$\frac{\partial}{\partial a}\mathbf{K} - \frac{\partial}{\partial k}\mathbf{A} + (\mathbf{K}\mathbf{A} - \mathbf{A}\mathbf{K}) = 0. \tag{3.4}$$

Denote the elements of $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Note, that if equation (3.3) is valid for two functions (say, for $\mathbf{V} = \hat{\mathbf{U}}_+$ and $\hat{\mathbf{W}}$), then it is valid for the third function $\hat{\mathbf{U}}_-$ due to (1.7). Substitute $\hat{\mathbf{U}}_+$ and $\hat{\mathbf{W}}$ into (3.3) and obtain four equations

$$\begin{aligned}
(\hat{U}_+)_a &= A_{11}\hat{U}_+ + A_{12}\hat{U}'_+, \\
(\hat{U}'_+)_a &= A_{21}\hat{U}_+ + A_{22}\hat{U}'_+, \\
\hat{W}_a &= A_{11}\hat{W} + A_{12}\hat{W}', \\
(\hat{W}')_a &= A_{21}\hat{W} + A_{22}\hat{W}',
\end{aligned}$$

where index a denotes the differentiation with respect to a. This set can be interpreted as two systems of linear algebraic equations with respect to A_{ij}. The solution can be written in the form

$$\begin{aligned}
A_{11} &= C_1/D, \\
A_{12} &= H/D, \\
A_{21} &= C_3/D, \\
A_{22} &= C_4/D,
\end{aligned} \tag{3.5}$$

where D is the determinant defined by (2.8),

$$C_1 = \begin{vmatrix} \hat{U}'_+ & (\hat{U}_+)_a \\ \hat{W}' & \hat{W}_a \end{vmatrix} = -\begin{vmatrix} \hat{U}'_- & (\hat{U}_-)_a \\ \hat{W}' & \hat{W}_a \end{vmatrix}, \tag{3.6}$$

$$H = \begin{vmatrix} (\hat{U}_+)_a & \hat{U}_+ \\ \hat{W}_a & \hat{W} \end{vmatrix} = -\begin{vmatrix} (\hat{U}_-)_a & \hat{U}_- \\ \hat{W}_a & \hat{W} \end{vmatrix}, \tag{3.7}$$

$$C_3 = \begin{vmatrix} \hat{U}'_+ & (\hat{U}'_+)_a \\ \hat{W}' & (\hat{W}')_a \end{vmatrix} = -\begin{vmatrix} \hat{U}'_- & (\hat{U}'_-)_a \\ \hat{W}' & (\hat{W}')_a \end{vmatrix}, \tag{3.8}$$

$$C_4 = \begin{vmatrix} (\hat{U}'_+)_a & \hat{U}_+ \\ (\hat{W}')_a & \hat{W} \end{vmatrix} = -\begin{vmatrix} (\hat{U}'_-)_a & \hat{U}_- \\ (\hat{W}')_a & \hat{W} \end{vmatrix}. \tag{3.9}$$

Now we are going to apply to the determinants (3.6) – (3.9) the same method that was applied in the previous section to the determinants D and E. Note, that we again have two representations for each determinant.

The derivatives $(\hat{U}_+)_a$, $(\hat{U}_-)_a$ and $(\hat{W})_a$ have asymptotics at $k \to \infty$ of the type $\sim k^{-1/2} \exp \pm iak$. It is obvious that differentiation with respect to a leaves the conditions of regularity unchanged, for example $(\hat{U}_+)_a$ is regular at $k = k_0$ and $\hat{W}_a$ changes its sign at this point. Therefore, we can repeat the argument from the proof of Theorem 1. Namely, each determinant has two branch points: $k = \pm k_0$. The determinants change their signs at $\pm k_0$. Functions $(\hat{U}_+)_a$, $(\hat{U}_-)_a$ have simple poles at $k = k_*$ Moreover, the determinants have algebraic growth at infinity. Applying Liouville theorem, we conclude that $\sqrt{k_0^2 - k^2} C_j$ and $\sqrt{k_0^2 - k^2} H$ are rational functions of k.

Here we implicitly used the preserving of monodromy data. In the opposite case it is impossible to find the solutions of the ODE possessing the required properties at the singular points *simultaneously.*

Calculate the value of $(\mathbf{V}')_a$ from (3.2) and (3.3):

$$\begin{aligned} \frac{\partial}{\partial a}\left(\frac{\partial}{\partial k}\mathbf{V}\right) &= \left(\frac{\partial}{\partial a}\mathbf{K}\right)\mathbf{V} + \mathbf{K}\frac{\partial}{\partial a}\mathbf{V} = \left(\frac{\partial}{\partial a}\mathbf{K}\right)\mathbf{V} + \mathbf{KAV}, \\ \frac{\partial}{\partial k}\left(\frac{\partial}{\partial a}\mathbf{V}\right) &= \left(\frac{\partial}{\partial k}\mathbf{A}\right)\mathbf{V} + \mathbf{A}\frac{\partial}{\partial k}\mathbf{V} = \left(\frac{\partial}{\partial k}\mathbf{A}\right)\mathbf{V} + \mathbf{AKV}. \end{aligned}$$

Two last relations are valid for two linearly independent $\mathbf{V}$ at almost each point (k, a). Therefore, they are valid for the matrices themselves. Combining these relations, we obtain (3.4).

3.2 Scalar form of evolution equation

One can suppose that equation (3.4) is useless because it contains 4 new unknown functions in addition to D and E. However, below we show that functions C_j and H can be easily calculated. Now we will express C_j by H, D and E and later find H using Liouville theorem.

Suppose that the determinant $H(k, a)$ is a known function (we shall calculate it below). Consider the definitions (3.6), (3.7), (3.8) and (2.8). Note that

$$\frac{\partial H(k, a)}{\partial k} = C_4(k, a) - C_1(k, a). \tag{3.10}$$

On the other hand,

$$\frac{\partial D(k,a)}{\partial a} = C_1(k,a) + C_4(k,a). \tag{3.11}$$

Therefore

$$C_1(k,a) = \frac{1}{2}\left(\frac{\partial D(k,a)}{\partial a} - \frac{\partial H(k,a)}{\partial k}\right), \tag{3.12}$$

$$C_4(k,a) = \frac{1}{2}\left(\frac{\partial D(k,a)}{\partial a} + \frac{\partial H(k,a)}{\partial k}\right). \tag{3.13}$$

Now consider the definition (3.8). Note that

$$\frac{\partial C_1}{\partial k} = C_3 + \left|\begin{array}{cc} \hat{U}''_+ & (\hat{U}_+)_a \\ \hat{W}'' & (\hat{W})_a \end{array}\right| =$$

$$C_3 + f\left|\begin{array}{cc} \hat{U}'_+ & (\hat{U}_+)_a \\ \hat{W}' & (\hat{W})_a \end{array}\right| + g\left|\begin{array}{cc} \hat{U}_+ & (\hat{U}_+)_a \\ \hat{W} & (\hat{W})_a \end{array}\right| = C_3 + fC_1 - gH. \tag{3.14}$$

Therefore

$$C_3 = \frac{1}{2}\left(\frac{\partial^2 D}{\partial k \partial a} - \frac{\partial^2 H}{\partial k^2}\right) - \frac{1}{2D}\frac{\partial D}{\partial k}\left(\frac{\partial D}{\partial a} - \frac{\partial H}{\partial k}\right) + \frac{EH}{D}. \tag{3.15}$$

Substitute (3.10), (3.12), (3.13) and (3.15) into the left-hand side of (3.4). Three elements of resulting matrix are identically equal to zero, and the fourth leads to the equation

$$\left(\frac{E}{D}\right)_a - \left(\frac{EH^2}{D^3}\right)' \frac{D}{H} + \frac{D}{2}\left(\frac{1}{D}\left(\frac{H' - D_a}{D}\right)'\right)' = 0, \tag{3.16}$$

where prime denotes partial differentiation with respect to k and index a denotes partial differentiation with respect to a.

Equation (3.16) is the scalar form of evolution equation (3.4).

Now we return to the calculation of $H(k,a)$. Note that $(k^2 - k_0^2)^{1/2} H(k,a)$ has the following properties as a function of k: it grows as k^{-1} at infinity, it has a simple pole at $k = k_*$ and it is regular at $k = \pm k_0$. All these properties follow from the properties of the elements of the determinant (3.7). There are no singular points except $k = k_*$, therefore $(k^2 - k_0^2)^{1/2} H(k,a)$ is equal to $1/(k - k_*)$ multiplied by a constant, depending on a. This constant can be calculated comparing the behavior of D and H as $k \to \infty$. From (2.12) and (2.14), at infinity $D(k) \sim Mk^{-3}$. Let the following estimations be valid in the upper half-plane of k:

$$\hat{U}_+ \sim \alpha k^{-3/2} e^{\mathrm{i}ka}, \qquad \hat{W} \sim \beta k^{-3/2} e^{-\mathrm{i}ka}.$$

Then

$$M = 2\alpha\beta \mathrm{i} a.$$

On the other hand, at infinity

$$H \sim 2\alpha\beta \mathrm{i} k^{-2}.$$

Comparing last relations, we obtain

$$H(k,a) = \frac{M}{a}\frac{1}{(k - k_*)(k^2 - k_0^2)^{1/2}}. \tag{3.17}$$

Now H can be substituted in (3.16) and evolution equation, containing only the coefficients of the polynomials P and Q can be obtained. However, in general case the calculations are rather tedious, so here we study only the case of normal incidence $k_* = 0$.

Substitute (3.17) into (3.16). The left-hand side of (3.16) is a rational function of k and we should find such functions $X(a)$, $Y(a)$, $\lambda(a)$, that tthe l.-h.s. of (3.16) is equal to zero for each k. We find the decomposition of the l.-h.s. of (3.16) into the partial fractions. The numerators of the partial fractions do not depend on k. These numerators must be equal to zero separately. After some algebraic manipulations, we obtain that (3.16) is fulfilled identically if and only if the relation (2.26) is valid and, besides

$$\frac{d\lambda}{da} = \frac{2\lambda^2 - k_0^2 + (k_0^2 - \lambda^2)X}{2a\lambda}, \tag{3.18}$$

$$\frac{dX}{da} = \frac{4a^2\lambda^4 + k_0^2(X-2)X}{2a\lambda^2}, \tag{3.19}$$

Now it is not necessary to find λ and X for each a from a complicated numerical procedure based on the verification of monodromy data. One can calculate λ and X for any specific $a = a_0$, solve nonlinear differential equations (3.18, 3.19) using $\lambda(a_0)$ and $X(a_0)$ as initial conditions and obtain λ and X for any a. Parameter Y then can be found from (2.26).

3.3 Some properties of evolution equations

In this subsection we shall study equations (3.4), (3.16) and (3.18, 3.19) as a particular case of (3.16).

Note the following features of evolution equations:

1. Equations (3.4) and (3.16) remain valid if parameter a is replaced by another parameter of the problem (say, k_*). In this case one must find a new formula for $H(k, k_*)$ instead of (3.17). Namely, a simple calculation yields

$$H(k, k_*) = \frac{M(k_* - \lambda_1)(k_* - \lambda_2)}{(k_0^2 - k_*^2)(k^2 - k_0^2)^{1/2}(k - k_0)^2}. \tag{3.20}$$

This formula can be substituted into (3.16) and new evolution equation can be obtained. This equation will describe the behaviour of the coefficients of the polynomials $P(k)$ and $Q(k)$ as functions of k_*. In this paper we are not going to perform these calculations. However, we should state, that the procedure of derivation of evolution equations can be performed with respect to *any parameter* of the problem.

2. In our proof of Theorem 2 we use the fact, that there are two solutions of the ODE (say, $\hat{U}_+$ and $\hat{U}_-$) that for each a have the following properties: $\hat{U}_+$ is a fundamental solution at k_0, $\hat{U}_-$ is a fundamental solution at $-k_0$, and their linear combination with constant coefficients (namely, $\hat{W}$) is a fundamental solution at $\pm k_0$. Besides, $\hat{U}_+$ and $\hat{U}_-$ are decaying as $k \to \pm i\infty$ respectively. The particular asymptotics of $\hat{U}_+$ and $\hat{U}_-$ at singular points and the values of the coefficients are not important. Therefore, the same proof is valid for the functions $\bar{U}_-$, $\bar{U}_+$. Moreover, $H(k, a)$ is given by (3.17), and equations (3.18, 3.19) are valid for the slit problem. System (3.18, 3.19) has the solutions, corresponding to the strip and slit problems. For each problem one must find correct initial conditions. More detailed study of the validity of Theorem 2 can be found in [15].

3. Consider the strip problem with normal incidence for large $k_0 a$. As it is known, in this case the scattered field can be approximately represented as a sum of the fields, diffracted by the edges of the strip, i.e. a sum of two Sommerfeld's solutions. The field diffracted from the right edge of the strip is described by the function (see [9])

$$\hat{U}^0_+(k) = \frac{\mathrm{i}e^{\mathrm{i}ka}\sqrt{k_0}}{k\sqrt{k_0+k}}, \tag{3.21}$$

and from the left edge by the function

$$\hat{U}^0_-(k) = -\frac{\mathrm{i}e^{-\mathrm{i}ka}\sqrt{k_0}}{k\sqrt{k_0-k}}. \tag{3.22}$$

Find an ODE such that functions $\hat{U}^0_\pm$ obey it. It is easy to note that this equation has the form (2.4) with the coefficients $f = D_0'/D_0$, $g = E_0/D_0$, where

$$D_0 = \begin{vmatrix} (\hat{U}^0_-)', & \hat{U}^0_- \\ (\hat{U}^0_+)', & \hat{U}^0_+ \end{vmatrix} = k_0 \frac{k_0 + 2\mathrm{i}a(k^2-k_0^2)}{k^2(k_0^2-k^2)^{3/2}}, \tag{3.23}$$

$$E_0 = \begin{vmatrix} (\hat{U}^0_-)', & (\hat{U}^0_-)'' \\ (\hat{U}^0_+)', & (\hat{U}^0_+)'' \end{vmatrix} =$$
$$k_0 \frac{15k_0 + 6\mathrm{i}a(k^2-3k_0^2) + 12a^2k_0(k^2-k_0^2) + 8\mathrm{i}a^3(k^2-k_0^2)^2}{4k^2(k_0^2-k^2)^{5/2}}. \tag{3.24}$$

The last expressions exactly match the anzats (2.12, 2.13).

Substituting (3.21, 3.22) into (2.25), we obtain equation of the form (2.25) with $\lambda = \lambda_0$, $X = X_0$, $Y = Y_0$, where

$$\lambda_0(a) = \left(k_0^2 + \frac{\mathrm{i}k_0}{2a}\right)^{1/2}, \tag{3.25}$$
$$X_0(a) = -2\mathrm{i}ak_0 + 3, \tag{3.26}$$
$$Y_0(a) = 3a\mathrm{i}k_0 - \frac{15}{4}. \tag{3.27}$$

The following fact is noticeable. Functions (3.25), (3.26), (3.27) obey equations (3.18, 3.19) and equation (2.26). This fact can be easily understood. Functions $\hat{U}^0_+$ and $\hat{U}^0_-$ are the fundamental solutions of the ODE at $\pm k_0$ and the proof of Theorem 2 can be applied to these functions. Therefore, evolution equations describe not only the exact solutions of the slit and the strip problems, but also zero-order approximations of these solutions.

Let be $\lambda(a) = \lambda_0(a) + \delta\lambda(a)$, $X(a) = X_0(a) + \delta X(a)$, where $\delta\lambda(a)$ and $\delta X(a)$ are small values. Substituting these expressions into (3.18, 3.19), we obtain approximate linearized evolution equations. Using standard technique, one can obtain that $\delta\lambda \sim \exp[\pm 2\mathrm{i}ak_0]$ and $\delta X \sim \exp[\pm 2\mathrm{i}ak_0]$. The estimations show, that these terms have the phase coefficient, corresponding to the next diffraction order, i.e. it describes diffraction of $\hat{U}^0_+$ on the left edge and $\hat{U}^0_-$ on the right edge.

Functions D_0 and E_0 can be used as zero-order approximations of D and E for large $k_0 a$. below we discuss how other terms of asymptotic series for D and E can be constructed.

4 Schwarzschild series and Liouville theorem

4.1 Formulation of diffraction problem for successive diffraction fields

In the previous section we proved, that it is necessary to find λ and X only for one particular value of a; their values for other a can be found from evolution equations. However, it is not an easy problem to find even one pair of $\lambda(a)$, $X(a)$. Here we describe a procedure to find approximations of λ and X for large $k_0 a$. We suppose below that k_0 has a small positive imaginary part.

From the point of view of physics, one can describe diffraction on a strip as a sequence of diffractions on the edges of the strip. In zero-order approximation the scattered field is a sum of the fields, diffracted independently on the half-lines $(-\infty, a)$, $(-a, \infty)$. The result of the first diffraction (function $\hat{U}_+^0 + \hat{U}_-^0$) has been concerned in the previous section. Wave field corresponding to the spectrum $\hat{U}_+^0$ satisfies boundary conditions (1.2), (1.3) everywhere, except the half-line $(-\infty, -a)$, so this field "does not know" about the left edge of the strip. When this field reaches the left edge, the secondary diffraction occurs. Note that the scatterer in this case is not the screen (the field satisfies boundary condition on the screen), but the absence of the screen, where the normal derivative of the field must be continuous. When we study the secondary diffraction, we suppose that the screen occupies the half-line $(-a, \infty)$ etc. So, the wave is successively scattered by the right and the left edge of the strip. The same process beginning with $\hat{U}_-^0$ must also be studied.

The spectrum of the total field (the sum of all iterations) satisfies the ODE (2.4). Moreover, zero-order approximation of the spectrum (functions $\hat{U}_+^0$ and $\hat{U}_-^0$) also satisfy the ODE of the same kind. One can suppose, that it is valid for each order of diffraction. However, *it is not true*. The second-order diffraction field does not satisfy the ODE with rational coefficients (at least we cannot prove that it does). But in spite of that, some combinations of determinants happen to have the same structure as $D(k)$ and $E(k)$. This (rather nontrivial) fact is the statement of Theorem 3 proven in the end of current section.

Represent the scattered field $u(x, y)$ for $y > 0$ as the sum

$$u(x, y) = -e^{\mathrm{i}k_0 y} + \sum_{n=0}^{\infty} (u_+^n(x, y) + u_-^n(x, y)), \tag{4.1}$$

where $u_\pm^n$ (except the zero-order terms $u_\pm^0$) is the result of diffraction of the field $u_\mp^{n-1}$ on the half-line (a, ∞) or $(-\infty, -a)$ respectively. the zero-order term compensates non-zero values of the normal derivative of the reflected wave on the lines (a, ∞) and $(-\infty, -a)$ (it must be equal to 0 in accordance with boundary condition (1.3)).

Note that the upper index nowhere denotes power in this and subsequent sections.

The first term in (4.1) is the reflected field. It is known, that there is no contribution corresponding to the reflected plane wave in the total scattered field, but such contribution presents in the solutions of half-line problems. So, the first term in (4.1) and the plane wave term in $u_+^0 + u_-^0$ compensate each other.

It is not a simple task, however to establish in what sense the series (4.1) is asymptotical. One can prove that angular spectrum of the series (4.1) taken on the segment $(-\pi/2+\varepsilon, \pi/2-\varepsilon)$

with respect to y-axis is asymptotical (ε is a small positive number). Keller's Geometrical Theory of Diffraction (GTD) is based on this fact.

For each $n \geq 0$ functions $\hat{U}^n_\pm$ obey the boundary conditions on the "big screens":

$$\begin{aligned} u^n_+(x,0) &= 0 \quad \text{for} \quad x < a, \\ u^n_-(x,0) &= 0 \quad \text{for} \quad x > -a. \end{aligned} \tag{4.2}$$

For $n \geq 1$ each term compensates the discontinuity of the normal derivative of the previous term on the line $y = 0$:

$$\begin{aligned} \frac{\partial u^n_+(x,0)}{\partial y} &= -\frac{\partial u^{n-1}_-(x,0)}{\partial y} \quad \text{for} \quad x > a, \\ \frac{\partial u^n_-(x,0)}{\partial y} &= -\frac{\partial u^{n-1}_+(x,0)}{\partial y} \quad \text{for} \quad x < -a. \end{aligned} \tag{4.3}$$

Zero-order term compensates the normal derivative of the reflected field:

$$\begin{aligned} \frac{\partial u^0_+(x,0)}{\partial y} &= \mathrm{i}k_0 \quad \text{for} \quad x < a, \\ \frac{\partial u^0_-(x,0)}{\partial y} &= \mathrm{i}k_0 \quad \text{for} \quad x > -a. \end{aligned} \tag{4.4}$$

Indeed, each term satisfies Helmholtz equation.

Thus, for each term we have a sequence of specific half-line diffractional problems (they must be supplemented with radiation conditions and Meixner conditions at the edges $x = \pm a$).

Each half-line problem can be solved using standard Wiener-Hopf method.

4.2 The solution of successive diffraction problems using Wiener-Hopf method

Denote Fourier transforms of $u^n_\pm$ by

$$\hat{U}^n_+(k) = \int_a^\infty e^{\mathrm{i}kx} u^n_+(x,0)dx, \tag{4.5}$$

$$\hat{U}^n_-(k) = \int_{-\infty}^{-a} e^{\mathrm{i}kx} u^n_-(x,0)dx. \tag{4.6}$$

Comparing the definitions of $\hat{U}_\pm$ (1.8), (1.9) with (4.5), (4.6), we conclude that

$$\hat{U}_\pm(k) = \sum_{n=0}^\infty \hat{U}^n_\pm(k). \tag{4.7}$$

Restrictions (4.2–4.4) can be reformulated in the form of Wiener-Hopf functional problems.

Zero-order diffraction terms obey the following analytic restrictions:

$$\begin{aligned} \hat{U}^0_\pm e^{\mp ika} \quad & \text{is analytic for} \quad \mathrm{Im}[\pm k] > 0, \\ \left(\hat{U}^0_\pm \sqrt{k_0^2 - k^2} e^{\mp \mathrm{i}ka} \pm \frac{k_0}{k}\right) \quad & \text{is analytic for} \quad \mathrm{Im}[\pm k] < 0 \text{ and } k = 0. \end{aligned} \tag{4.8}$$

Note that functions $\hat{U}^0_\pm$ have simple pole at $k = 0$. Other terms $\hat{U}^n_\pm$ are regular at this point.

For all $n \geq 1$ regularity restrictions are

$$\begin{aligned} \hat{U}^n_\pm e^{\mp \mathrm{i}ka} \quad & \text{is analytic for} \quad \mathrm{Im}[\pm k] > 0 \text{ and } k = 0, \\ (\hat{U}^n_\pm + \hat{U}^{n-1}_\mp)\sqrt{k_0^2 - k^2} e^{\mp \mathrm{i}ka} \quad & \text{is analytic for} \quad \mathrm{Im}[\pm k] < 0 \text{ except } k = 0. \end{aligned} \tag{4.9}$$

Besides, functions $\hat{U}^n_\pm$ must grow no faster than $k^{-1}\exp\{\pm \mathrm{i}ak\}$ at infinity. It follows from Meixner conditions.

Skipping the details (which are discussed in [9]), we represent the solution of Wiener-Hopf problems (4.8, 4.9) in the form

$$\hat{U}^0_\pm(k) = \pm \frac{\mathrm{i}e^{\pm \mathrm{i}ak}\sqrt{k_0}}{k\sqrt{k_0 \pm k}} \tag{4.10}$$

$$\hat{U}^n_\pm(k) = -\frac{e^{\pm \mathrm{i}ka}}{\sqrt{k_0 \pm k}} F_\pm[\hat{U}^{n-1}_\mp(k) e^{\mp \mathrm{i}ka}\sqrt{k_0 \pm k}], \tag{4.11}$$

where $F_\pm$ are integral operators of Cauchy type

$$F_\pm[V(k)] = \pm \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm} \frac{V(\tau) d\tau}{\tau - k}. \tag{4.12}$$

Contours of integration $\gamma_\pm$ coincide with the real axis everywhere except the vicinity of $\tau = 0$. Contour γ_+ passes below $\tau = 0$, contour γ_- passes above $\tau = 0$.

The iterative procedure close to described above has been proposed by D.S.Jones [10]. He calculated approximately the terms $\hat{U}^1_\pm$ (in our notations), which provide ultimately good approximation for the solution even for the case of relatively small values of $k_0 a$.

Operators $F_\pm$ are known [9] to perform an additive decomposition of function V. For example,

$$V(k) = F_+[V(k)] + (V(k) - F_+[V(k)]),$$

where the first term is regular above the contour γ_+ and the second is analytical below this line. Operator F_+ in the form (4.12) defines $\hat{U}^n_+$ only above the contour γ_+. All other values of $\hat{U}^n_+$ must be found using analytic continuation. In the next section we shall obtain some useful properties of $F_\pm$.

For $n = 0$ representation (4.10) coincides with (3.21, 3.22).

Directly from the formulation of the functional problems one can find that functions $\hat{U}^m_\pm$ are regular at $k = \pm k_0$, functions $\hat{U}^n_\pm + \hat{U}^{n-1}_\mp$ change signs at $k = \mp k_0$ (see (4.8, 4.9)). This property will be used in the next subsection to prove Theorem 3. Integral representation (4.11) will be used in Section 5 for numerical calculations.

4.3 Representation of the coefficients of the ODE in terms of $\hat{U}^n_\pm$

Let us represent the determinants $D(k)$ and $E(k)$ in the form

$$D = \begin{vmatrix} \hat{U}'_-, & \hat{U}_- \\ \hat{U}'_+, & \hat{U}_+ \end{vmatrix} \qquad E = \begin{vmatrix} \hat{U}'_-, & \hat{U}''_- \\ \hat{U}'_+, & \hat{U}''_+ \end{vmatrix} \tag{4.13}$$

Substituting (4.7) into (4.13), we obtain representation of D and E in the form of formal series

$$\begin{aligned} D &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \begin{vmatrix} (\hat{U}^m_-)', & \hat{U}^m_- \\ (\hat{U}^{n-m}_+)', & \hat{U}^{n-m}_+ \end{vmatrix} \right), \\ E &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \begin{vmatrix} (\hat{U}^m_-)', & (\hat{U}^m_-)'' \\ (\hat{U}^{n-m}_+)', & (\hat{U}^{n-m}_+)'' \end{vmatrix} \right). \end{aligned} \tag{4.14}$$

Below we prove that not only the whole series, but *each term* of the series has a simple structure.

Theorem 3 *Finite sums*

$$D_n(k) = \sum_{m=0}^{n} \begin{vmatrix} (\hat{U}^m_-)', & \hat{U}^m_- \\ (\hat{U}^{n-m}_+)', & \hat{U}^{n-m}_+ \end{vmatrix} \tag{4.15}$$

and

$$E_n(k) = \sum_{m=0}^{n} \begin{vmatrix} (\hat{U}^m_-)', & (\hat{U}^m_-)'' \\ (\hat{U}^{n-m}_+)', & (\hat{U}^{n-m}_+)'' \end{vmatrix} \tag{4.16}$$

are rational functions of k multiplied by $\sqrt{k_0^2 - k^2}$. Moreover, functions $D_n(k)$ and $E_n(k)$ have the form of (2.12), (2.13).

Consider the function $D_n(k)$ for some $n > 0$. (the consideration of $E_n(k)$ is similar). Introduce the notation

$$|V_1, V_2| \equiv \begin{vmatrix} V'_1, & V_1 \\ V'_2, & V_2 \end{vmatrix} \tag{4.17}$$

for any functions $V_1(k)$ and $V_2(k)$. Obvious properties

$$\begin{aligned} |V_1, V_2| &= -|V_2, V_1|, \\ |V_1, V_1| &= 0, \\ |V_1, V_2 + V_3| &= |V_1, V_2| + |V_1, V_3| \end{aligned} \tag{4.18}$$

follow from (4.17).

Note that the sum $\sum_{m=0}^{n} |\hat{U}^m_-, \hat{U}^{n-m}_-|$ is equal to zero, because the terms with indexes m and $n-m$ compensate each other.

Rewrite (4.15) in the form

$$D_n = \sum_{m=0}^{n} |\hat{U}^m_-, \hat{U}^{n-m}_+| = \sum_{m=0}^{n} |\hat{U}^m_-, \hat{U}^{n-m}_+| + \sum_{m=0}^{n-1} |\hat{U}^m_-, \hat{U}^{n-m-1}_-| =$$

$$\sum_{m=0}^{n-1} |\hat{U}_-^m, \hat{U}_+^{n-m} + \hat{U}_-^{n-m-1}| + |\hat{U}_-^n, \hat{U}_+^0| \tag{4.19}$$

Note that each determinant $|\hat{U}_-^m, \hat{U}_+^{n-m} + \hat{U}_-^{n-m-1}|$ has the following property: it has no branch points in the lower half-plane except $k = -k_0$ and it changes its sign at this point (see (4.9)). Beside that, $|\hat{U}_-^m, \hat{U}_+^{n-m} + \hat{U}_-^{n-m-1}|$ has algebraic growth in the lower half-plane of k. The term $|\hat{U}_-^n, \hat{U}_+^0|$ has the same properties (note, that $\hat{U}_+^0$ is known algebraic function).

Adding the sum $\sum |\hat{U}_+^m, \hat{U}_+^{n-m-1}|$ to D_n, one can prove, that D_n changes its sign at $k = k_0$ and has algebraic growth in the upper half-plane. Using Liouville theorem, we find that $\sqrt{k_0^2 - k^2} D_n$ is a rational function. A more accurate study of singular points of D_n and E_n and their behavior at infinity shows that these functions have the form of (2.12), (2.13):

$$D_n(k) = \frac{P_n(k)}{(k^2 - k_0^2)^{3/2}(k - k_*)^2}, \qquad E_n(k) = \frac{Q_n(k)}{(k^2 - k_0^2)^{5/2}(k - k_*)^3},$$

where P_n and Q_n are the polynomials of order 2 and 5 respectively. Zero order functions D_0 and E_0 are given by (3.23), (3.24).

One possible way to calculate the polynomials P_n, Q_n for arbitrary n is to calculate D_n and E_n at several points k using the explicit integral representation (4.11). In the following section we propose another method for calculating D_1 and E_1. Meanwhile, this method gives an independent proof of Theorem 3 for the particular case of $n = 1$.

5 Method of calculations and numerical results

5.1 Elementary properties of integral representation (4.11)

For the numerical calculations below we will use the truncated series (4.14):

$$D(k) \approx D_0(k) + D_1(k), \qquad E(k) \approx E_0(k) + E_1(k). \tag{5.1}$$

Now we are focusing on calculation of D_1 and E_1. Direct substitution of the integral representation (4.11) into the definitions (4.3) and (4.4) of D_1 and E_1 leads to ugly formulas containing integrals with parameters. At a glance it is not clear why D_1 and E_1 have the form of r.-h.s. of (2.12, 2.13). In the following subsection we obtain a convenient representation of $(\hat{U}_\pm^1)'$, which yields reasonable expressions for D_1 and E_1.

Let us establish here some elementary properties of operators $F_\pm$, defined in the previous section.

1. It is obvious that $F_\pm$ are linear operators, so for a constant c and arbitrary V, V_1, V_2

$$F_\pm[cV(k)] = cF_\pm[V(k)], \qquad F_\pm[V_1(k) + V_2(k)] = F_\pm[V_1(k)] + F_\pm[V_2(k)].$$

Here we do not discuss the class of functions, to which the operators $F_\pm$ can be applied. However, there is no doubt that all our functions are "good" in this sense because of exponential factors.

2. For a wide class of functions V

$$(F_\pm[V])' = F_\pm[V'] \tag{5.2}$$

This property can be proven by integration by parts:

$$(F_\pm[V(k)])' = \pm\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pm}\frac{V(\tau)d\tau}{(\tau-k)^2} = \mp\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pm}V(\tau)d\left(\frac{1}{\tau-k}\right) = \pm\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pm}\frac{V'(\tau)d\tau}{\tau-k}.$$

3. For each k_1 not lying on contour $\gamma_\pm$

$$F_\pm\left[\frac{V}{k-k_1}\right] = \frac{F_\pm[V]}{k-k_1} + \frac{\mathcal{F}_\pm(V,k_1)}{k-k_1}, \tag{5.3}$$

where

$$\mathcal{F}_\pm(V,k_1) = \mp\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pm}\frac{V(\tau)d\tau}{\tau-k_1}. \tag{5.4}$$

This property follow from an elementary relation

$$\frac{1}{(\tau-k_1)(\tau-k)} = \frac{1}{k-k_1}\left(\frac{1}{\tau-k} - \frac{1}{\tau-k_1}\right).$$

Note, that $\mathcal{F}_\pm(V,k_1)$ does not depend on k.

Equation (5.3) can be interpreted as follows. Operator F_+ (4.11) provides the additive decomposition of function V into two terms, one of which is analytic above the contour γ_+, and another is analytic in the lower one:

$$V(k) = V_+(k) + V_-(k); \qquad V_+(k) = F_+[V(k)].$$

Let us decompose the same way the function $V/(k-k_1)$. Decomposition of the form

$$\frac{V}{k-k_1} = \frac{V_+}{k-k_1} + \frac{V_-}{k-k_1}$$

is not valid because of an undesired pole at $k = k_1$, which either belongs to the first term in the upper half-plane or to the second term in the lower half-plane. However, this pole can be easily subtracted from the corresponding term. The value $-\mathcal{F}_+(V,k_1)$ is the residue of either the first or the second term at k_1.

5.2 Calculation of $D_1(k)$ and $E_1(k)$

For calculation of D_1 we need a convenient representation of $\hat{U}^1_\pm$ instead of (4.11). Using the integral representation (4.11) and the property (5.2), we obtain

$$(\hat{U}^1_\pm)' = \left(\pm\mathrm{i}a - \frac{1}{2(k\pm k_0)}\right)\hat{U}^1_\pm - \frac{e^{\pm\mathrm{i}ka}}{\sqrt{k_0\pm k}}\,F_\pm\left[\left(\hat{U}^0_\mp e^{\mp\mathrm{i}ka}\sqrt{k_0\pm k}\right)'\right]. \tag{5.5}$$

Using explicit formula (4.10) for $\hat{U}^0_{\pm}$, we note that

$$\left(\hat{U}^0_{\mp} e^{\mp ika}\sqrt{k_0 \pm k}\right)' = \left(\mp 2\mathrm{i}a - \frac{1}{k} - \frac{1}{2(k \mp k_0)} + \frac{1}{2(k \pm k_0)}\right) \hat{U}^0_{\mp} e^{\mp ika}\sqrt{k_0 \pm k}, \tag{5.6}$$

Substituting (5.6) into (5.5) and applying (5.3), we obtain

$$(\hat{U}^1_{\pm})' = \left(\pm \mathrm{i}a - \frac{1}{k} - \frac{1}{2(k \pm k_0)}\right) \hat{U}^1_{\pm} \pm \frac{\mathrm{i}e^{\pm ika}\sqrt{k_0}}{\sqrt{k_0 \pm k}} \left[\frac{\mathcal{F}^1_{\pm}}{2(k \pm k_0)} - \frac{\mathcal{F}^2_{\pm}}{2(k \mp k_0)} - \frac{\mathcal{F}^3_{\pm}}{k}\right], \tag{5.7}$$

where

$$\begin{aligned}
\mathcal{F}^1_{\pm} &= \mathcal{F}_{\pm}\left(\frac{e^{\mp 2ika}\sqrt{k_0 \pm k}}{k\sqrt{(k_0 \mp k)}}, \mp k_0\right), \\
\mathcal{F}^2_{\pm} &= \mathcal{F}_{\pm}\left(\frac{e^{\mp 2ika}\sqrt{k_0 \pm k}}{k\sqrt{(k_0 \mp k)}}, \pm k_0\right), \\
\mathcal{F}^3_{\pm} &= \mathcal{F}_{\pm}\left(\frac{e^{\mp 2ika}\sqrt{k_0 \pm k}}{k\sqrt{(k_0 \mp k)}}, 0\right).
\end{aligned}$$

Using the definition of $\mathcal{F}_{\pm}$, we note that

$$\mathcal{F}^j_+ = -\mathcal{F}^j_- \equiv \mathcal{F}_j \quad \text{for} \quad j = 1, 2, 3.$$

Equation (5.7) can be used for calculation of $(\hat{U}^2_{\pm})$ e.t.c. Using this technique, one can obtain differential equation of order n for any $\hat{U}^n_{\pm}$.

Note that (5.7) is an inhomogeneous differential equations of order 1 ($\mathcal{F}_j$ are constants). This equation has some noticeable features. the series (4.7) is not asymptotic series for arbitrary k and some special methods must be used for constructing the solution near $k = \pm k_0$ (see [9]). However, equation (5.7) involves three constants, which are *asymptotically small!* Namely, $\mathcal{F}_1$ is of order $(k_0 a)^{-1/2} \exp \mathrm{i}k_0 a$; $\mathcal{F}_2$ and $\mathcal{F}_3$ are of order $(k_0 a)^{-3/2} \exp \mathrm{i}k_0 a$ (it can be proven by deforming contour $\gamma_{\pm}$ into the loop, coming from $\pm \mathrm{i}\infty$ and circling $\pm k_0$). Later we will see that D_1 and E_1 are the linear combinations of $\mathcal{F}_j$, so D_1 and E_1 are exponentially small relatively to D_0, E_0. More detailed investigation shows that series (4.14) is asymptotical for each k.

Note also that equation (5.7) is valid on the whole Riemann surface of $\hat{U}^1_{\pm}$, and representation (4.11) is valid only above γ_+ for $\hat{U}^1_+$ and below γ_- for $\hat{U}^1_-$.

Formula (5.7) is too simple to be new, but the author is not aware of any references on similar results.

Equation (5.7) enables to rewrite $(\hat{U}^1_{\pm})'$ and $(\hat{U}^1_{\pm})'$ in terms of $\hat{U}^1_{\pm}$. Substitute (5.7)into the definitions of D_1 and E_1. Performing simple calculations, we see, that the terms containing $\hat{U}^1_{\pm}$ vanish and functions D_1, E_1 become expressed in terms of $\mathcal{F}_j$:

$$D_1 = \frac{k_0[k^2(\mathcal{F}_1 - \mathcal{F}_2 - 2\mathcal{F}_3) + 2k_0^2\mathcal{F}^3]}{k^2(k_0^2 - k^2)^{3/2}}, \tag{5.8}$$

$$E_1 = -\frac{k_0 q_1(k)}{4k^2(k_0^2 - k^2)^{5/2}}, \tag{5.9}$$

where

$$q_1(k) = 2k_0^2(-5+2\mathrm{i}ak_0)\mathcal{F}_1 + 2(3k_0^2+3\mathrm{i}ak_0^3)\mathcal{F}_2 - 2(-7k_0^2+4\mathrm{i}ak_0^3+4a^2k_0^4)\mathcal{F}_3 +$$
$$k^2[(-3+8\mathrm{i}ak_0+4a^2k_0^2)\mathcal{F}_1 + (3-4a^2k_0^2)\mathcal{F}_2 + 2(3-4\mathrm{i}ak_0-8a^2k_0^2)\mathcal{F}_3] +$$
$$k^4(-4a^2\mathcal{F}_1 + 4a^2\mathcal{F}_2 + 8a^2\mathcal{F}_3).$$

Expressions (5.8, 5.9) have the form of (2.12, 2.13). This fact is an agreement with Theorem 3 for the particular case of n=1. Note, that this result is obtained without applying Liouville theorem.

Formula (5.4) provides an explicit integral representation for the values of $\mathcal{F}_j$ for each a and k_0, so (5.8), (5.9) with (3.23), (3.24) can be used for approximate numerical calculation of D and E according to (5.1).

Since $\mathcal{F}_j$ are asymptotically small, D_1 and E_1 are asymptotically small relatively to D_0 and E_0.

All the results above concerned the *strip* problem. One can repeat point by point this procedure for the *slit* problem as well. All the calculations are similar, so below we write down only the final expressions for the "slit" D_0, E_0, D_1, E_1.

$$D_0 = \frac{k_0 - 2\mathrm{i}a(k^2-k_0^2)}{k^2k_0(k_0^2-k^2)^{3/2}}, \tag{5.10}$$
$$E_0 = -\frac{15k_0 - 6\mathrm{i}a(k^2-3k_0^2) + 12a^2k_0(k^2-k_0^2) - 8\mathrm{i}a^3(k^2-k_0^2)^2}{4k_0k^2(k_0^2-k)^{5/2}}, \tag{5.11}$$

$$D_1 = \frac{k^2(\mathcal{F}_1-\mathcal{F}_2+2\mathcal{F}_3) - 2k_0^2\mathcal{F}_3}{k_0k^2(k_0^2-k^2)^{3/2}}, \tag{5.12}$$
$$E_1 = \frac{q_1(k)}{4k_0k^2(k_0^2-k^2)^{5/2}}, \tag{5.13}$$

where

$$q_1(k) = 2k_0^2(3-2\mathrm{i}ak_0)\mathcal{F}_1 - 2k_0^2(5+2\mathrm{i}ak_0)\mathcal{F}_2 - 2k_0^2(7+4\mathrm{i}ak_0-4a^2k_0^2)\mathcal{F}_3 +$$
$$k^2((3-4a^2k_0^2)\mathcal{F}_1 + (4a^2k_0^2-3-8\mathrm{i}ak_0)\mathcal{F}_2 + (6+8\mathrm{i}ak_0-16a^2k_0^2)\mathcal{F}_3) +$$
$$k^4(4a^2\mathcal{F}_1 - 4a^2\mathcal{F}_2 + 8a^2\mathcal{F}_3),$$

$$\mathcal{F}_1 = \mathcal{F}_+\left(\frac{e^{\mp 2\mathrm{i}ka}\sqrt{k_0\mp k}}{k\sqrt{(k_0\pm k)}}, \mp k_0\right),$$
$$\mathcal{F}_2 = \mathcal{F}_+\left(\frac{e^{\mp 2\mathrm{i}ka}\sqrt{k_0\mp k}}{k\sqrt{(k_0\pm k)}}, \pm k_0\right),$$
$$\mathcal{F}_3 = \mathcal{F}_+\left(\frac{e^{\mp 2\mathrm{i}ka}\sqrt{k_0\mp k}}{k\sqrt{(k_0\pm k)}}, 0\right).$$

Note, that $\mathcal{F}_1$ now has asymptotical estimation $(k_0a)^{1/2}\exp[\mathrm{i}k_0a]$.

5.3 Numerical results

Our purpose here is to show that equations (2.4) in the form (2.25) and (3.18), (3.19) provide reasonable results, which are in good agreement with the GTD and Jones' approximations.

In the calculations using our method we use use truncations (5.1) for estimation of D and E.

Consider the strip problem with normal incidence. Using (5.8), (5.9) we calculate the parameters $X(a)$ and $\lambda^2(a)$. We choose $k_0 = 1$. The results are presented in Fig. 2 (dots); the solid line represents the solution of evolution equation (3.18, 3.19). The pair (X, λ) obtained from (5.8, 5.9) for $k_0 a = 10$ has been used as initial conditions for the evolution equation. One can see that the agreement between two different methods of calculation is very good for this case.

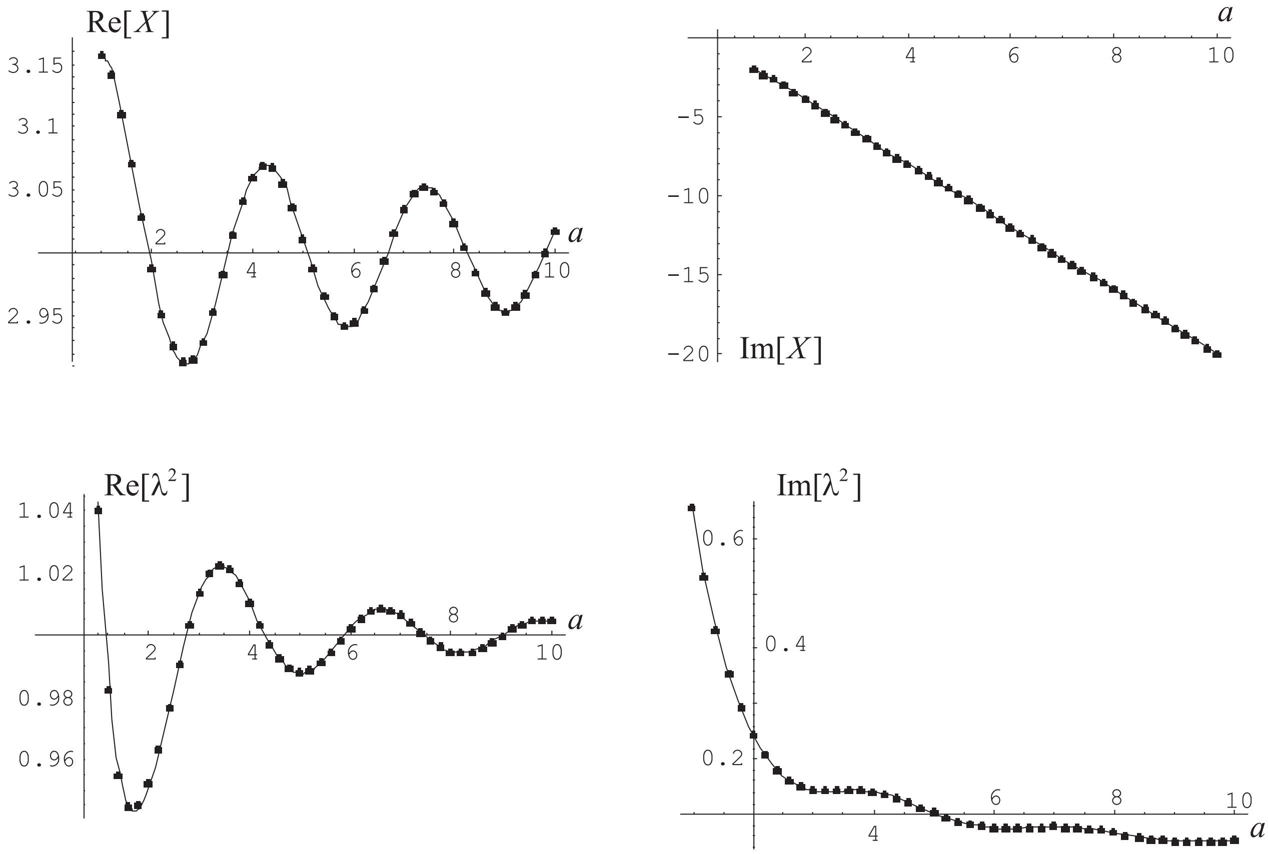

Figure 2: Values of λ^2 and X for the strip problem. Points are calculated using first-order diffraction approximation, solid line is the solution of the evolution equations.

Now we compare the solutions of ODE (2.25) with the GTD approximation. We use the following representation of GTD formula ([5]):

$$\hat{W}_{\mathrm{GTD}}(k) = -\frac{\mathrm{i}\sqrt{k_0}}{k}\left(\frac{e^{ika}}{\sqrt{k_0+k}} - \frac{e^{-ika}}{\sqrt{k_0-k}}\right) +$$

$$\frac{(\mathrm{i}k_0)^{1/2} e^{2\mathrm{i}ak_0}}{8\sqrt{\pi}(ak_0)^{3/2}}\left(\frac{e^{ika}}{(k_0+k)^{3/2}} - \frac{e^{-ika}}{(k_0-k)^{3/2}}\right) \tag{5.14}$$

We also use for comparison the zero-order approximation $\hat{W}^0 = -\hat{U}^0_+ - \hat{U}^0_-$, where $\hat{U}^0_\pm$ are defined by (4.10).

Instead of Jones' approximate formulas we use his exact integral representation for first-order diffraction term. It takes several seconds to perform such direct calculations using a computer of moderate computational abilities.

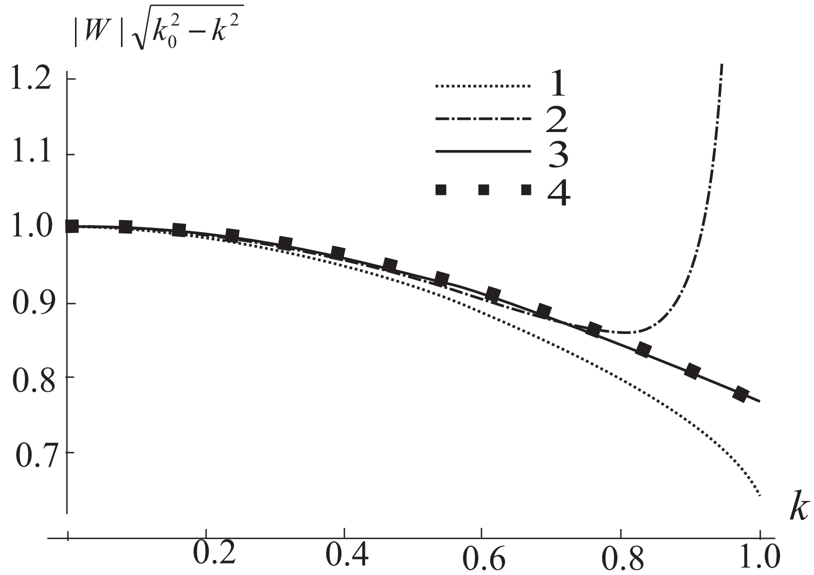

Figure 3: Normalized diffraction patterns for $k_0 a = 1$ (strip problem). 1 — Zero-order diffraction approximation; 2 — GTD approximation; 3 — the solution of the ODE (the coefficients are calculated using the first-order diffraction approximation); 4 — Jones' approximation (two first diffraction terms).

Fig. 3 represents the graphs of the absolute values of normalized $\hat{W}_0\sqrt{k_0^2-k^2}$, $\hat{W}_{\mathrm{GTD}}\sqrt{k_0^2-k^2}$, $\hat{W}\sqrt{k_0^2-k^2}$ and $\hat{W}_J\sqrt{k_0^2-k^2}$ (Jones') (curves 1, 2, 3, 4 respectively) All functions are multiplied by constants, such that their values at $k=0$ are equal to 1. Normalized function $\hat{W}$ is the solution of the ODE (2.25) calculated under the initial conditions $\hat{W}'(0) = 0$, $\hat{W}(0) = 1/k_0$. Parameters λ, X are taken from the results presented in Fig. 2 (dots). One can see that $\hat{W}$ is close to the GTD approximation everywhere except the proximity of $k = k_0$, where the GTD theory is not valid.

One can see that Jones' approximation is very close to our solution everywhere. It is not surprising because we used the same first-order diffraction terms to calculate the approximaion of the coefficients of the ODE. However, it is not true that Jones' solution obeys the eqution (2.4). One can show that this solution obeys a homogeneous equation of order 4.

Parameter $k_0 a$ is chosen to be equal to 1, since for larger $k_0 a$ the curves are too close to each other.

Fig. 4 and Fig. 5 represent the same results as Fig. 2 and Fig. 3 but for the slit problem. For this case

$$\bar{W}_{\mathrm{GTD}} = \frac{1}{\sqrt{k_0}k}\left(\frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}} - \frac{e^{-\mathrm{i}ka}}{\sqrt{k_0+k}}\right) +$$

$$\frac{\mathrm{i}^{-3/2}e^{2\mathrm{i}k_0a}}{\sqrt{\pi a}k_0}\left(\frac{e^{\mathrm{i}ak_0}}{\sqrt{k_0-k}(k_0+k)}+\frac{e^{-\mathrm{i}ak_0}}{\sqrt{k_0+k}(k_0-k)}\right) \tag{5.15}$$

Figures 5a, 5b and 5c correspond to $k_0a = 3$, 8 and 20 respectively. Figure 5b can be compared with [6].

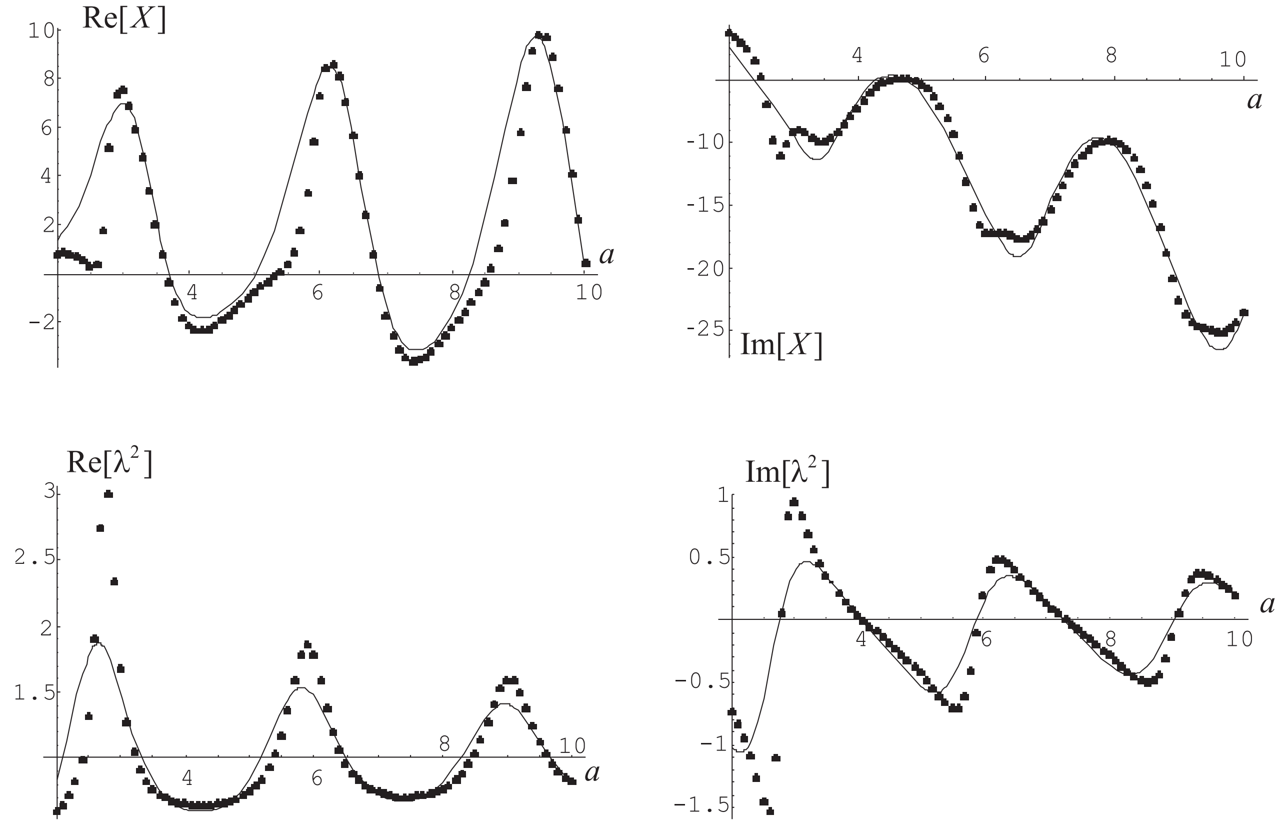

Figure 4: Values of λ^2 and X for the slit problem.

One can note that the difference between the parameters X and λ calculated using the approximate formulas and the evolution equation is bigger than that of strip problem. It can explained by the fact that the magnitude of the edge to edge interaction for the slit is greater than for the strip. Namely, the magnitude of the field, radiated by the edge of the slit and reaching another edge is of order $(k_0a)^{-1/2}\exp\{2\mathrm{i}k_0a\}$. Such magnitude for the strip problem is of order $(k_0a)^{-3/2}\exp\{2\mathrm{i}k_0a\}$.

One possible measure of the validity of the solution for the slit problem is the value of $\bar{W}\sqrt{k_0^2-k^2}$ at $k = \pm k_0$. For exact solution it must be equal to zero. One can see that for relatively small $k_0a = 3$ the solution of (2.25) is reasonably good in this sense.

6 Conclusions

1. Above we have represented a new approach to a specific functional problem emerging in a diffraction problem. This approach is, roughly speaking, in reducing the functional problem to a boundary problem for some ODE with rational coefficients. One can note that in its current form our method can hardly be named practical for solving the problem of diffraction by a strip; simplier methods give results that are numerically satisfactory. So it would be better to say that here we only prove some theorems concerning the functional problems; the practical benefits can be enjoyed only after the significant improvements of the method.

The main source of numerical difficulties is the presence of many unknown parameters including the location of some singular points of the ODE. Fortunately, there are some reasons to hope that the structure of the ODE can be significantly simplified. One possible way to simplify the ODE is to use the idea of Craster [16], who proposed to search the solutions of some specific ODEs in the form of linear combinations of solutions of simplier ODEs. As we can see now, the ODE (2.4) admits such a transformation. Another way is to develop the technique of summation of Schwarzchild's series. It can be made with the help of the trick used here for approximate calculation of the coefficients of the ODE. A detailed study of the series shows that the set of ODEs can be derived directly from the series and this set has a simplier structure. The author is going to continue the work in these two direction and hopes that finally an effective tool for diffraction problems will be developed.

2. The ideas discussed in the paper hardly can be interesting if they can be applied only to such a particular problem as diffraction on an ideal strip. We have some ideas how to apply these ideas to a wider class of problems. Sommerfeld proposed to study some 2D diffraction problems as acoustic propagation on multi-sheet "Riemann" surfaces [13]. *We think that the method proposed above works for diffraction problems that can be interpreted as propagation on the surfaces with finite number of sheets.*

Namely, the problem studied above is propagation on a surface with 2 sheets and 2 branch points (the edges). The simpliest generalization is to study the diffraction on a set of strips located in one plane (2 sheets and $2N$ branch points). The analysis made above can be applied to this problem almost unchanged. A more difficult structure is a juncture of 2 strips at right angle. This structure is equivalent to propagation on 4-sheet surface with 5 branch points. A special technique (see [17]) can be used for deriving the functional problem for this case. Of course, the work on the interesting and complicated problems mentioned here can be performed only after the development of effective methods of solving the ODEs.

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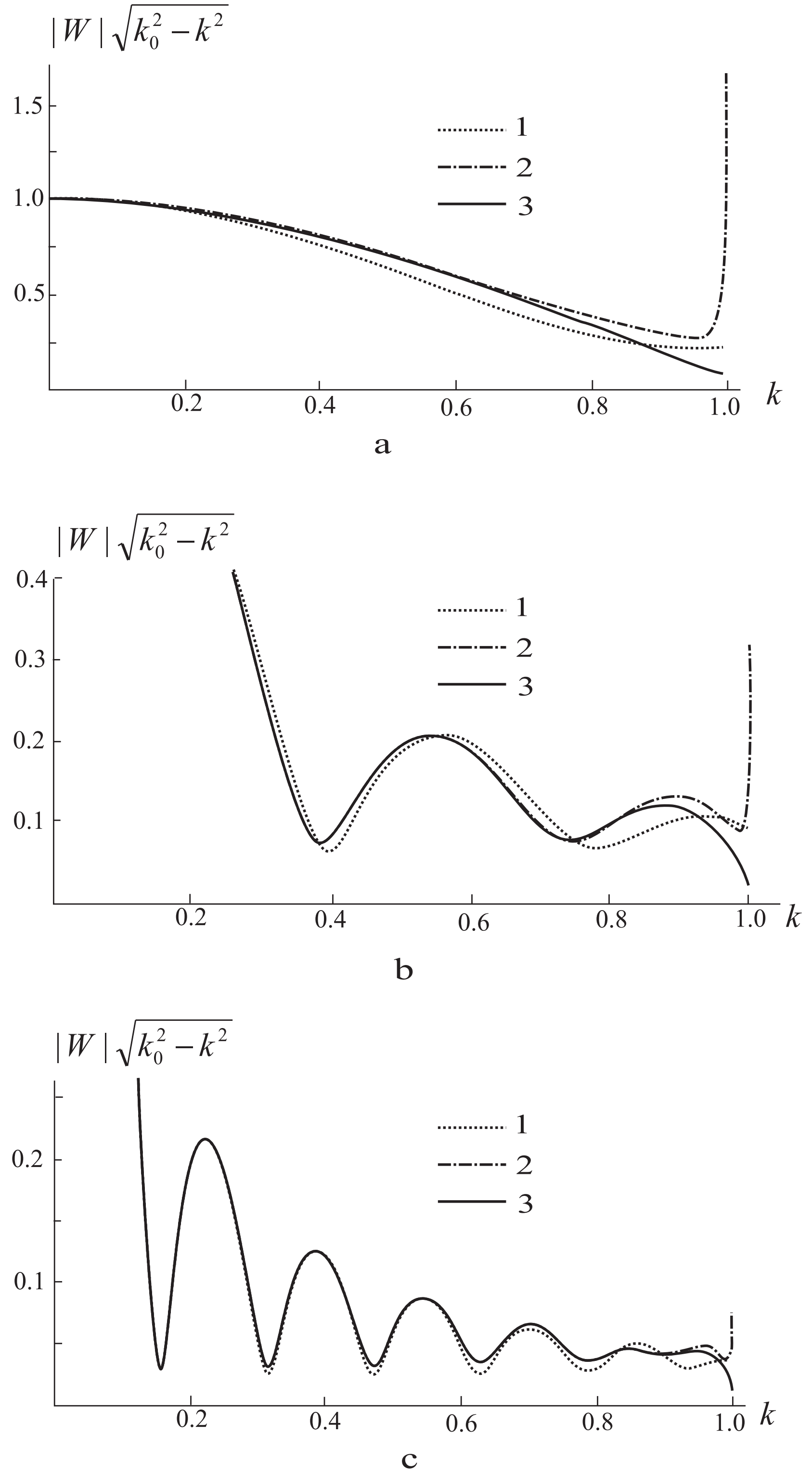

Figure 5: Normalized diffraction patterns for the slit problem. a) $k_0a = 3$, b) $k_0a = 8$, c) $k_0a = 20$. 1 — Zero-order diffraction approximation; 2 — GTD approximation; 3 — the solution of the ODE.