An Extension of Wiener-Hopf Method: Ordinary Differential Equations Associated with Diffraction Problems

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The problem of plane wave scattering on a strip or a set of strips situated in one plane is under consideration. A functional equation of Wiener-Hopf type with analytical restrictions on unknown functions is derived. It is shown that the solution of the problem (the spectrum of the scattered field) is a solution of an ordinary differential equation (ODE). The coefficients of the ODE are known up to several numerical constants. The restrictions enabling to determine the constants are discussed. Thus, the problem of diffraction of strips is reduced to the problem of finding the numerical constants and solving the ODE instead of solving the integral equation.

Introduction

The problem of diffraction of a plane harmonic wave on an infinite strip with Dirichlet boundary condition is studied. This problem can be solved explicitly in elliptic coordinates. However, this solution is not satisfactory for numerical calculations since the series converge slow for large wave size of the strip. This problem had been excessively studied in sixties [1, 2] in order to build a satisfactory asymptotics. An iterative procedure for solving the boundary integral equations was constructed for this problem.

Some attempts have been performed to apply Wiener-Hopf method to the problem of diffraction on a strip. However, these attempts faced a serious obstacle. Wiener-Hopf equations, constructed for the problems of diffraction on a strip or a slit involve an entire function with known order of growth [3]. No effective method to represent an entire function with known properties has been proposed. An approximate Wiener-Hopf method has been developed for obtaining asymptotic solution. As a result the elegance of Wiener-Hopf method was lost.

Here we introduce a method applicable to solving the mixed boundary-value problem, which can be reduced to a Wiener-Hopf functional equation, involving unknown entire functions. The idea of the method is the following. The behavior of the unknown functions on the complex plane can be easily studied. Some properties of unknown functions seem to be similar to typical properties of solutions of a homogenous ordinary differential equation (ODE). It is proven that the solutions actually obey an ODE with rational coefficients. A detailed analysis of singular points shows that the coefficients are determined up to a finite number of numerical parameters. The restrictions on unknown parameters can be obtained.

Problem formulation and derivation of functional equation

Consider a classical problem with mixed boundary conditions. Let the equation

$$\Delta u + k_0^2 u = 0 \tag{1}$$

be fulfilled for the function $u(x, y)$ in the half-plane $y > 0$, $-\infty < x < \infty$. Time dependence of all values is chosen as $e^{-\mathrm{i}\omega t}$, i.e. a plane wave propagating in the positive direction os x axis has the form $e^{\mathrm{i}k_0 x}$.

Boundary conditions are:

$$u(x, 0) = -e^{-\mathrm{i}k_* x} \qquad \text{for} \qquad -a < x < a. \tag{2}$$

and

$$\frac{\partial u(x, 0)}{\partial y} = 0 \qquad \text{for} \qquad -\infty < x < -a \quad \cup \quad a < x < \infty. \tag{3}$$

We are supposing that there are no waves coming from infinity and no growing terms (the field satisfies Sommerfeld's conditions at infinity). We also expect that the field and its derivative possess a known asymptotics in the vicinity of the points $-a$ and a. The asymptotics is taken from the exact solution of the problem of diffraction on a semi-infinite screen. This restriction corresponds to Meixner's condition.

The problem (1, 2, 3) is equivalent to the problem of diffraction on an infinite strip (or on a segment for 2D section). The incident field is a plane wave coming at the angle ψ to y-axis, such that $k_* = k_0 \sin \psi$.

The field can be presented in the form of Fourier integral

$$u(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{W}(k) e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2 - k^2} y} dk, \tag{4}$$

where the contour of integration coincides with the real axis almost everywhere and passes below the point k_0 and above the point $-k_0$.

Representation (4) can be used for calculation of $y-$derivative of the field on $x-$axis. Applying Fourier transform and taking into account (3), we obtain

$$\hat{W}(k) = \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_{-a}^{a} \frac{\partial u(x, 0)}{\partial y} e^{\mathrm{i}kx} dx. \tag{5}$$

Applying Fourier transform to function $u(x, 0)$ and taking into account (2), we obtain

$$\hat{U}_+(k) + \hat{U}_-(k) + \hat{W}(k) = 0, \tag{6}$$

where

$$\hat{U}_+(k) = \int_{a}^{\infty} u(x, 0) e^{\mathrm{i}kx} dx + \frac{\mathrm{i}e^{\mathrm{i}(k - k_*)a}}{k - k_*}, \tag{7}$$

$$\hat{U}_-(k) = \int_{-\infty}^{-a} u(x, 0) e^{\mathrm{i}kx} dx - \frac{\mathrm{i}e^{-\mathrm{i}(k - k_*)a}}{k - k_*}, \tag{8}$$

A similar equation has been obtained in [3].

Note that due to (7, 8, 5) the following properties of unknown functions are valid: $\hat{U}_+(k)$ is defined and has no singularities in the upper complex half plane k; $\hat{U}_-(k)$ is defined and has no singularities in the lower half plane; $\sqrt{k_0^2 - k^2}\hat{W}(k)$ is an entire function of k. Note also that there are no waves coming from the infinity, therefore $\hat{U}_+(k)$ has no singularities on the positive real half-axis (except $k = k_*$) and $\hat{U}_-(k)$ has no singularities on the negative real half-axis (except $k = k_*$). At $k = k_*$ both $\hat{U}_+(k)$ and $\hat{U}_-(k)$ have a simple pole at k_*. Note that $\hat{U}_+(k)$ is a regular function at $k = k_0$ and $\hat{U}_-(k)$ is a regular function at $k = -k_0$ (in the opposite case there must be cylindrical waves coming from infinity).

Using (6) one can build analytical continuation of $\hat{U}_+(k)$ and $\hat{U}_-(k)$ to a Riemann's surface over the complex plane. For our purposes we need only the prolongation of $\hat{U}_+(k)$ into the lower half-plane and of $\hat{U}_-(k)$ into the upper one from the real axis. Note that the function $\hat{U}_+(k)$ obtained from this continuation is analytical on the complex plane cut along the line $(-\infty, -k_0)$ (except the pole at k_*); $\hat{U}_-$ is analytical on the complex plane cut along the line (k_0, ∞) (except the pole at k_*).

Accordingly to the exact solution of a simpler problem of diffraction on a semi-infinite screen [3], we are looking for the functions $\hat{U}_+(k)$, $\hat{U}_-(k)$, $\hat{W}(k)$ with the following asymptotic expansions in the upper and lower half-planes:

$$
\begin{aligned}
\hat{U}_+(k) &\sim \frac{e^{\mathrm{i}ka}}{k^{3/2}} \sum_{n=0}^{\infty} \frac{a_n}{k^n} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\
\hat{U}_-(k) &\sim \frac{e^{-\mathrm{i}ka}}{k^{3/2}} \sum_{n=0}^{\infty} \frac{b_n}{k^n} \quad \text{for} \quad \mathrm{Im}[k] < 0, \\
\hat{W}(k) &\sim \frac{e^{-\mathrm{i}ka}}{k^{3/2}} \sum_{n=0}^{\infty} \frac{c_n}{k^n} \quad \text{for} \quad \mathrm{Im}[k] > 0, \\
\hat{W}(k) &\sim \frac{e^{\mathrm{i}ka}}{k^{3/2}} \sum_{n=0}^{\infty} \frac{d_n}{k^n} \quad \text{for} \quad \mathrm{Im}[k] < 0,
\end{aligned}
\tag{9}
$$

These asymptotics can be illustrated as follows. Let r and ϕ be local polar coordinates in the proximity of the point $x = a$ and let the direction $\phi = 0$ correspond to the negative x direction. Then the total field (the incident plane wave plus the scattered field u) can pe presented as a series in Bessel's functions

$$
u_{tot}(r, \phi) = \sum_{n=1}^{\infty} q_n J_{n/2}(k_0 r) \sin(n\phi/2).
$$

This representation leads to (9).

Another (Meixner's) argument takes into account the energy flow from the vertexes of the segment $(-a, a)$.

Equation (6) with asymptotics (9) and restrictions on analyticity of functions $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$ form the functional equation under consideration.

Differential equation associated with the functional equation

Suppose that the solution of the problem is known, i.e. functions $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$ possessing the properties mentioned above have been found. Let us find a linear homogenous ordinary differential equation (ODE) of the second order

$$V''(k) + f(k)V'(k) + g(k)V(k) = 0, \tag{10}$$

such that the functions $\hat{U}_+(k)$, $\hat{U}_-(k)$ and $\hat{W}(k)$ are its solution. Prime corresponds to the derivation in k.

It is obvious that we can demand that only two functions obey the equation, the third obeys it due to (6).

Note that one can find a differential equation, such that arbitrary two functions obey it. Let for example $\hat{U}_+(k)$ and $\hat{W}(k)$ be such functions. Than the pair of equations is valid:

$$\begin{aligned} f(k)\hat{U}'_+(k) + g(k)\hat{U}_+(k) &= -\hat{U}''_+(k), \\ f(k)\hat{W}'(k) + g(k)\hat{W}(k) &= -\hat{W}''(k). \end{aligned} \tag{11}$$

This set can be treated as a system of linear algebraic equations with respect to $f(k)$ and $g(k)$. Its solution is given by the formulas

$$f(k) = -\frac{D'(k)}{D(k)}, \tag{12}$$

$$g(k) = -\frac{E(k)}{D(k)}, \tag{13}$$

where $D(k)$ and $E(k)$ are the determinants

$$D(k) = \begin{vmatrix} \hat{U}'_+(k), & \hat{U}_+(k) \\ \hat{W}'(k), & \hat{W}(k) \end{vmatrix} = -\begin{vmatrix} \hat{U}'_-(k), & \hat{U}_-(k) \\ \hat{W}'(k), & \hat{W}(k) \end{vmatrix}, \tag{14}$$

$$E(k) = \begin{vmatrix} \hat{U}'_+(k), & \hat{U}''_+(k) \\ \hat{W}'(k), & \hat{W}''(k) \end{vmatrix} = -\begin{vmatrix} \hat{U}'_-(k), & \hat{U}''_-(k) \\ \hat{W}'(k), & \hat{W}''(k) \end{vmatrix}. \tag{15}$$

Each determinant has its "first representation" (through $\hat{U}_+$ and $\hat{W}$) and "second representation" (through $\hat{U}_-$ and $\hat{W}$). Note that this fact is of great importance because the analytical restrictions on $\hat{U}_+$ and $\hat{U}_-$ are different.

Note that

$$D'(k) = \begin{vmatrix} \hat{U}''_+(k), & \hat{U}_+(k) \\ \hat{W}''(k), & \hat{W}(k) \end{vmatrix}.$$

The following argument proves that the determinants $D(k)$ and $E(k)$ have a simple structure on their Riemann surfaces and they belong to a very narrow class of functions. The idea of the method is very close to classical Wiener-Hopf method.

Consider the determinant $D(k)$. In the upper half plane of the argument it has growth of order $\sim O(k^{-3})$ due to the first representation. It has the same order of growth in the lower half plane due to the second representation. Besides, $D(k)$ only changes its sign if the branch point k_0 is circled by the argument (due to the first representation). The same fact is valid for the point $-k_0$ due to the second representation. It means that the function $\sqrt{k_0^2 - k^2} D(k)$ is single-valued on the whole complex plane.

As it follows from the definition of functions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$, function $\sqrt{k_0^2 - k^2} D(k)$ has some singularities on the complex plane. It has the pole of the second order at $k = k_*$ and simple poles at the points k_0 and $-k_0$. There are no other singularities. Taking into account the known order of growth of the function $D(k)$, one can conclude that

$$\sqrt{k_0^2 - k^2} D(k) = M \frac{(k - \lambda_1)(k - \lambda_2)}{(k^2 - k_0^2)(k - k_*)^2}, \tag{16}$$

where M, λ_1 and λ_2 are unknown constants depending on k_0, a and k_*.

Similar reasoning yields to the following representation for $E(k)$:

$$\sqrt{k_0^2 - k^2} E(k) = N \frac{P(k)}{(k^2 - k_0^2)^2 (k - k_*)^3}, \tag{17}$$

where

$$P(k) = k^5 + p_4 k^4 + p_3 k^3 + p_2 k^2 + p_1 k + p_0.$$

In order to find the coefficients $f(k)$ and $g(k)$ one must find parameters λ_1, λ_2, 5 coefficients of the polynomial $P(k)$ and N/M.

Note that $f(k)$ and $g(k)$ are *rational functions* of k, which are known up to 8 constant parameters.

Restrictions on unknown parameters

There are two types of restrictions posed on unknown parameters of coefficients. Some of them are local and obtained from the asymptotics in the singular points of the ODE. The others are non-local and represent the monodromy data of the solutions of the equation.

Consider the points from the set k_*, λ_1, λ_2. Each point has two integer exponents, so the difference between exponents is also integer. It is well known, that there can be logarithmic terms at such points. But the solutions $\hat{U}_+$, $\hat{U}_-$ and $\hat{W}$ have no logarithmic singularities. For each point there is a condition obtained in a standard way that assures that there is no logarithmic term. So, we have 3 conditions.

Besides, we study the point $k = \infty$. Two restrictions can be found using known asymptotics of the solutions at infinity.

If the parameters obey the restrictions mentioned above, then at each point there is a pair of solutions with required asymptotics. One asymptotic must belong to $\hat{W}(k)$, another — to $\hat{U}_+(k)$ or $\hat{U}_-(k)$. But there is no warranty that all asymptotics (at different singular points) that must below for example to $\hat{W}(k)$ actually belong to one solution of ODE. So one must model solutions of ODE along the lines connecting singular points and check whether asymptotic expansions match at the ends of the lines. Three restrictions can be obtained this way.

EIGENWAVE PROBLEM FOR A SRTIP WAVEGUIDER

In this section we discuss the problem for a strip waveguider. It is shown that the ODE appearing in our consideration can be reduced to Mathieu's equations.

We are looking for travelling modes along the strip, therefore they must have exponential decay in the cross-section plane axes. Therefore we must consider the initial equation in the form of

$$\Delta u - \lambda^2 u = 0 \tag{18}$$

instead of (1). Parameter λ^2 is equal to $\beta^2 - k_0^2$, where β is a wavenumber in the direction along the strip (orthogonal to x- and y-axes).

Construct the equation (10) for this problem. Note that there is no incident wave, therefore $\hat{U}_+(k)$ and $\hat{U}_-(k)$ have no singularities except $k = \pm \mathrm{i}\lambda$. Therefore the power of growth of $D(k)$ is 2 less than in the previous sections and the power of growth of $E(k)$ is 3 less than in the previous sections. There is only one parameter that must be found numerically. Equation (10) can be rewritten in the form:

$$V''(k) + \frac{3k}{k^2 + \lambda^2} V'(k) + (a^2 + \frac{C}{k^2 + \lambda^2}) V(k) = 0, \tag{19}$$

where C is the unknown constant. It must be determined from the restrictions 1 and 2 from the Section 5. Note that in fact 2 parameters must be determined as the solution of the eigenfunction problem: C and β (or λ), thus two restrictions are not renundant.

Perform a substitution of independent and dependent variables:

$$k = \mathrm{i}\lambda \cos\varphi, \tag{20}$$

$$V(k) = \frac{v(\varphi)}{\sin\varphi}. \tag{21}$$

A differential equation can be obtained for $v(\varphi)$ directly from (19):

$$\frac{d^2 v}{d\varphi^2} + (\lambda^2 a^2 \cos^2\varphi - C) v = 0. \tag{22}$$

Note that (22) is Mathieu's equation that appear when the equation (18) is solved in elliptic coordinates. C is the constant of separation of variables. Equation for function s in another variable (say, ξ) in elliptic coordinates is

$$\frac{d^2 s}{d\xi^2} - (\lambda^2 a^2 \operatorname{ch}^2\xi - C) s = 0. \tag{23}$$

This equation is equivalent to (22) if ξ is formally substituted by $\mathrm{i}\varphi$.

Note that the eigenvalue problem in elliptic coordinates is equivalent to restrictions 1 and 2 from Section 5 for ODE (10).

Diffraction on a set of strips

Let equation (1) be fulfilled in the half plane $y > 0$, $-\infty < x < \infty$. Boundary conditions for $y = 0$ are

$$u(x, 0) = -e^{-\mathrm{i}k_* x} \tag{24}$$

for $a_1 < x < b_1 \quad \cup \quad a_2 < x < b_2 \quad \cup \quad \ldots \quad \cup \quad a_n < x < b_n$

and

$$\frac{\partial u(x, 0)}{\partial y} = 0 \tag{25}$$

for $-\infty < x < a_1 \quad \cup \quad b_1 < x < a_2 \quad \cup \quad \ldots \quad \cup \quad b_n < x < \infty$.

This problem is equivalent to the problem of diffraction of a plane wave on a set of n strips located in one plane.

The method used above leads to functional equation

$$\hat{U}_-(k) + \hat{W}_1(k) + \hat{U}_1(k) + \ldots + \hat{U}_{n-1}(k) + \hat{W}_n(k) + \hat{U}_+(k) = 0, \tag{26}$$

where

$$\begin{aligned}
\hat{W}_m(k) &= \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_{a_m}^{b_m} \frac{\partial u(x, 0)}{\partial y} e^{\mathrm{i}kx} dx, \\
\hat{U}_+(k) &= \int_{b_n}^{\infty} u(x, 0) e^{\mathrm{i}kx} dx + \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)b_n}}{k - k_*}, \\
\hat{U}_-(k) &= \int_{-\infty}^{a_1} u(x, 0) e^{\mathrm{i}kx} dx - \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)a_1}}{k - k_*}, \\
\hat{U}_m(k) &= \int_{b_m}^{a_{m+1}} u(x, 0) e^{\mathrm{i}kx} dx + \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)b_{m+1}}}{k - k_*} - \frac{\mathrm{i}e^{\mathrm{i}(k-k_*)a_m}}{k - k_*},
\end{aligned}$$

One can find a differential equation of order $2n$, such that all the functions in (26) are the solutions of it. It is necessary to exclude $\hat{U}_-$ or $\hat{U}_+$ from the set of functions at (26) and solve a set of linear equations with respect to the coefficients of ODE. Methods similar to used above can be applied and one can find that all the coefficients of equation are rational functions with a limited set of singularities. The procedure of finding the unknown parameters is beyond this paper.

Conclusion

It is shown for the problem of diffraction on a single strip that the function $\hat{W}(k)$ (the spectrum of the scattered field) is a solution of an ODE of second order. The coefficients of the ODE are rational functions of k, known up to 8 parameters (complex numbers). There are 5 analytical restrictions on these parameters (all restrictions follow from analysis of solutions near the singular points) and 3 conditions that must be checked numerically.

For the problem of scattering on a set of n strips it is shown that all components of the solution are solutions of an ODE of order $2n$ with rational coefficients.

Therefore a wide class of diffraction problems are reduced to ODE problems. Namely, one must find numerically a set of parameters and then find the solution of corresponding ODE to find the exact solution of the problem. This procedure seems to be more effective, than the traditional one (solving the integral equation).

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