Pseudo-differential operators for embedding formulae

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Abstract

A new method is proposed for deriving embedding formulae in 2-D diffraction problems. In contrast to the approach developed in [7], which is based on a differential operator, here a pseudo-differential, i.e., a non-local operator is applied to the wave field. Using this non-local operator a new embedding formula is derived for scattering by a single wedge. The formula has uniform structure for all opening angles, including angles irrational with respect to π; the earlier theory, [7], was valid only for rational angles.

1 Introduction

For a general 2-D diffraction problem, with piecewise linear scatterers, the main unknown to be determined is the diffraction coefficient, which is a function depending both on the angle of incidence and the angle of observation/scattering. An embedding formula represents the diffraction coefficient in the form of a combination of several auxiliary functions each of which has a smaller number of arguments. Possible choices for these auxiliary functions are those created by edge Green's functions, i.e. the directivities of multipole sources located at the edges of the scatterer; these then depend just on a single angle, the scattering angle. Historically, an embedding formula was first introduced in [1] for the problem of acoustic diffraction by a strip and other contemporaneous applications were to diffraction by a penny-shaped crack in an elastic solid [2]. More recently the embedding technique has been developed for more complicated structures [3, 4, 5, 6, 7].

The authors have proposed a simple derivation of an embedding formula [7] for scattering by polygonal shapes. The method is based on applying a differential operator to the wave field. Unfortunately, the operator can only be applied to scatterers containing rational (with respect to π) opening angles. To be precise the obstacle should be composed of several polygons, whose neighbouring sides subtend interior angles equal to $\pi q_j / p_j$, where q_j, p_j are integers. The denominators p_j play an important role in the method, since the order of the *embedding differential operator* should be a multiple of all p_j. Application of an operator of high order forces the use of a requisite high order edge Green's function and the resulting embedding formulae consist of summations [7]. It means that, for example, for a single wedge, the form of the directivity provided by embedding formula is very different for angles equal to, say, 2π and $15\pi/8$. Moreover, this "traditional" embedding cannot be applied to irrational (with respect to π) opening angles. Such a situation is not particularly satisfactory since the known exact solution for scattering by a single wedge possesses a simple "embedding" structure irrespective of the opening angle.

Our current paper revisits the wedge scattering problem and reveals the connection between the wedge solution and the embedding formula. Namely, we demonstrate that it can also be solved by applying a new embedding method based solely on pseudo-differential operators. We introduce a class of pseudo-differential operators, study their properties, and show that the embedding differential operators introduced in [7], for rational angles, are just a special case of this more general class of operators. We also find that the pseudo-differential operators are harder to apply to, say, polygonal scatterers, than the differential ones and so the results can, at present, only be directly used for the case of a single wedge.

2 Formulation of the problem

Here we consider a sample scattering geometry, which is a wedge occupying the sectorial area $0 < \varphi < \Phi$, $0 < r < \infty$. Cartesian coordinates are introduced, such that the positive x direction corresponds to $\varphi = 0$, and the positive y direction to $\varphi = \pi/2$. The field in the wedge obeys the Helmholtz equation

$$\Delta u + k_0^2 u = 0 \tag{1}$$

with the time dependence of all variables having the form $e^{-\mathrm{i}\omega t}$ which is omitted henceforth. The boundary conditions along the wedge faces could belong to any of three commonly used types (i.e. Dirichlet, Neumann or impedance), but, for brevity and definiteness, we shall present the approach only for Dirichlet conditions. The edge (Meixner's) and radiation conditions are formulated in the usual way [7].

The wedge is assumed to be illuminated by an incident plane wave

$$u^{\mathrm{in}} = e^{-\mathrm{i}k_0 r\cos(\varphi-\psi)}, \tag{2}$$

where ψ is the angle of incidence, such that $0 < \psi < \Phi$. As is typical for a diffraction problem, the field is decomposed into the geometrical part consisting of the incident and reflected waves, and the scattered field, which is described by the far-field asymptotics

$$u = D(\varphi,\psi)\frac{e^{\mathrm{i}k_0 r - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 r}}, \tag{3}$$

where D is the diffraction coefficient or equivalently the directivity. The main task is to find the directivity which, for the wedge problem, is known for all basic boundary conditions. Deriving the embedding formula for a wedge allows us to explore the application of the pseudo-differential operator and check whether it does indeed replicate the exact solution.

3 The pseudo-differential operator

We study operators whose general form is as follows:

$$\mathbf{K}[u](x, y) = \iint u(x', y')K(x' - x, y' - y)dx'\,dy'. \tag{4}$$

In the operators used here the kernel is a distribution localized on some contour Γ encircling the origin. In our case the distribution is a sum of a Dirac delta-function and its derivative with respect to the normal to Γ. Thus, the distribution is defined as a functional

$$\mathbf{K}[w](\mathbf{r}') = \int_\Gamma (K'(l)w(\mathbf{r} + \mathbf{r}') + K''(l)\partial_n w(\mathbf{r} + \mathbf{r}'))dl, \tag{5}$$

where w is an arbitrary smooth test function, l is a coordinate along the contour Γ, $\mathbf{r} = \mathbf{r}(l)$ is the radius vector of a point on Γ having coordinate l, $n = n(l)$ is the unit vector normal to Γ, and K', K'' are the amplitudes of the delta-function and its derivative.

We now specify the contour Γ and the functions K' and K''. The contour is taken to be a loop encircling the origin (see Fig. 1). The straight parts of the loop are stretched close to the positive x half-axis, and the circular part has vanishingly small radius. Note that the positive x half-axis is parallel to a wedge face. Such a contour is a generalization of the integral of some function along the positive x half-axis if the function has a non-integrable singularity at the origin. The circular part of the integral plays the role of regularization of the integrals related to the straight parts.

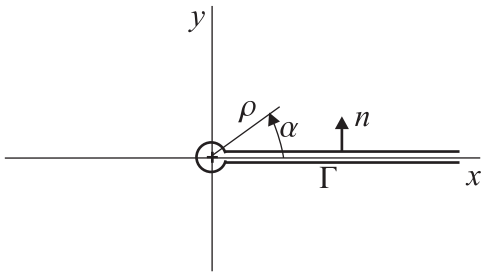

Figure 1: Contour Γ

The functions K' and K'' are defined as follows:

$$K'(l) = -\partial_n U_\mu(\mathbf{r}(l)), \qquad K''(l) = U_\mu(\mathbf{r}(l)). \tag{6}$$

Here $U_\mu(x,y)$ is a function defined in the vicinity of the contour:

$$U_\mu(r) = H_\mu^{(1)}(k_0\rho)\cos[\mu(\pi-\alpha)], \tag{7}$$

in which ρ and α are polar coordinates of the vector $\mathbf{r}$, i.e.

$$\mathbf{r} = (\rho\cos\alpha, \rho\sin\alpha), \qquad \rho = \rho(l), \alpha = \alpha(l).$$

We assume that the coordinate α is continuous on the contour and takes values from the interval $(0, 2\pi)$. The operator $\mathbf{K}$ depends on the continuous real parameter μ, so we shall denote this operator as $\mathbf{K}_\mu$. In the case of a single wedge μ should be chosen to be equal to $m\pi/\Phi$, where $m = 1, 2, 3\dots$ The simplest embedding formula is obtained for $m = 1$.

We now discuss some immediate properties of the operator $\mathbf{K}_\mu$. The integral in (4) is a convolution, therefore it commutes with the differentiations with respect to the coordinates and with the Helmholtz operator. The trigonometric function in (7) is chosen to have an obvious symmetry $U_\mu(\rho,\alpha) = U_\mu(\rho, 2\pi-\alpha)$. This choice enables one to eliminate the integral of $\partial_n u$ along the straight parts of the contour, since the normal derivatives of u on two straight branches are opposite to each other.

The integral (5) has a recognizable Green's form

$$\int_\Gamma (U_\mu \partial_n w - w\,\partial_n U_\mu) dl$$

(note that U_μ is itself a solution of the same Helmholtz equation), therefore the path can be deformed provided the singularity of the function U_μ is not crossed. This possibility can be used for continuation of $\mathbf{K}_\mu[u]$ as follows. Let $u(x,y)$ be the wave field in a wedge area. An immediate application of (4, 5) to find $\mathbf{K}_\mu[u](\mathbf{r})$ is possible only when the contour $\Gamma + \mathbf{r}$ lies completely in the wedge area. Obviously, this is true only for the points with the polar angle φ lying between 0 and π. To continue $\mathbf{K}_\mu[u]$ use another contour, for example the one shown in Fig. 2 in the right. The figure shows the areas where the operator is defined by corresponding contours. Note that in the area where both contours are applicable, the values of the operators defined by them are equal to each other, i.e. the contour in the right provides a continuation of the operator. It is quite clear that one can continue the operator $\mathbf{K}_\mu[u]$ into any point of the sectorial area.

4 Properties of the operator

We formulate the properties of the operator $\mathbf{K}_\mu$ in the form of several propositions. In this section we study $\mathbf{K}_\mu[u]$ as a wave field in the wedge. Proposition 1 states that this function obeys the Helmholtz equation. Proposition 2 establishes a connection with the previous work by the authors [7] related to the differential operators. Proposition 3 demonstrates a symmetry of the operator. Although it is introduced to satisfy conditions on the face $\varphi = 0$, it can be converted into

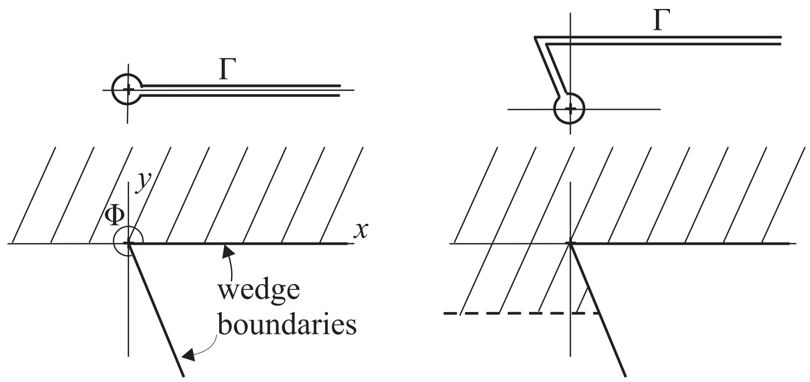

Figure 2: Transformation of the contour Γ for continuation of $\mathbf{K}_\mu[u]$

a similar operator for the face $\varphi = \Phi$. Proposition 4 shows how the operator acts on the incident wave. Propositions 5 and 6 concern the boundary conditions obeyed by $\mathbf{K}_\mu[u]$. Proposition 7 is about the radiation condition and the directivity, and, finally, Propositions 8 and 9 establish the edge asymptotics of $\mathbf{K}_\mu[u]$.

Proposition 1 *The operator $\mathbf{K}_\mu$ maps solutions of the Helmholtz equation into solutions of Helmholtz equation.*

This fact follows from the commutativity between $\mathbf{K}_\mu$ and differentiations with respect to the spatial coordinates.

Proposition 2 *If $\mu = n$, and n is a positive integer, then*

$$\mathbf{K}_n[u] = 4e^{\mathrm{i}(n-1)\pi/2} T_n\left(\frac{\mathrm{i}}{k_0}\partial_x\right), \tag{8}$$

where T_n is a Tchebyshev polynomial.

Proof The relation

$$u = \frac{\mathrm{i}}{4} K_0[u], \tag{9}$$

follows from the Green's theorem. The function $\mathbf{K}_\mu[u](\mathbf{r}')$ in a small vicinity of some point can be rewritten in the following form:

$$\mathbf{K}_\mu[u](\mathbf{r}') = \int_{\Gamma^*} [U_\mu(\mathbf{r}-\mathbf{r}')\partial_n u(\mathbf{r}) - u(\mathbf{r})\partial_n U_\mu(\mathbf{r}-\mathbf{r}')]dl, \tag{10}$$

where the contour Γ^* is fixed for all points of the vicinity (in the case of integer μ the contour Γ^* can be chosen as a circle of non-zero radius with the centre at $\mathbf{r}'$). Obviously,

$$\partial_{x'}\mathbf{K}_\mu[u](\mathbf{r}') = -\int_{\Gamma^*} [U'_\mu(\mathbf{r}-\mathbf{r}')\partial_n u(\mathbf{r}) - u(\mathbf{r})\partial_n U'_\mu(\mathbf{r}-\mathbf{r}')]dl, \tag{11}$$

where $U'_\mu = \partial_x U_\mu$.

Applying the operator $T_n(\mathrm{i}k_0^{-1}\partial_x)$ to (9), and using (11) several times, we find that

$$T_n\left(\frac{\mathrm{i}}{k_0}\partial_x\right) u = \frac{\mathrm{i}(-1)^n}{4}\int_{\Gamma^*}[U_0^*(\mathbf{r}-\mathbf{r}')\partial_n u(\mathbf{r}) - u(\mathbf{r})\partial_n U_0^*(\mathbf{r}-\mathbf{r}')]dl, \qquad (12)$$

where

$$U_0^* = T_n\left(\frac{\mathrm{i}}{k_0}\partial_x\right) U_0 \qquad (13)$$

From the definition of U_μ, i.e. from (7), and the properties of Hankel functions it is easy to establish the identity

$$T_n\left(\frac{\mathrm{i}}{k_0}\partial_x\right)[U_0] = e^{\mathrm{i}\pi n/2}U_n. \qquad (14)$$

Combining (12) and (14), we obtain (8).

Note that the operator in the right-hand side is up to a multiplicative and an additive constant equal to the operator introduced in [7] for rational angles. Thus, Property 2 establishes a connection between the results obtained earlier for pure differential operators and the results obtained in the present article.

Proposition 3 *Let $\mu > 1/2$ and introduce a field u that obeys the Helmholtz equation and the radiation condition in some angular area. Then*

$$\mathbf{K}_\mu[u] = -\bar{\mathbf{K}}_\mu[u] \qquad (15)$$

where $\bar{\mathbf{K}}_\mu$ is the integral operator belonging to the class (4), (5), (6) with the contour $\bar\Gamma$ shown in Fig. 3. The kernel of $\bar{\mathbf{K}}_\mu$ is given by the formula

$$U_\mu(\rho,\bar\alpha) = H_\mu^{(1)}(k_0\rho)\cos[\mu(\pi-\bar\alpha)]. \qquad (16)$$

The variable $\bar\alpha$ is equal to $\alpha - \pi/\mu$ and it takes values from 0 to 2π.

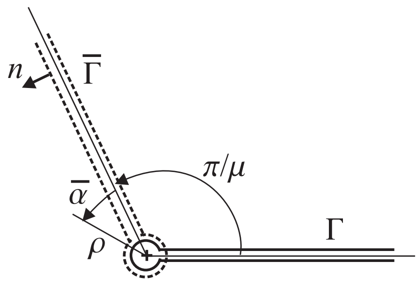

Figure 3: Transformation of the contour of integration

Proof The new contour can be obtained from the old one by deformation. The integral along the large arcs emerging during the transformation can be neglected due to the radiation condition.

Property 3 can be used to study $\mathbf{K}_\mu[u]$ on the face $\varphi = \Phi$.

Proposition 4 *Let the function $v(x, y)$ be a plane wave coming from direction ψ, i.e.*

$$v(r, \varphi) = \exp\{-\mathrm{i}k_0 r \cos(\varphi - \psi)\} \tag{17}$$

with $0 < \psi < 2\pi$. Then

$$\mathbf{K}_\mu[v](x, y) = G_\mu(\psi)\, v(x, y), \tag{18}$$

$$G_\mu(\psi) = 4e^{-\mathrm{i}(\mu+1)\pi/2} \cos[\mu(\pi - \psi)]. \tag{19}$$

Proof The form of the relation (18) follows from linearity and translation invariance, and thus one needs to prove only (19). Consider the whole plane, i.e. let there be no wedge boundaries and take $(x, y) = (0, 0)$.

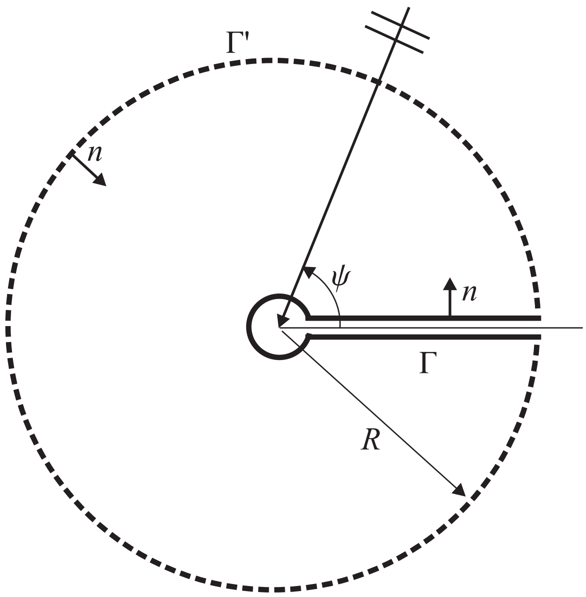

Figure 4: Contour Γ'

Close the contour Γ by connecting its ends by an arc of large radius Γ' (see Fig. 4). The integral along the total contour $\Gamma' + \Gamma$ is equal to zero. Thus,

$$\mathbf{K}_\mu[v](0, 0) = -\lim_{R\to\infty} \int_0^{2\pi} [U_\mu(R, \phi)\partial_n v(R, \phi) - v(R, \phi)\partial_n U_\mu(R, \phi)] R d\phi. \tag{20}$$

Estimate the integral in the right by applying the stationary phase method. A standard consideration shows that the main term of the integral is obtained by integration over a small vicinity of the point $\phi = \psi$:

$$\mathbf{K}_\mu[v](0, 0) = 2\sqrt{\frac{2k_0 R}{\pi}} \exp\left\{-\mathrm{i}\frac{\pi}{2}(\mu + 1) - \mathrm{i}\frac{\pi}{4}\right\} \cos[\mu(\pi - \psi)] \times$$

$$\int_{\psi-\epsilon}^{\psi+\epsilon} \exp\{\mathrm{i}k_0 R(1-\cos(\phi-\psi))\}d\phi + o(R^0). \tag{21}$$

An estimation of the main term of the integral gives the formula (19). Note that taking the limit $R \to \infty$ eliminates all other terms, i.e. although the asymptotic argument is used, formula (19) is exact.

We derive a generalization of (18) for complex angles of incidence ψ; the formula can be analytically continued from the real segment $0 < \psi < 2\pi$ to the area where the integral (4) converges, i.e. where $\mathrm{Im}(\cos\psi) < 0$. This area is shown in Fig. 5.

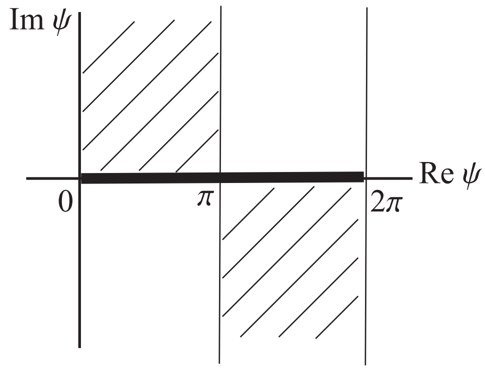

Figure 5: Area, in which formula (18) is valid

If $\mathrm{Im}(\cos\psi) < 0$, but either $\mathrm{Re}(\psi) < 0$ or $\mathrm{Re}(\psi) > 2\pi$, one should use periodicity and bring the angle into the strip $0 < \mathrm{Re}(\psi) < 2\pi$, i.e. a general form of (19) looks like

$$G_\mu(\psi) = 4e^{-\mathrm{i}(\mu+1)\pi/2}\cos(\mu(\pi - \psi + 2\pi[\mathrm{Re}(\psi)/(2\pi)])), \tag{22}$$

and the square brackets in the last expression denote the integer part.

Proposition 5 *Let the field u obey the Helmholtz equation and the Dirichlet boundary condition $u = 0$ at the face $\varphi = 0$. Then*

$$\mathbf{K}_\mu[u] = 0 \tag{23}$$

on the face $\varphi = 0$.

Proof Unlike the situation with a pure differential operator, now this statement is not obvious due to the presence of an integral over a small arc encircling the singularity. First, it is necessary to define the value $\mathbf{K}_\mu[u]$ on the wedge face. For this, one should be able to take the integral over a small part of the contour, which lies outside the boundary. Thus a smooth continuation of the field to some strip outside the boundary is required. In our case, the smooth continuation can be easily obtained by the reflection method, i.e. the field is obtained by antisymmetrical reflection across the boundary.

Finally, the field is antisymmetric with respect to the boundary, and function U_μ is symmetrical. Therefore, the integral (4) is equal to zero.

Note that the same property can also be proved for Neumann and impedance boundary conditions. For this, one applies to the field the corresponding operator (i.e. ∂_y or $\partial_y + \text{const}$) and takes into account that this operator commutes with $\mathbf{K}_\mu$.

Proposition 6 *Let the field obey the Helmholtz equation, radiation condition and boundary conditions*

$$u(r,0) = -e^{-\mathrm{i}k_0 r\cos\psi}, \qquad\qquad u(r,\Phi) = 0, \tag{24}$$

with

$$\mu = \pi m/\Phi, \qquad m = 0,1,2,\dots \tag{25}$$

Then on both faces of the wedge

$$(\mathbf{K}_\mu - G_\mu(\psi))\,[u] = 0. \tag{26}$$

Proof In the vicinity of the face $\varphi = 0$ one can decompose the field into a sum of a plane wave and a field obeying the condition $u = 0$ at the boundary. For both terms the condition (26) is fulfilled. To prove (26) on the face $\varphi = \Phi$ we apply Proposition 3.

Proposition 7 *Let u satisfy the radiation condition. Then $\mathbf{K}_\mu[u]$ also satisfies the radiation condition. If the directivity of u is given by (3) then the directivity of $\mathbf{K}_\mu[u]$ is given by*

$$D(\varphi) \xrightarrow{\mathbf{K}_\mu} D'(\varphi) = 4e^{-\mathrm{i}(\mu+1)\pi/2}\cos(\mu\varphi)\,D(\varphi) \tag{27}$$

Proof Consider the field $\mathbf{K}[u](R,\varphi)$ for some fixed φ and $R \to \infty$. Transform the integration contour as shown in Fig. 6, i.e. make the straight parts of the contour have angle φ with the x-axis.

Fix the point (R,φ), at which the function $\mathbf{K}_\mu[u]$ is calculated. Let (x,y) be the coordinates along which the integration is held (i.e. the point (x,y) runs along the contour Γ). Represent the field $u(x,y)$ near contour Γ as a sum

$$u = u_0 + u_1,$$

where u_0 is a plane wave having an appropriate amplitude:

$$u_0(x,y) = D(\varphi)\frac{e^{-\mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}}\exp\{\mathrm{i}k_0(x\cos\varphi + y\sin\varphi)\}, \tag{28}$$

and u_1 is the remainder.

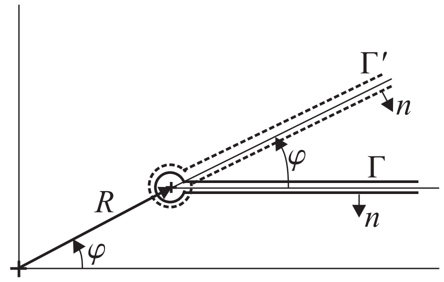

Figure 6: Transformation of the contour of integration for establishing the radiation condition

The value $\mathbf{K}_\mu[u_0](R,\varphi)$ has been calculated in Proposition 4, and it is given by

$$\mathbf{K}_\mu[u_0](R,\varphi) = D(\varphi)\frac{e^{\mathrm{i}k_0R-\mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}}4e^{-\mathrm{i}(\mu+1)\pi/2}\cos(\mu\varphi). \tag{29}$$

Consider $\mathbf{K}_\mu[u_1](R,\varphi)$. Using the standard far-field asymptotic expansion for u it is not difficult to show that

$$\mathbf{K}_\mu[u_1](R,\varphi) = O(R^{-3/2}), \tag{30}$$

i.e. the contribution of u_1 is asymptotically small comparatively to (29).

The same estimations can be done for the function $\mathbf{K}_\mu[u']$, where

$$u' = (\cos\varphi\,\partial_x + \sin\varphi\,\partial_y)u.$$

Comparison of the asymptotic decompositions for $\mathbf{K}_\mu[u]$ and $\mathbf{K}_\mu[u']$ gives the radiation condition. The relation (29) gives (27).

Proposition 8 *Let*

$$v(r,\varphi) = J_\nu(k_0 r)e^{\pm\mathrm{i}\nu\varphi}, \qquad \nu > 0. \tag{31}$$

If $0 < \mu < \nu$,

$$K_\mu[v](r,\varphi) = O(1) \qquad \textit{as } r\to 0. \tag{32}$$

If $\nu < \mu$ *then the field near the origin behaves as follows:*

$$K_\mu[v](r,\varphi) = -2\sin(\pi\nu)H^{(1)}_{\mu-\nu}(k_0 r)e^{\pm\mathrm{i}(\nu-\mu)\varphi} + O(1) \qquad \textit{as } r\to 0. \tag{33}$$

If $\nu = \mu$ *then near the origin the field behaves as follows:*

$$K_\mu[v](r,\varphi) = -\sin(\pi\nu)H^{(1)}_0(k_0 r) + O(1) \qquad \textit{as } r\to 0. \tag{34}$$

Proof Consider only the case of sign "+" in the exponent in (31). The other sign can be taken into account by mirror reflection $y\to -y$. Moreover, consider only the values $-\pi/2 < \varphi < \pi/2$. Other values can be considered by deformation of the integration contour Γ.

Using the integral formula for Bessel functions represent the function v as a linear combination of plane waves:

$$v(r,\varphi) = J_\nu(k_0 r)e^{\mathrm{i}\nu\varphi} = \frac{1}{2\pi}\int_\gamma e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}e^{\mathrm{i}\nu(\theta-\pi/2)}d\theta \tag{35}$$

where contour γ is shown in Fig. 7.

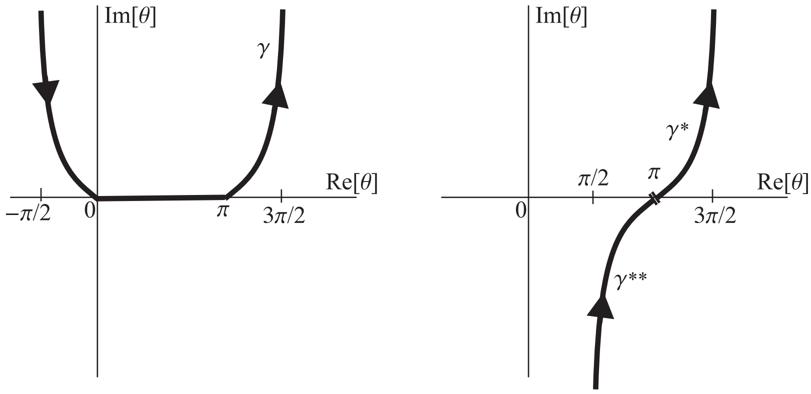

Figure 7: Integration contours for Proposition 7

Apply operator $\mathbf{K}_\mu$ to (35):

$$\mathbf{K}_\mu[v](r,\varphi) = \frac{1}{2\pi}\int_\gamma \mathbf{K}_\mu[e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}]e^{\mathrm{i}\nu(\theta-\pi/2)}d\theta \tag{36}$$

To calculate the r.-h.s. use formula (18):

$$\mathbf{K}_\mu[v](r,\varphi) = \frac{1}{2\pi}\int_\gamma G(\theta+\pi)e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}e^{\mathrm{i}\nu(\theta-\pi/2)}d\theta \tag{37}$$

Note that due to (22) function $G(\theta+\pi)$ is not continuous on the contour γ. Thus, the integral can be decomposed as follows:

$$\mathbf{K}_\mu[v] = I_1 + I_2, \tag{38}$$

where

$$I_1 = \frac{1}{2\pi}\int_\gamma \cos(\mu\theta)e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}e^{\mathrm{i}\nu(\theta-\pi/2)}d\theta =$$

$$2e^{-\mathrm{i}\pi/2}[J_{\nu+\mu}(k_0 r)e^{\mathrm{i}(\nu+\mu)\varphi} + e^{-\mathrm{i}\pi\mu}J_{\nu-\mu}(k_0 r)e^{\mathrm{i}(\nu-\mu)\varphi}], \tag{39}$$

$$I_2 = \frac{1}{2\pi}\int_{\gamma^*} (\cos(\mu\theta-2\pi\mu)-\cos(\mu\theta))e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}e^{\mathrm{i}\nu(\theta-\pi/2)}d\theta \tag{40}$$

Here the contour γ^* goes from π to $3\pi/2 + \mathrm{i}\infty$. Local asymptotics of I_1 can be found in textbooks, while I_2 should be estimated. To make the estimations, introduce the contour γ_{**} (see Fig. 7). Note that for any real η

$$\frac{1}{\pi}\int_{\gamma^*+\gamma^{**}} \cos(\mu\theta)e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}e^{\mathrm{i}\eta(\theta-\pi/2)}d\theta = H_\eta^{(2)}(k_0 r)e^{\mathrm{i}\eta\varphi} \tag{41}$$

The integral over γ^* converges for $r = 0$ if $\eta > 0$, the integral over γ^{**} converges for $r = 0$ if $\eta < 0$. Therefore it is possible to estimate I_2 up to a term, which is limited as $r \to 0$. Detailed but elementary estimations provide (34).

Proposition 9 *Let*

$$v(r,\varphi) = \varphi J_0(k_0 r)$$

then as $r \to 0$

$$\mathbf{K}_\mu[v](r,\varphi) = 2\pi\sin(\mu\varphi)H_\mu^{(1)}(k_0 r) + O(1). \tag{42}$$

Proof The following representation should be used:

$$v(r,\varphi) = \frac{1}{2\pi}\int_\gamma e^{\mathrm{i}k_0 r\cos(\theta-\varphi)}\theta d\theta - \frac{\pi}{2}H_0^{(2)}(k_0 r). \tag{43}$$

Then operator $\mathbf{K}_\mu$ is applied, and the integrals are estimated as it is done above.

Propositions 8 and 9 give some information about local asymptotics of $\mathbf{K}_\mu[v]$ provided that the local expansion of v is known. It is easy to show that if v is expanded as a series of terms (31) with different ν in some vicinity of the origin then local behaviour of $\mathbf{K}_\mu[v]$ is determined by a corresponding series of terms (33) or (34). The proof is based on the fact that if function w is equal to zero for $r < \epsilon$ for some $\epsilon > 0$ and is bounded for $r \geq \epsilon$ then $\mathbf{K}_\mu[w]$ is smooth and bounded near the origin.

Note that in the relations (33), (34), (42) we defined the asymptotics up to a term, which is bounded as $r \to 0$. Contrary to the usual consideration, the terms, which are $O(1)$ do not necessarily obey Meixner's condition. For example, a more detailed form of (34) is as follows:

$$K_\mu[v] = -\sin(\pi\nu)H_0^{(1)}(k_0 r) \pm \frac{2\sin(\pi\nu)}{\pi}\varphi + \text{Meixner's terms}. \tag{44}$$

The term proportional to φ obeys Helmholtz equation, and it is bounded, but it does not obey Meixner's condition, since $|\nabla u|^2$ is not integrable.

5 An embedding formula for an irrational angle

The operator $\mathbf{K}_\mu + \text{const}$ does not display all the desired properties of an embedding operator when applied to the total field or to the scattered field. This situation differs from that of [7]. That is why here we have to split the initial diffraction problem into two auxiliary ones. Namely, we let the total field be represented as follows:

$$u = u^{\mathrm{in}} + u^{\mathrm{I}} + u^{\mathrm{II}}, \tag{45}$$

where u^{I} is the field obeying the following inhomogeneous Dirichlet boundary conditions at the faces of the wedge:

$$u^{\mathrm{I}}(r,0) = -e^{-\mathrm{i}k_0 r\cos\psi}, \qquad u^{\mathrm{I}}(r,\Phi) = 0, \tag{46}$$

i.e. the excitation is set only on the first face of the wedge. Respectively, u^{II} obeys complementary boundary conditions:

$$u^{\mathrm{II}}(r,0) = 0, \qquad u^{\mathrm{II}}(r,\Phi) = -e^{-\mathrm{i}k_0 r\cos(\Phi-\psi)}, \tag{47}$$

Both u^{I} and u^{II} must also obey the edge and radiation conditions. We set Φ to be equal to π/μ and μ to be an irrational number.

Consider the component u^{I} of (45); we shall need the asymptotic expansion of this function near the edge. This expansion can be constructed as follows: first, expand the function $-e^{-\mathrm{i}k_0 r\cos\psi}$ into a Bessel series:

$$-e^{-\mathrm{i}k_0 r\cos\psi} = \sum_{n=0}^{\infty} a_n J_n(k_0 r)$$

where the coefficients a_n can be explicitly calculated. We then use the following ansatz for the function u^{I}:

$$u^{\mathrm{I}}(r,\varphi) = \frac{\varphi-\Phi}{\Phi} J_0(k_0 r) + \sum_{n=1}^{\infty} \frac{a_n}{\sin(\Phi n)} J_n(k_0 r)\sin(n(\Phi-\varphi)) + w(r,\varphi), \tag{48}$$

where $w(r,\varphi)$ obeys the Helmholtz equation, homogeneous Dirichlet boundary conditions on the faces and Meixner's condition at the edge, i.e.

$$w(r,\varphi) = \sum_{n=1}^{\infty} b_n J_{\mu n}(k_0 r)\sin(\mu n\varphi), \tag{49}$$

where b_n are unknown coefficients. To construct the ansatz (48) we use the fact that $\varphi J_0(k_0 r)$ is itself a solution of the Helmholtz equation.

Now we consider the function

$$W \equiv (\mathbf{K}_\mu - G_\mu(\psi))\,[u^{\mathrm{I}}]$$

and according to the properties of the operator $\mathbf{K}_\mu$, this function W obeys the Helmholtz equation, radiation condition, and homogeneous Dirichlet boundary conditions on the faces of the wedge. Consider the singular terms of W. Due to Propositions 8 and 9

$$W = \frac{2\pi}{\Phi}\sin(\mu\varphi) H_\mu^{(1)}(k_0 r) + O(1) \tag{50}$$

as $r \to 0$. The remainder, i.e. the second term, in (50) obeys the Helmholtz equation, Dirichlet boundary conditions at the faces, and radiation conditions. By constructing the general Meixner's series for it, and taking into account the boundedness, we conclude that this remainder obeys the Meixner's condition. Therefore, by the uniqueness theorem, this component of the solution should be identically equal to zero. Thus, the following relation is valid:

$$\left(\mathbf{K}_\mu - 4e^{-\mathrm{i}(\mu+1)\pi/2}\cos(\mu(\pi-\psi))\right)[u^{\mathrm{I}}] = \frac{2\pi}{\Phi}\sin(\mu\varphi)H_\mu^{(1)}(k_0 r). \tag{51}$$

We denote the directivity of u^{I} to be D^{I}. Finding the directivities of the right- and left-hand sides of (51), we obtain that

$$D^{\mathrm{I}}(\varphi,\psi) = \frac{\mathrm{i}\mu\sin(\mu\varphi)}{\cos(\mu\varphi)-\cos(\mu(\pi-\psi))}. \tag{52}$$

Similarly, if we consider the component u^{II}, where now the operator producing the embedding formula has the form $\mathbf{K}_\mu - D_\mu(2\pi-\psi)$, the result for the directivity is

$$D^{\mathrm{II}}(\varphi,\psi) = -\frac{\mathrm{i}\mu\sin(\mu\varphi)}{\cos(\mu\varphi)-\cos(\mu(\pi+\psi))}. \tag{53}$$

The sum of (52) and (53) then gives the directivity of the scattered field. A direct check can be performed showing that this directivity is exactly the classical solution for a wedge problem.

6 Conclusions

A connection between the classical wedge solution and the embedding procedure is revealed using a pseudo-differential embedding operator. The properties of the operator are studied. Clearly it is encouraging that this operator both reduces to the known form for rational wedge angles, [7], and generates the known wedge solution. However, the new operator has a disadvantage, namely, for a complicated scatterer (any scatterer different from a simple wedge) it does not preserve boundary conditions on more than one face. Thus, the powerful technique developed for differential embedding operators in [7] cannot be directly generalized. However, if the field is studied on a branched surface without reflecting boundaries, then the application of the pseudo-differential operator gives interesting results. However, that will be the subject of another paper.

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