To the problem of diffraction on a slit: Some properties of Schwarzschild's series

A.V.Shanin

Abstract

We study the diffraction series (Schwarzschild's series) to solve the problem of diffraction on a slit with ideal boundary conditions. Uing this series we derive the representation obtained earlier by M.Williams and the differential equations for the unknown functions.

Introduction

The problem of diffraction on a strip or a slit is studied since the end of the 19th century. After Sommerfeld had obtained an explicit solution of a half-plane diffraction problem a great number of researchers tried to find such simple and compact solution for the strip (Sommerfeld himself was among these researchers). Unfortunately, for a long time an only exact solution was the one in the form of the Fourier series in ellyptic coordinates [1]. This solution is not satisfactory for many needs and it can be compared with the representation of the Sommerfeld problem solution in the form of the Fourier decomposition in parabolic coordinates — almost nobody uses it. Some attempts to obtain a convenient exact solution has been made and most of them were unsuccessfull. The review of these works was made by Luneburg [2].

A bright work on diffraction on a strip or a slit was published by M. Williams in 1982 [3]. He used the method proposed earlier by Latta [4] for studying the integral equation of a certain type and reduced the integral equation to a pair of ordinary differential equations.

Another method, which also leads to an ordinary differential equation was peroposed by the author last year [5]. This method utilizes an extention of Wiener-Hopf method for the problem with entire functions. The solutions in both [3] and [5] are expressed in terms of specific boundary-value problems for ordinary differential equations, whose the coefficients depend on several unknown parameters. Unfortunately, in [3] and [5] the solutions are represented in very different forms and it is difficult to compare them or convert one into another. Even the asymptotic estimation of the unknown parameters is not a simple task.

Here we are going to present another technique applicable to the slit and hopefully to some more general problems. The process of diffraction on a slit is treated as a series of successive diffractions on the edges of the slit. Each diffraction can be easily described using the Sommerfeld's half-line solution. Probably, Schwarzschild was the first who proposed this idea and studied the series in [6], so we call the solution Schwarzschild's series. The series was used for obtaining approximate results by many authors (in fact, the GTD and PTD theories are based on this idea). An interesting interpretation of the diffraction series can be found in the book [7].

In the current paper we develop a special technique for transformation of the diffraction series enabling one to obtain *exact nontrivial results*. The main results are the derivation of teh Williams representation describing the behaviour of the diffraction diagram as a function of the angle of incidence, and obtaing the differential equations for unknown functions.

Using the proposed technique we obtain both the results of [3] and [5], thereby providing an independent verification of them. Besides, we find all undetermined values of both works in the form of the asymptotic series, each term of which can be directly calculated in quadratures (originally, both papers referred to some specific boundary-value problems).

1 Problem formulation and the solution in the form of diffraction series

1.1 Problem formulation

Consider 2-dimensional problem of diffraction of a plane wave on an ideal screen with a slit. Namely, let the Helmholtz equation

$$\Delta u + k_0^2 u = 0 \tag{1.1}$$

be valid on the (x, y) plane. The boundary conditions are

$$u(x, 0) = 0 \qquad \text{for} \quad |x| > a. \tag{1.2}$$

the slit occupies the segment $-a < x < a$ (see Figure 1).

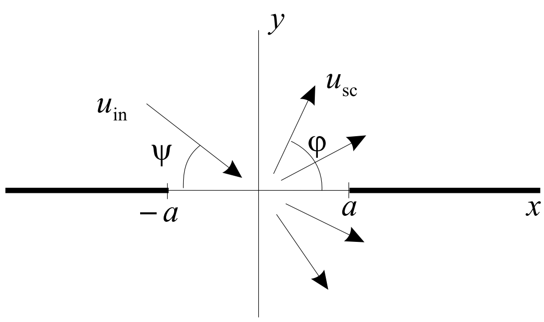

Figure 1: Geometry of the problem

We choose the time dependence of all values as $e^{-\mathrm{i}\omega t}$, where $\omega = k_0 c$, and c is the sound velocity.

Let the incident plane wave fall from the upper half-plane $y > 0$ and has the form:

$$u_{\mathrm{in}} = e^{-\mathrm{i}k_* x - \mathrm{i}\sqrt{k_0^2 - k_*^2}\, y}, \tag{1.3}$$

where $k_* = -k_0 \cos\psi$, ψ is the angle of incidence (shown in Figure 1). Let the total field in the upper half-plane be the sum of the incident field u_{in}, the reflected field

$$u_{\mathrm{r}} = -e^{-\mathrm{i}k_* x + \mathrm{i}\sqrt{k_0^2 - k_*^2}\, y} \tag{1.4}$$

and the scattered field u_{sc}. The field in the lower half-plane is equal to u_{sc}.

It is clear that the field u_{sc} is symmetrical, i.e. $u_{\mathrm{sc}}(x,y) = u_{\mathrm{sc}}(x,-y)$ and its y-derivative must have the discontinuity at $y=0$, $|x|<a$ equal to $2\mathrm{i}\sqrt{k_0^2-k_*^2}e^{-\mathrm{i}k_*x}$.

1.2 Diffraction series

We find it convenient to consider the combination $u_{\mathrm{sc}} + u_{\mathrm{r}}$ in the upper half-plane. For this combination we have the following boundary conditions at $y=+0$:

$$u_{\mathrm{sc}}(x,+0) + u_{\mathrm{r}}(x,+0) \quad = \quad -e^{-\mathrm{i}k_*x} \qquad \text{for} \quad |x|>a, \tag{1.5}$$

$$\frac{\partial[u_{\mathrm{sc}}(x,+0) + u_{\mathrm{r}}(x,+0)]}{\partial y} \quad = \quad 0 \qquad \text{for} \quad |x|>a. \tag{1.6}$$

Besides, the Meixner's conditions must be valid at the edges of the slit and the radiation conditions must be satisfied at infinity.

Consider the process of diffraction as a process successive diffractions on the parts of the screen, i.e. represent the field in the form of a series

$$u_{\mathrm{sc}} + u_{\mathrm{r}} = u_1 + u_2 + u_{12} + u_{21} + u_{121} + u_{212} + \dots \tag{1.7}$$

where u_1 and u_2 are the zero-order diffraction terms (the results of the diffraction of the incident plane wave on the separate edges of the slit); index 1 corresponds to the diffraction on the edge $x=a$, index 2 corresponds to the diffraction on the edge $x=-a$.

The terms u_{12} and u_{21} correspond to diffraction field of order 1, i.e. the result of the diffraction of the terms u_1 on the edge 2 and the term u_2 on the edge 1. The sequence of successive diffraction on the edges can be continued, thus giving the terms $u_{1212\dots21}$. Let all the sequence of symbols $1212\dots21$ be the "index" of the term. This sequence has the following properties: the symbols are aletring; the left symbol corresponds to the first act of diffraction.

We should note that such indexing is renundant. Instead of writing down the whole sequence of the acts of diffraction, one could fix the first or the last act and the number of the acts. We shall introduce compact notations a bit later. Now we should say that the notation system introduced here can be very useful in description of diffraction on more complicated structures, such as the sets of strips, joints e.t.c.

Introduce the following shorter notations:

$u^n_{1\dots}$: the sequence in the index starts with 1 and has the length $n+1$;

$u^n_{\dots1}$: the last symbol of the sequence is 1 and the length is $n+1$;

$u^n_{1\dots1}$: the first and the last symbols are 1's (it is obvious that n is even in this case);

We have the following boundary conditions at $y=0$ tor the zero-order terms:

$$\frac{\partial u_1}{\partial y} = 0 \text{ for } x<a, \qquad u_1 = -e^{-\mathrm{i}k_*x} \text{ for } x>a, \tag{1.8}$$

$$\frac{\partial u_2}{\partial y} = 0 \text{ for } x>-a, \qquad u_2 = -e^{-\mathrm{i}k_*x} \text{ for } x<-a. \tag{1.9}$$

Other terms obey the following boundary conditions:

$$\frac{\partial u^n_{\dots1}}{\partial y} = 0 \text{ for } x<a, \qquad u^n_{\dots1} = -u^{n-1}_{\dots2} \text{ for } x>a, \tag{1.10}$$

$$\frac{\partial u^n_{\dots 2}}{\partial y} = 0 \text{ for } x > -a, \qquad u^n_{\dots 2} = -u^{n-1}_{\dots 1} \text{ for } x < -a. \tag{1.11}$$

Beside the boundary conditions, these terms indeed satisfy the Helmholtz equation, edge conditions and radiation conditions at infinity.

1.3 The solution of the sequence of diffraction problems using Wiener-Hopf method

Introduce the following noation for the diffraction terms:

$$\bar{U}^n_{\dots 1}(k) = \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_a^\infty \frac{\partial u^n_{\dots 1}(x, +0)}{\partial y} e^{\mathrm{i}kx} dx \tag{1.12}$$

$$\bar{U}^n_{\dots 2}(k) = \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_{-\infty}^{-a} \frac{\partial u^n_{\dots 2}(x, +0)}{\partial y} e^{\mathrm{i}kx} dx \tag{1.13}$$

The integrals are defined correctly for all n except $n = 0$. The zero-order terms contain the non-decaying contribution of u_{r}. That is why the formulae (1.12), (1.13) must be modified as follows:

$$\bar{U}_1(k) = \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_a^\infty \left[\frac{\partial u_1(x, +0)}{\partial y} + \mathrm{i}\sqrt{k_0^2 - k_*^2} e^{-\mathrm{i}k_* x} \right] e^{\mathrm{i}kx} dx + \frac{\mathrm{i}\sqrt{k_0^2 - k_*^2}}{\sqrt{k_0^2 - k^2}} \frac{e^{\mathrm{i}k_* a}}{k - k_*}, \tag{1.14}$$

$$\bar{U}_2(k) = \frac{\mathrm{i}}{\sqrt{k_0^2 - k^2}} \int_{-\infty}^{-a} \left[\frac{\partial u_2(x, +0)}{\partial y} + \mathrm{i}\sqrt{k_0^2 - k_*^2} e^{-\mathrm{i}k_* x} \right] e^{\mathrm{i}kx} dx - \frac{\mathrm{i}\sqrt{k_0^2 - k_*^2}}{\sqrt{k_0^2 - k^2}} \frac{e^{-\mathrm{i}k_* a}}{k - k_*}, \tag{1.15}$$

Taking into account the boundary conditions (1.8)–(1.11), we obtain the inverse Fourier transformation in the form:

$$u^n_{\dots} = -\frac{1}{2\pi} \int_{\gamma_\mp} \bar{U}^n_{\dots} e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2 - k^2} y} dk, \tag{1.16}$$

where the contours $\gamma_\pm$ are shown in Figure 2. Contour γ_- must be chosen for the terms $u^n_{\dots 1}$; contour γ_+ must be chosen for $u^n_{\dots 2}$. This choice quarantees the validity of the radiation conditions and the correct account of the pole, corresponding to the incident wave.

Note that for the Fourier transformation we use the contours shown in Figure 2a, and for the asymptotic estimation the deformed contours shown in Figure 2b can be used.

Unknown functions $\bar{U}^n_{\dots}$ can be found using the Wiener-Hopf method [8]. Skipping the detail, we write down here the final form of the solutions.

The zero-order terms are defined by

$$\bar{U}_1(k) = A_1 \frac{e^{\mathrm{i}ka}}{\sqrt{k_0 - k}(k - k_*)}, \qquad \bar{U}_2(k) = A_2 \frac{e^{-\mathrm{i}ka}}{\sqrt{k_0 + k}(k - k_*)}, \tag{1.17}$$

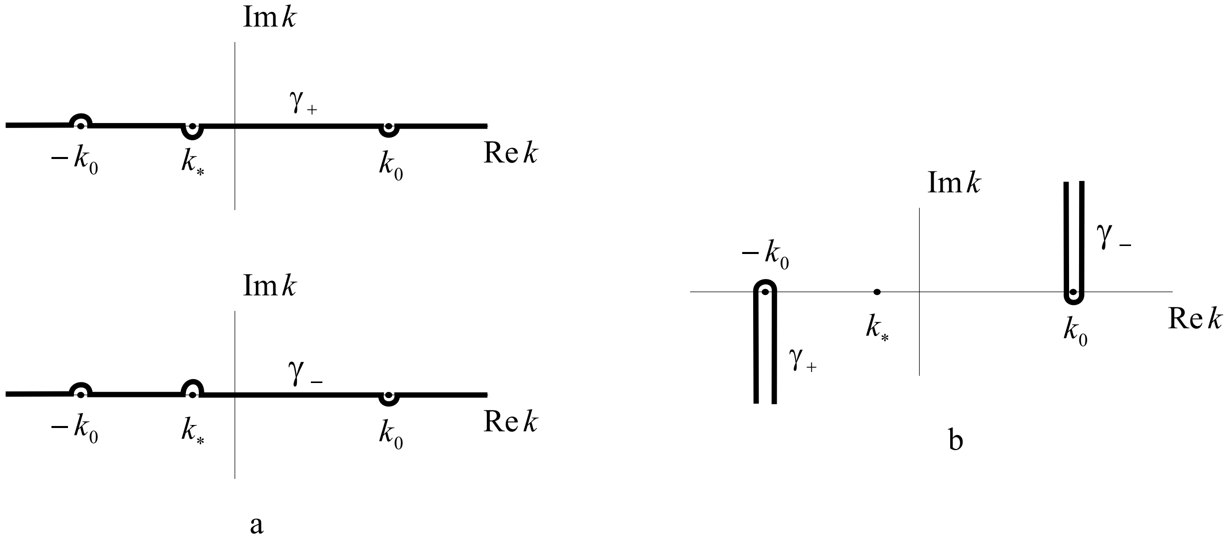

Figure 2: The contours of integration

the coefficients $A_{1,2}$ are equal to

$$A_1 = \mathrm{i}e^{-\mathrm{i}k_* a}\sqrt{k_0 - k_*}, \qquad A_2 = -\mathrm{i}e^{\mathrm{i}k_* a}\sqrt{k_0 + k_*}. \tag{1.18}$$

The terms of higher orders are defined recursively as follows:

$$\bar{U}^n_{...1} = -\frac{e^{\mathrm{i}ka}}{\sqrt{k_0 - k}} F_+[e^{-\mathrm{i}ka}\sqrt{k_0 - k}\,\bar{U}^{n-1}_{...2}], \tag{1.19}$$

$$\bar{U}^n_{...2} = -\frac{e^{-\mathrm{i}ka}}{\sqrt{k_0 - k}} F_-[e^{\mathrm{i}ka}\sqrt{k_0 + k}\,\bar{U}^{n-1}_{...1}]. \tag{1.20}$$

F_+ and F_- are the operators for the additive decomposition into the componets, regular in the upper and lower half-planes of the complex variable k. They are defined as

$$F_\pm[V(k)] = \pm\frac{1}{2\pi\mathrm{i}} \int_{\gamma_\pm} \frac{V(\tau)d\tau}{\tau - k}, \tag{1.21}$$

One should note that the spectral functions $\bar{U}^n_{...}$ depend not only on the variable k, but on k_* as well. The last is the parameter related to the angle of incidence. Below we indicate this dependence explicitly. The sum of all terms $\bar{U}^n_{...}(k, k_*)$ is proportional to the far-field directivity of the scattered field $f(\varphi, \psi)$, where $k_0 \cos\psi = -k_*$, $k_0 \cos\varphi = -k$.

$$f(k, k_*) \sim \sqrt{k_0^2 - k^2}\bar{U}(k_0^2 - k^2) = \sqrt{k_0^2 - k^2} \sum_{n=0}^{\infty} \left(\bar{U}^n_{...1}(k, k_*) + \bar{U}^n_{...2}(k, k_*)\right). \tag{1.22}$$

2 Preliminary results

2.1 The auxiliary functions G

Avoiding recursion, one can write

$$\bar{U}^n_{\dots 1}(k,k_*) = \begin{cases} \dfrac{A_1 e^{\mathrm{i}ka}}{\sqrt{k_0-k}} F_+[\beta_+(k) F_-[\beta_-(k) \dots F_-[\beta_-(k) \dfrac{1}{k-k_*}] \dots]] & \text{for even } n, \\ \dfrac{A_2 e^{\mathrm{i}ka}}{\sqrt{k_0-k}} F_+[\beta_+(k) F_-[\beta_-(k) \dots F_+[\beta_+(k) \dfrac{1}{k-k_*}] \dots]] & \text{for odd } n, \end{cases} \tag{2.1}$$

$$\bar{U}^n_{\dots 2}(k,k_*) = \begin{cases} \dfrac{A_2 e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} F_-[\beta_-(k) F_+[\beta_+(k) \dots F_+[\beta_+(k) \dfrac{1}{k-k_*}] \dots]] & \text{for even } n, \\ \dfrac{A_1 e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} F_-[\beta_-(k) F_+[\beta_+(k) \dots F_-[\beta_-(k) \dfrac{1}{k-k_*}] \dots]] & \text{for odd } n. \end{cases} \tag{2.2}$$

Each expression contains n pairs of the brackets.

Functions $\beta_\pm(k)$ are defined as

$$\beta_+(k) = e^{-2\mathrm{i}ka} \frac{\sqrt{k_0-k}}{\sqrt{k_0+k}}, \qquad \beta_-(k) = e^{2\mathrm{i}ka} \frac{\sqrt{k_0+k}}{\sqrt{k_0-k}}. \tag{2.3}$$

They emerge when the coefficients of the half-line problems are factorized.

Below the expressions for the diffraction terms shall be obtained, in which the dependence on k and k_* is separated. For this we introduce the auxiliary functions $G^n_{\dots 1}(k)$ and $G^n_{\dots 2}(k)$, whose properties are close to that of the functions $\bar{U}^n_{\dots 1}(k,k_*)$ and $\bar{U}^n_{\dots 2}(k,k_*)$, considered as the functions of k. Auxiliary functions do not depend on k_*. Later we shall obtain the representation of the functions $\bar{U}^n_{\dots}$ in the form of the linear combinations of functions $G^n_{\dots}$ with rational coefficients.

Introduce the auxiliary functions as

$$G_1(k) = G_2(k) \equiv 1 \tag{2.4}$$

$$G^n_{\dots 1}(k) = F_+[\beta_+(k) G^n_{\dots 2}(k)], \tag{2.5}$$

$$G^{n+1}_{\dots 2}(k) = F_-[\beta_-(k) G^n_{\dots 1}(k)]. \tag{2.6}$$

The structure of the indexes if the functions G is the same as that of $\bar{U}$, i.e., the indexes are the sequences of the symbols 1 and 2, corresponding to the sequence of the operators $F_\pm$.

To compare the structure of the functions $\bar{U}$ (as the functions of the variable k) and G, we consider as the example $\bar{U}_{12121}$ and G_{12121}:

$$\bar{U}_{12121}(k,k_*) = A_1 \frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}} F_+[\beta_+ F_-[\beta_- F_+[\beta_+ F_-[\beta_- \frac{1}{k-k_*}]]]], \tag{2.7}$$

$$G_{12121}(k) = F_+[\beta_+ F_-[\beta_- F_+[\beta_+ F_-[\beta_-]]]]. \tag{2.8}$$

It is clear that someone has to get rid of the fraction $1/(k-k_*)$ in the operands in order to express $\bar{U}_{12121}$ in terms of G. Below we propose a procedure to perform this. The procedure utilizes some specific properties of the operators $F_\pm$.

In the end of this section we should note that within this paper we do not consider the questions of the convergence and asymptotic estimation of the diffraction terms. Such estimations can be performed easily by using cnown methods and deforming the contours of integration. The author has made these calculations and happy to inform the readers that one can choose an imaginary value of k_0, such that all the series are convergent and all formulae are valid. If one uses a usual procedure of analytic continuation in k_0, then the validity of all relations for arbitrary k_0 will be established.

2.2 Elementary properties of the operators $F_\pm$

In this section we shall study some properties of the operators $F_\pm$. This properties will be used for manipulations with the diffraction series, namely for excluding the dependence on k_* and for differentiation.

Properties 1 and 2 are obvious and they are listed here only to avoid misunderstanding below. The first one expresses the linearity of the operators; the second one expresses the invariance with respect to translations along the real axis.

The third property is less evident; all tricks below are based on this property.

1. It is clear that $F_\pm$ are linear operators, i.e. for arbitrary constant c and arbitrary functions $V(k)$, $V_1(k)$, $V_2(k)$

$$F_\pm[cV(k)] = cF_\pm[V(k)], \qquad F_\pm[V_1(k) + V_2(k)] = F_\pm[V_1(k)] + F_\pm[V_2(k)].$$

Here we do not discuss to what class of fucntions the operators $F_\pm$ can be applied correctly. However, we have no doubt that in our case all functions are "good" in this sence. The exponent factors provide the necessary decay.

2. For arbitrary function V (belonging to a wide class)

$$(F_\pm[V])' = F_\pm[V'] \tag{2.9}$$

This property can be proved by means of integration by parts:

$$(F_\pm[V(k)])' = \pm\frac{1}{2\pi \mathrm{i}}\int_{\gamma_\pm}\frac{V(\tau)d\tau}{(\tau-k)^2} = \mp\frac{1}{2\pi \mathrm{i}}\int_{\gamma_\pm} V(\tau)d\left(\frac{1}{\tau-k}\right) = \pm\frac{1}{2\pi \mathrm{i}}\int_{\gamma_\pm}\frac{V'(\tau)d\tau}{\tau-k}.$$

3. For arbitrary k_1 not lying on the contours $\gamma_\pm$, and an arbitrary function $V = V(k)$

$$F_\pm\left[\frac{V}{k-k_1}\right] = \frac{F_\pm[V]}{k-k_1} + \frac{\mathcal{F}_\pm(V,k_1)}{k-k_1}, \tag{2.10}$$

where

$$\mathcal{F}_\pm(V,k_1) = \mp\frac{1}{2\pi \mathrm{i}}\int_{\gamma_\pm}\frac{V(\tau)d\tau}{\tau-k_1}. \tag{2.11}$$

The last property follows from the elementary relation:

$$\frac{1}{(\tau-k_1)(\tau-k)} = \frac{1}{k-k_1}\left(\frac{1}{\tau-k} - \frac{1}{\tau-k_1}\right).$$

Note that the value of $\mathcal{F}_\pm(V, k_1)$ does not depend on k, i.e. it is a constant in k.

Equation (2.11) can be interpreted as follows. The operator F_+ performs the decomposition of the function V into two terms: $V = V_+ + V_-$, where the first term is regular above the contour γ_+, and the second is regular below this contour. Try to decompose the same way the function $V/(k-k_1)$. The decomposition

$$\frac{V}{k-k_1} = \frac{V_+}{k-k_1} + \frac{V_-}{k-k_1}$$

has almost all required properties, but one of the terms possesses an undisired pole at $k = k_1$. This pole belongs to either the first or the second term, depending on the half-plane (upper or lower with respect to γ_+), to which k_1 belongs. However, the undesired pole can be easily subtracted. Due to this, the following formulae are valid:

$$\mathcal{F}_+(V, k_1) = \begin{cases} -V_+(k_1), & k_1 \text{ lies above the contour } \gamma_+ \\ V(k_1) - V_+(k_1), & k_1 \text{ lies below the contour } \gamma_+ \end{cases} \tag{2.12}$$

$$\mathcal{F}_-(V, k_1) = \begin{cases} -V_-(k_1), & k_1 \text{ lies below the contour } \gamma_- \\ V(k_1) - V_-(k_1), & k_1 \text{ lies above the contour } \gamma_- \end{cases} \tag{2.13}$$

Note that the values $\mathcal{F}(\dots)$ are finite, therefore if k_1 coincides with a singularitie of either V_+ or V_-, then a corresponding limit must be taken in the relations above.

3 Transformations of the diffraction series

3.1 Representation of the diffraction terms as combinations of G

Transform the expression (2.7) as follows. Apply the property (2.10) first to the internal operator, then to the next one, e.t.c. In each case choose k_1 equal to k_*. To explain this transformation consider the example. Using (2.10) transform the term $\bar{U}_{21}$:

$$\bar{U}_{21} = -\frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}} F_+[e^{-\mathrm{i}ka}\sqrt{k_0-k}\bar{U}_2(k)] = -A_2 \frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}} F_+\left[\frac{\beta_+(k)}{k-k_*}\right] =$$

$$-A_2 \frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}} \left\{ \frac{G_{21}(k)}{k-k_*} + \frac{\mathcal{F}_+(\beta_+, k_*)}{k-k_*} \right\} \tag{3.1}$$

Note that $\mathcal{F}_+(\beta_+, k_*)$ is a constant with respect to the variable k. The value of this constant can be determined using the relation (2.12):

$$\mathcal{F}_+(\beta_+, k_*) = -G_{21}(k_*). \tag{3.2}$$

Using (3.1), one can transform the next term, namely $\bar{U}_{212}$:

$$\bar{U}_{212} = A_2 \frac{e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} F_-\left[\beta_-(k) \frac{G_{21}(k) - G_{21}(k_*)}{k-k_*}\right] =$$

$$A_2 \frac{e^{-\mathrm{i}ka}}{\sqrt{k_0+k}(k-k_*)} \left\{ G_{212}(k) - G_{21}(k_*)G_{12}(k) - G_{212}(k_*) + G_{21}(k_*)G_{12}(k_*) \right\}. \tag{3.3}$$

One can see that this process can be continued, thus yielding the representations of the terms $\bar{U}_{2121}$, $\bar{U}_{21212}$ ...Let us formulate the general result in the form of the following theorem:

Theorem 1 *The diffraction terms $\bar U_{\dots}$ can be represented as follows:*

$$\begin{aligned}\bar U^n_{\nu\dots 1}(k,k_*) &= (-1)^n A_\nu \frac{e^{\mathrm{i}ka}}{\sqrt{k_0-k}(k-k_*)}\sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*)\,G^m_{\dots 1}(k),\\ \bar U^n_{\nu\dots 2}(k,k_*) &= (-1)^n A_\nu \frac{e^{-\mathrm{i}ka}}{\sqrt{k_0+k}(k-k_*)}\sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*)\,G^m_{\dots 2}(k),\end{aligned} \tag{3.4}$$

where the first index $\nu = 1,2$ in the sequence either coincides with the last one μ (in the case of even n), or it does not coincide with μ (if n is odd). The values $g^n_{\dots}$ do not depend on k.

The indexes of the symbols $g^n_{\dots}$ are similar to the indexes of $\bar U$ and G, i.e. the lower indexes are the sequences of altering symbols 1 and 2, and the upper index is the length of the sequence minus 1. In the case of symbols g this sequence cannot be directly interpreted as the successive diffraction acts.

The indexes if formula (3.4) are chosen according the following example. Let in the left-hand side there is the value $\bar U_{12121}$. Then in the right-hand side there stand the products $g_1 G_{12121}$, $g_{12}G_{2121}$, $g_{121}G_{121}$, $g_{1212}G_{21}$ and $g_{12121}G_1$. One can see that the initial sequence 12121 is split into two parts all possible ways. One part stands with g, another one stands with G. The last symbol of the first sequence must coincide with the first symbol of the second sequence. The sum of the lengths of the parts must be greater by 1 than the length of the initial sequence. Note that the sum has the structure of convolution with respect to the indexes.

The recursive formulae for the coefficients are

$$g_1(k_*) = g_2(k_*) \equiv 1, \tag{3.5}$$

$$g^{n+1}_{\nu\dots\mu}(k_*) = -\sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*) G^{m+1}_{\dots\mu}(k_*). \tag{3.6}$$

We imply that the indexes $\mu,\nu = 1,2$ coincide when n is odd and do not coincide in the opposite case.

Theorem 1 and the formulae (3.5, 3.6) can be proved by induction. The basic statement ($n = 0$) can be obtained by comparing (3.4) with (1.17).

Suppose that the statement (3.4) is fulfilled for some n. Using this fact, let us calculate $\bar U^{n+1}_{\nu\dots 1}$:

$$\bar U^{n+1}_{\nu\dots 1}(k,k_*) = \frac{A_\nu(-1)^{n+1}e^{\mathrm{i}ka}}{\sqrt{k-k_0}} F_+\left[e^{-2\mathrm{i}ka}\frac{\sqrt{k_0-k}}{\sqrt{k_0+k}(k-k_*)}\sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*)G^m_{\dots 2}(k)\right] =$$

$$\frac{A_\nu(-1)^{n+1}e^{\mathrm{i}ka}}{\sqrt{k-k_0}} \sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*) F_+\left[\frac{\beta_+(k)G^m_{\dots 2}(k)}{k-k_*}\right]. \tag{3.7}$$

Each term in the sum can be transformed using (2.10), thus giving

$$\bar U^{n+1}_{\dots 1}(k,k_*) = \frac{A_\nu(-1)^{n+1}e^{\mathrm{i}ka}}{\sqrt{k-k_0}} \sum_{m=0}^{n} g^{n-m}_{\nu\dots}(k_*) \frac{G^{m+1}_{\dots 1}(k) - G^{m+1}_{\dots 1}(k_*)}{k-k_*}. \tag{3.8}$$

Similar expresiions can be obtained for $\bar{U}^{n+1}_{\dots 2}$. Relations obtained verify the formulae (3.4) and (3.5)–(3.6).

Let us discuss the result of Theorem 1. Each term of the diffraction series depends on k and k_*. The relation (3.4) represents the diffraction terms as combinations of functions depending on one variable: function $G^n_{\dots}$ depend only on k, and functions $g^n_{\dots}$ depend only on k_*.

Some interesting properties of the coefficients $g^n_{\dots}(k_*)$ will be studied in the Appendix.

3.2 The expression for the sum of the diffraction series

We obtained above the representation for each diffraction term. Here we use this representation for simplifying the whole diffraction series. While doing this, we use implicitly the fact that the sum (3.4) has the form of a discrete convolution with respect to the index.

Represent the diffraction series in the form

$$\bar{U}(k,k_*) = \sum_{\text{even}\, n} \bar{U}^n_{1\dots 1}(k,k_*) + \sum_{\text{odd}\, n} \bar{U}^n_{2\dots 1}(k,k_*) + \sum_{\text{even}\, n} \bar{U}^n_{2\dots 2}(k,k_*) + \sum_{\text{odd}\, n} \bar{U}^n_{1\dots 2}(k,k_*). \quad (3.9)$$

All sums are taken over nonnegative integer n. Applying the result of Theorem 1 and grouping the terms for different G, we obtain the representation:

$$\bar{U}(k,k_*) = \frac{1}{k-k_*}\left[(A_1 g_{1-1}(k_*) - A_2 g_{2-1}(k_*)) \left(\frac{G_{1-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} - \frac{G_{1-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} \right) + \right.$$
$$\left. (A_1 g_{1-2}(k_*) - A_2 g_{2-2}(k_*)) \left(\frac{G_{2-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} - \frac{G_{2-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} \right) \right], \quad (3.10)$$

where the value with the index $\mu - \nu$ denotes the sum of all corresponding values with the indexes starting with μ and finishing with ν, i.e.

$$G_{1-1} = G_1 + G_{121} + G_{12121} + \dots \qquad G_{2-2} = G_2 + G_{212} + G_{21212} + \dots \quad (3.11)$$
$$G_{1-2} = G_{12} + G_{1212} + G_{121212} + \dots \qquad G_{2-1} = G_{21} + G_{2121} + G_{212121} + \dots \quad (3.12)$$
$$g_{1-1} = g_1 + g_{121} + g_{12121} + \dots \qquad g_{2-2} = g_2 + g_{212} + g_{21212} + \dots \quad (3.13)$$
$$g_{1-2} = g_{12} + g_{1212} + g_{121212} + \dots \qquad g_{2-1} = g_{21} + g_{2121} + g_{212121} + \dots \quad (3.14)$$

The properties of such sums are described in the Appendix. Here we list some of these properties necessary for the further transformations of the diffraction series.

It follows from (A.23)–(A.26) that

$$g_{1-1}(k_*) = G_{2-2}(k_*)/N(k_*), \quad (3.15)$$
$$g_{2-1}(k_*) = -G_{2-1}(k_*)/N(k_*), \quad (3.16)$$
$$g_{2-2}(k_*) = G_{1-1}(k_*)/N(k_*), \quad (3.17)$$
$$g_{1-2}(k_*) = -G_{1-2}(k_*)/N(k_*), \quad (3.18)$$

where $N(k_*)$ is the determinant

$$N(k_*) = \begin{vmatrix} G_{1-1}(k_*) & G_{2-1}(k_*) \\ G_{1-2}(k_*) & G_{2-2}(k_*) \end{vmatrix}. \quad (3.19)$$

It is shown in the Appendix that

$$N(k_*) \equiv 1. \tag{3.20}$$

Note that the far-field directivity is proportional to $\sqrt{k_0^2 - k^2}\bar{U}$. Using the definition of the coefficients $A_{1,2}$ and the properties (3.15–3.20), we transform (3.10) to the form

$$f(k, k_*) \sim \sqrt{k_0^2 - k^2}\bar{U}(k) = \mathrm{i}\frac{\sqrt{k_0^2 - k^2}\sqrt{k_0^2 - k_*^2}}{k - k_*}\times$$

$$\left[\left(\frac{G_{1-1}(k_*)e^{\mathrm{i}k_* a}}{\sqrt{k_0 - k_*}} - \frac{G_{1-2}(k_*)e^{-\mathrm{i}k_* a}}{\sqrt{k_0 + k_*}}\right)\left(\frac{G_{2-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0 - k}} - \frac{G_{2-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0 + k}}\right) - \right.$$
$$\left.\left(\frac{G_{2-1}(k_*)e^{\mathrm{i}k_* a}}{\sqrt{k_0 - k_*}} - \frac{G_{2-2}(k_*)e^{-\mathrm{i}k_* a}}{\sqrt{k_0 + k_*}}\right)\left(\frac{G_{1-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0 - k}} - \frac{G_{1-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0 + k}}\right)\right]. \tag{3.21}$$

I.e., the far-field directivity becomes expressed in the form

$$f(k, k_*) = \frac{V(k)W(k_*) - W(k)V(k_*)}{k - k_*}, \tag{3.22}$$

where V and W are some functions depending on one variable. Similar representation for the far-field directivity was obtained (probably, for the first time) by Williams [3].

Note that such a representation cannot be unique. Any transformation of the form

$$V^*(k) = c_1 V(k) + c_2 W(k), \tag{3.23}$$
$$W^*(k) = c_2 V(k) + c_1 W(k) \tag{3.24}$$

for arbitrary constant $c_1 \neq c_2$ leads to another representation of the same form. As we shall see below, for some values of these constants, the representation (3.21) can be transformed into Williams' formula.

4 Differentiation of the diffraction series

The formula (3.21) looks to be useful, but calculation of the function $G_{\mu-\nu}$ using their definition is not a simple task. Only recursive relations involving integral representations (2.4)–(2.6) have been proposed for this above.

Below we show that the functions $G^n_{\dots}(k)$ and the infinite sums of these functions $G_{\mu-\nu}(k)$ obey ordinary differential equations with rational in k coeffieicents. The orders of the equations for $G^n_{\dots}(k)$ are equal to n, and the order of the equations for $G_{\mu-\nu}(k)$ is equal to 2.

4.1 Differentiation of the functions $G^n_{\dots}(k)$

Let the prime denote the differentiation with respect to k. Applying the relation (2.9) to the definition of the functions G, we obtain the relarion

$$(G^{n+1}_{\nu\dots1})' = F_+[\beta'_+ G^n_{\nu\dots2}] + F_+[\beta_+ (G^n_{\nu\dots2})']. \tag{4.1}$$

Similar relation can be obtained for $(G^{n+1}_{\nu\dots2})'$.

Note that the logarithmic derivatives of the functions $\beta_\pm(k)$ are rational functions of k, in other words, the differentiation of the functions $\beta_\pm(k)$ is equivalent to multiplication of these functions by some known rational functions:

$$(\beta_\pm(k))' = \left(\mp 2\mathrm{i}a \pm \frac{1}{2(k-k_0)} \mp \frac{1}{2(k+k_0)} \right) \beta_\pm(k). \tag{4.2}$$

We described above a method of elimination of a rational function under the operator $F_\pm$. In combination with the relation (4.1) this method enables to represent $(G^n_{\dots}(k))'$ in the form of a linear combination of the functions $G^m_{\dots}(k)$ with $m \le n$. The structure of the formulae, which can be obtained using this method, is more complicated than that of (3.4):

Theorem 2 *The derivatives of $G^n_{\dots}(k)$ with respect to k are given by*

$$(G^n_{\nu\dots\mu}(k))' = \left(r_\nu - r_\mu - \frac{p_\mu}{k-k_0} - \frac{m_\mu}{k+k_0} \right) G^n_{\nu\dots\mu}(k) + \sum_{m=0}^{n} \left(\frac{p^{n-m}_{\nu\dots}}{k-k_0} + \frac{m^{n-m}_{\nu\dots}}{k+k_0} \right) G^m_{\dots\mu}(k). \tag{4.3}$$

The coefficients p, m do not depend on k.

The indexes of p, m are similar to the indexes of g and other diffraction values.

The recursive relations for the coefficients are

$$r_1 = \mathrm{i}a, \qquad r_2 = -\mathrm{i}a, \tag{4.4}$$

$$p_1 = -1/4, \qquad p_2 = 1/4, \tag{4.5}$$

$$m_1 = 1/4, \qquad m_2 = -1/4, \tag{4.6}$$

$$p^{n+1}_{\nu\dots\bar\mu} = \sum_{m=0}^{n} p^{n-m}_{\nu\dots} \mathcal{F}_\pm(\beta_\pm G^m_{\dots\mu}, k_0) - p_\mu \mathcal{F}_\pm(\beta_\pm G^n_{\dots\mu}, k_0), \tag{4.7}$$

$$m^{n+1}_{\nu\dots\mu} = \sum_{m=0}^{n} m^{n-m}_{\nu\dots} \mathcal{F}_\pm(\beta_\pm G^m_{\dots\mu}, -k_0) - m_\mu \mathcal{F}_\pm(\beta_\pm G^n_{\dots\mu}, -k_0). \tag{4.8}$$

The signs $\pm$ in two last formulae depend on μ: it is "+" for $\mu = 2$ and "-" for $\mu = 1$. The symbol $\bar\mu$ denotes 2 for $\mu = 1$ and it is equal to 1 for $\mu = 2$.

This theorem can be easily proved by induction using (4.1) and (4.2).

The coefficients p and m possess some interesting algebraic properties; they are studied in the Appendix.

Let us discuss some coroolaries of Theorem 2. The equation (4.3) for $n = 1$ is an inhomogeneous linear ordinary differential equation for $G_{12}(k)$ and $G_{21}(k)$ with rational coefficients and rational right-hand side. These equations are valid on the vhole complex plane (unlike the represenations (2.5), (2.6), which are valid either above γ_+ or below γ_-, respectively).

For finding the functions G_{121} we take the equations (4.3) for G_{121} and G_{21}. So, we have a closed system of 2 differential equations. Generally, for each $G^n_{\dots}$ we have a system of n inhomogeneous ordinary differential equations. This system must be supplied by appropriate boundary conditions. These conditions must include the restrictions on the behaviour of unknown functions at infinity and at $k = \pm k_0$. Here we are not going to perform any practical calculations, so we do not explore this question.

Note that using Theorem 1, the diifferential equations can be obtained for the values $\bar U^n_{\dots}$.

4.2 Differentiation of the infinite sums

We introduced the series $G_{\nu-\mu}(k)$ by (3.11), (3.12)). These series are auxiliary functions for calculation of the far-field directivity. Let us calculate the derivatives of these series. Using Theorem 2 we obtain:

$$(G_{\nu-\mu}(k))' = \left(r_\nu - r_\mu - \frac{p_\mu}{k-k_0} - \frac{m_\mu}{k+k_0} \right) G_{\nu-\mu}(k) +$$

$$\left(\frac{p_{\nu-1}}{k-k_0} + \frac{m_{\nu-1}}{k+k_0} \right) G_{1-\mu}(k) + \left(\frac{p_{\nu-2}}{k-k_0} + \frac{m_{\nu-2}}{k+k_0} \right) G_{2-\mu}(k), \tag{4.9}$$

where the values $p_{\mu-\nu}$ and $m_{\mu-\nu}$ are introduced similarly to $g_{\mu-\nu}$:

$$p_{1-1} = p_1 + p_{121} + p_{12121} + \dots \qquad p_{2-2} = p_2 + p_{212} + p_{21212} + \dots \tag{4.10}$$

$$p_{1-2} = p_{12} + p_{1212} + p_{121212} + \dots \qquad p_{2-1} = p_{21} + p_{2121} + p_{212121} + \dots \tag{4.11}$$

$$m_{1-1} = m_1 + m_{121} + m_{12121} + \dots \qquad m_{2-2} = m_2 + m_{212} + m_{21212} + \dots \tag{4.12}$$

$$m_{1-2} = m_{12} + m_{1212} + m_{121212} + \dots \qquad m_{2-1} = m_{21} + m_{2121} + m_{212121} + \dots \tag{4.13}$$

Depending on the values of ν and μ, the equation (4.9) has four realizations. These 4 equations can be split into two pairs. One of these pairs is composed by the equations for $(G_{1-1})'$ and $(G_{2-1})'$, another pair is composed by two other equations. Each pair forms a closed system of two homogeneous differential equations with rational coefficients. The coefficients of the equations are representaed in the form of asymptotic series (4.10)–(4.13). Each term in these series can be calculated using the recursive formulae involving integral representations. In the Appendix we derive more convenient representations for the coefficients.

Boundary conditions for these equations are composed by the restrictions on the behaviour of the functions $G_{\nu-mu}$ at infinity and at the singular points $\pm k_0$.

4.3 Connection with the results obtained earlier

The contents of the ciurrent paper is closely connected with the works by Williams [3] and Shanin [5]. These authors also explore the possibility to find the solution using ordinary differential equations. Here we propose an interpretation of the results of these two works in terms of the current paper. Moreover, we obtain the unknown parameters from [3] and [5] in the form of asymptotic series.

Let us start with [3]. Consider the functions

$$\begin{aligned} V(k) &= G_{1-2}(k_0) \left(\frac{G_{2-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} - \frac{G_{2-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} \right) + \\ &\quad G_{1-1}(k_0) \left(\frac{G_{1-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} - \frac{G_{1-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} \right), \end{aligned} \tag{4.14}$$

$$\begin{aligned} W(k) &= G_{1-1}(k_0) \left(\frac{G_{2-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} - \frac{G_{2-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} \right) + \\ &\quad G_{2-1}(k_0) \left(\frac{G_{1-1}(k)e^{\mathrm{i}ka}}{\sqrt{k_0-k}} - \frac{G_{1-2}(k)e^{-\mathrm{i}ka}}{\sqrt{k_0+k}} \right). \end{aligned} \tag{4.15}$$

These functions coincide with Williams' "fundamental solutions" up to constant coefficients. Note that $V(k) = W(-k)$, and derive the differential equations for V and W. Using (4.9), we obtain in Williams' notations

$$V' - \mathrm{i}a \cosh\frac{\theta}{2} V + \mathrm{i}a \sinh\frac{\theta}{2} W = \frac{1}{2}\frac{\Gamma W - V}{k_0 + k}. \tag{4.16}$$

This equation is precisely the equation from [3]. The constants are defined as

$$\cosh\frac{\theta}{2} = \frac{(G_{2-1}(k_0))^2 + (G_{1-1}(k_0))^2}{(G_{2-1}(k_0))^2 - (G_{1-1}(k_0))^2}, \tag{4.17}$$

$$\Gamma = -2(m_{1-2} + m_{2-1}). \tag{4.18}$$

Consider now the results of [5]. The main statement of this work is that the functions

$$\bar{U}_+(k) \equiv \sum_{n=0}^{\infty} \bar{U}^n_{\dots 1}(k, k_*) \qquad \text{and} \qquad \bar{U}_-(k) \equiv \sum_{n=0}^{\infty} \bar{U}^n_{\dots 2}(k, k_*)$$

are two different solutions of an ordinary differential equation of order 2 with rational coefficients with respect to the variable k.

Let this equation have the form

$$U'' = X(k)U' + Y(k)U, \tag{4.19}$$

where prime corresponds to differentiation with respect to k. The coefficients X and Y can be represented as (see [5]):

$$X = D'/D, \qquad Y = E/D, \tag{4.20}$$

where D and E are the determinants

$$D = \begin{vmatrix} (\bar{U}_+)' & \bar{U}_+ \\ (\bar{U}_-)' & \bar{U}_- \end{vmatrix} \qquad E = \begin{vmatrix} (\bar{U}_+)' & (\bar{U}_+)'' \\ (\bar{U}_-)' & (\bar{U}_-)'' \end{vmatrix}. \tag{4.21}$$

Usin the technique of diffraction series transformation, $\bar{U}_+$ and $\bar{U}_-$ can be written in the form

$$\bar{U}_+(k, k_*) = \frac{e^{\mathrm{i}ka}}{\sqrt{k_0 - k}(k - k_*)} \left[C_1 G_{1-1}(k) + C_2 G_{2-1}(k)\right], \tag{4.22}$$

$$\bar{U}_-(k, k_*) = -\frac{e^{-\mathrm{i}ka}}{\sqrt{k_0 + k}(k - k_*)} \left[C_2 G_{2-2}(k) + C_1 G_{1-2}(k)\right],, \tag{4.23}$$

where

$$C_1 = A_1 g_{1-1} - A_2 g_{2-1}, \qquad C_2 = A_1 g_{1-2} - A_2 g_{2-2}. \tag{4.24}$$

Using Theorem 1 and Theorem 2, we can write down the expressions for the determinants D and E in general case, but they are rather ugly. Consider the particular case of the normal incidence $k_* = 0$. In this case the calculations are less complicated. The result is

$$D = \frac{-2\mathrm{i}a(k^2 - k_0^2) + k_0(1 + 4p_{2-2} + 2p_{1-2} - 2p_{2-1})}{(k_0^2 - k^2)^{3/2} k^2}, \tag{4.25}$$

$$E = \frac{Q}{4(k_0^2 - k^2)^{7/2} k^2}, \tag{4.26}$$

$$Q = 8\mathrm{i}a^3(k^2 - k_0^2)^3 - 4a^2 k_0 (k^2 - k_0^2)^2 (3 + 2p_{1-2} + 12p_{2-2} - 2p_{2-1}) -$$
$$2\mathrm{i}a(k^2 - k_0^2)(k_0^2(9 - 4(p_{1-2})^2 + 48(p_{2-2})^2 +$$
$$p_{2-2}(24 - 16p_{2-1}) - 8p_{2-1} - 4(p_{2-1})^2 + 8p_{1-2}(2p_{2-2} + p_{2-1})) +$$
$$k^2(-3 + 4(p_{1-2})^2 - 4p_{2-1} + 4(p_{2-1})^2 + 4p_{1-2}(-1 + 2p_{2-1}))) +$$
$$k_0(1 + 2p_{1-2} + 4p_{2-2} - 2p_{2-1})(k_0^2(15 - 4(p_{1-2})^2 + 8p_{2-2} + 16(p_{2-2})^2 - 4p_{2-1} -$$
$$4(p_{2-1})^2 + 4p_{1-2}(-1 + 2p_{2-1})) + k^2(-15 + 4(p_{1-2})^2 + 4p_{2-1} + 4(p_{2-1})^2 + 4p_{1-2}(1 + 2p_{2-1}))).$$

The ansatz for the determinants D and E is the same as in [5]. Comparing (4.9) with the equations obtained in [5], we note that the solution of the main equation can be represented as a linear combination of the solutions of simlier equations having similar form. These auxiliary equations contain less singular points and less monodromy restrictions are imposed on the solutions. This fact can be interpreted in terms of the work [9], but this topic needs a detailed investigation.

5 Conclusions

The following results are obtained in this paper:
— The relation (2.10) is derived. It is used for simplification and differentiation of the diffraction series. This property enables to express the diffraction terms and their derivatives trough the auxiliary functions G, which depend on a single variable.
— It is shown that the coefficients g, p and m satisfy unobvious algebraic relations. The most important of them are (A.23)–(A.26), (A.28)–(A.31), (A.32), (A.45).
— The represention (3.21) was obtained. Ordinary differential equations (4.9) for the auxiliary functions were derived.
— Comparison with the results of [3] [5] was performed. Asymptotic series for the unknown coefficients of these works was obtained.

Now let us say a few words about the motivation of the current work. Some results were obtained as the development of the ideas from [5]. It was necessary to find convenient expressions for D and E for numerical calculations. From the other hand, the representation (3.21) does not follow from [5]. This representation cannot be ignored because this representation simplifies the calculations dramatically. We also suppose that this representation is a particular case of a more general theoretical result.

We must confess that the problem of diffraction by a strip or a slit is not very interesting itself. It is pleasant to obtain new result in such intensively studied problem, but the primary target is developing the method applicable to a wider class of diffraction problems. We hope that the technique developed here can be generalized on the problems that can be interpreted as wave propagation on non-schlicht surfaces in the sence of Sommerfeld. We remind that he treated diffraction by a half-line as propagation on a 2-sheet "Riemann" surface.

As the examples of such problems we mention here 2D problem of diffraction on a set of ideal segments, which lie in one line.

The approach developed in [5] can be applied to such problems, but the practical calculations are extremely difficult because of a great number of unknown parameters and monodromy restrictions. We hope to obtain the representations like (3.21) and differential equation of the form (4.9) for these problems.

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Appendix. Some properties of the coefficients g, p, m

The coefficients $g^n_{\dots}$, $p^n_{\dots}$ and $m^n_{\dots}$ have been introduced using the recursive relations. These formulae involve the values $\mathcal{F}_\pm(\beta_\pm G^m_{\dots}, k_1)$, where k_1 can be equal to $\pm k_0$ or to k_*. The properties of the coefficients follow either from the properties of the properties of the operators $\mathcal{F}$ or from the properties of the structure of the recursive relations. Here we shall study the properties of the second sort. The values $\mathcal{F}_\pm(\beta_\pm G^m_{\dots}, k_1)$ within our study can be substituted bya an arbitrary set of complex numbers.

a) q-type and h-type sequences

Let the symbols $f^n_{\dots}$ denote an arbitrary set of values, having the indexes peculiar to the diffraction terms (i.e. f_{21}, f_{212}, $f^n_{\dots 1}$ e.t.c.). We do not use the symbols with $n = 0$, namely f_1 and f_2.

Consider the recursive relations of two types. the sequence obeying the relations of the form

$$q_1 = q_2 = 1, \tag{A.1}$$

$$q^{n+1}_{\nu\dots\mu} = \sum_{m=0}^{n} q^{n-m}_{\nu\dots} f^{m+1}_{\dots\mu} \tag{A.2}$$

we shall call a q-type sequence (or simply a q-sequence), associated with the set $f^n_{\dots}$. For example, if

$$f^n_{\nu\dots} = -G^n_{\nu\dots}(k_*), \tag{A.3}$$

then

$$g^n_{\nu\dots}(k_*) = q^n_{\nu\dots} \tag{A.4}$$

(see (3.5)–(3.6)). Beside this example, below we introduce the q-sequences that are necessary for calculating of the coefficients p and m. These sequences are associated with some other sets of values f.

The sequence of the second type (h-sequences) obey the recursive relations

$$h_1 = 1/2, \qquad h_2 = -1/2, \tag{A.5}$$

$$h^{n+1}_{\nu\dots\mu} = \sum_{m=0}^{n} h^{n-m}_{\nu\dots} f^{m+1}_{\dots\mu} - h_\mu f^{n+1}_{\nu\dots\mu}. \tag{A.6}$$

The coefficients p and m (see Theorem 2) are the sequences of h-type. If the sequence $h^n_{\dots}$ is associated with the set $f^n_{\nu\dots} = \mathcal{F}_\pm(\beta_\pm G^{n-1}_{\nu\dots}, k_0)$, then

$$p^n_{\nu\dots} = -\frac{h^n_{\nu\dots}}{2}. \tag{A.7}$$

If the sequence is associated with $f^n_{\nu\dots} = \mathcal{F}_\pm(\beta_\pm G^{n-1}_{\nu\dots}, -k_0)$, then

$$m^n_{\nu\dots} = \frac{h^n_{\nu\dots}}{2}. \tag{A.8}$$

Consider the following example. Let us calculate the coefficients q_{12}, q_{121}, q_{1212}, q_{12121}, using the relation using the relations (A.1)–(A.2):

$$\begin{aligned}
q_{12} &= f_{12}, && \text{(A.9)}\\
q_{121} &= f_{12}f_{21} + f_{121}, && \text{(A.10)}\\
q_{1212} &= f_{12}f_{21}f_{12} + f_{121}f_{12} + f_{12}f_{212} + f_{1212}, && \text{(A.11)}\\
q_{12121} &= f_{12}f_{21}f_{12}f_{21} + f_{121}f_{12}f_{21} + f_{12}f_{212}f_{21} + f_{1212}f_{21} + \\
&\quad f_{12}f_{21}f_{121} + f_{121}f_{121} + f_{12}f_{2121} + f_{12121}. && \text{(A.12)}
\end{aligned}$$

Some of these expressions can be simplified, but here we prefer to avoid simplification.

Consider the elements of an h-sequence. Following the rules (A.5)–(A.6), write down the expressions for h_{12121} and h_{21212}:

$$\begin{aligned}
h_{12121} &= f_{12}f_{21}f_{12}f_{21} + f_{12}f_{212}f_{21} + f_{1212}f_{21} + f_{12}f_{21}f_{121} + f_{12}f_{2121}.\\
h_{21212} &= -(f_{21}f_{12}f_{21}f_{12} + f_{21}f_{121}f_{12} + f_{2121}f_{12} + f_{21}f_{12}f_{212} + f_{21}f_{1212})
\end{aligned} \tag{A.13}$$

b) The properties of infinite sums of q- and h-sequences

Introduce the values

$$\begin{aligned}
f_{1-1} &= f_{121} + f_{12121} + f_{1212121} + \dots && \text{(A.14)}\\
f_{1-2} &= f_{12} + f_{1212} + f_{121212} + \dots && \text{(A.15)}\\
f_{2-1} &= f_{21} + f_{2121} + f_{212121} + \dots && \text{(A.16)}\\
f_{2-2} &= f_{212} + f_{21212} + f_{2121212} + \dots && \text{(A.17)}
\end{aligned}$$

i.e., the sum $f_{\nu-\mu}$ contains all values with the indexes starting with ν and finishing with μ. Note that unlike (3.11) the sums (A.14) and (A.17) do not contain f_1 and f_2.

Introduce the sums $q_{\nu-\mu}$ and $h_{\nu-\mu}$ analogously to (3.13) and (3.14).

Using the recursive relations, one can obtain the following formulae

$$\begin{aligned}
q_{1-1}f_{1-1} + q_{1-2}f_{2-1} &= q_{1-1} - 1, && \text{(A.18)}\\
q_{1-1}f_{1-2} + q_{1-2}f_{2-2} &= q_{1-2}, && \text{(A.19)}\\
q_{2-1}f_{1-1} + q_{2-2}f_{2-1} &= q_{2-1}, && \text{(A.20)}\\
q_{2-1}f_{1-2} + q_{2-2}f_{2-2} &= q_{2-2} - 1. && \text{(A.21)}
\end{aligned}$$

These relations can be rewritten in the matrix form and solved with respect to q:

$$\begin{pmatrix} q_{1-1} & q_{1-2} \\ q_{2-1} & q_{2-2} \end{pmatrix} = \begin{pmatrix} 1-f_{1-1} & -f_{1-2} \\ -f_{2-1} & 1-f_{2-2} \end{pmatrix}^{-1}, \tag{A.22}$$

or in the components

$$q_{1-1} = \frac{1-f_{2-2}}{N^*}, \tag{A.23}$$

$$q_{2-1} = \frac{f_{2-1}}{N^*}, \tag{A.24}$$

$$q_{1-2} = \frac{f_{1-2}}{N^*}, \tag{A.25}$$

$$q_{2-2} = \frac{1-f_{1-1}}{N^*},, \tag{A.26}$$

where N^* is the determinant

$$N^* = \begin{vmatrix} 1-f_{1-1} & -f_{1-2} \\ -f_{2-1} & 1-f_{2-2} \end{vmatrix}. \tag{A.27}$$

Consider the sequences q and h associated with the same set $f^n_{\dots}$. One can see that

$$h_{2-1} = -f_{2-1}q_{1-1}, \tag{A.28}$$

$$h_{1-2} = f_{1-2}q_{2-2}, \tag{A.29}$$

$$h_{1-1} = f_{1-2}q_{2-1}, \tag{A.30}$$

$$h_{2-2} = -f_{2-1}q_{1-2}. \tag{A.31}$$

Using (A.28)–(A.31) and (A.23)-(A.26), we obtain an important equation

$$h_{1-1} = -h_{2-2} = \frac{f_{1-2}f_{2-1}}{N^*}. \tag{A.32}$$

Taking into account (A.7) and (A.8), we obtain:

$$p_{1-1} = -p_{2-2}, \qquad m_{1-1} = -m_{2-2}. \tag{A.33}$$

c) Calculation of the determinants N and N^*

Let us show that the determinant $N(k)$ defined by (3.19) is identically equal to 1. First, let us show that it is a constant, i.e., does not depend on k.

Note that the formula (3.19) defines N as a function of k_*, but the variable k_* in the definition can be replaced by any symbol, e.g. by k.

Differentiate $N(k)$ with respect to k. Using the relations (4.3) and after elementary transformations, we obtain

$$(N(k))' = \left(\frac{p_{1-1}+p_{2-2}}{k-k_0} + \frac{m_{1-1}+m_{2-2}}{k+k_0} \right) (G_{1-1}G_{2-2} - G_{2-1}G_{1-2}). \tag{A.34}$$

Taking into account the relations (A.33) we conclude that N' is identically equal to 0; it means that $N(k)$ is a constant.

To determine this constant one can study the behaviour of the determinant at infinity. Note that all functions $G^n_{\dots}(k)$ for $n > 0$ tend to zero like $\sim |k|^{-1}$ for large real k. Therefore, G_1G_2 is the main term of $N(k)$. Thus,

$$G_{1-1}(k)G_{2-2}(k) - G_{1-2}(k)G_{2-1}(k) \equiv 1. \tag{A.35}$$

Now consider the determinant N^* defined by (A.27). This determinant is used for calculating the coefficients m. We chose $f^m_{\nu\dots} = \mathcal{F}_\pm(\beta_\pm G^{m-1}_{\nu\dots}, -k_0)$ as the set of values f.

According to the properties of the operators $\mathcal{F}$,

$$f^m_{\dots 2} = -G^m_{\dots 2}(-k_0) = -\lim_{\tau\to -k_0} G^m_{\dots 2}(\tau), \tag{A.36}$$

$$f^m_{\dots 1} = \lim_{\tau\to -k_0} [\beta_-(\tau)G^{m-1}_{\dots 2}(\tau) - G^m_{\dots 1}(\tau)]. \tag{A.37}$$

Therefore

$$N^* = \lim_{\tau\to -k_0} [N(\tau) + \beta_-(\tau)(G_{2-2}(\tau)G_{1-2}(\tau) - G_{1-2}(\tau)G_{2-2}(\tau))] \equiv 1. \tag{A.38}$$

Taking into account the formulae (A.23)–(A.26) and (A.23)–(A.26), we conclude that

$$m_{2-1} = -\frac{f_{2-1}(1-f_{2-2})}{2}, \tag{A.39}$$

$$m_{1-2} = \frac{f_{1-2}(1-f_{1-1})}{2}, \tag{A.40}$$

$$m_{1-1} = \frac{f_{1-2}f_{2-1}}{2}, \tag{A.41}$$

$$m_{2-2} = -\frac{f_{1-2}f_{2-1}}{2}. \tag{A.42}$$

The properties of the coefficients p can be obtained from the properties of the coefficients m by using the symmetry properties:

$$p_{\mu-\nu} = m_{\nu-\mu}. \tag{A.43}$$

d) Extra properties of the elements of h-sequences and the properties of the finite sums of the functions G

We showed above that for the infinite sums of each h-sequence, the identity $h_{1-1} = -h_{2-2}$ is valid. Here we show that a stronger statement is valid, namely for any even n

$$h^n_{1\dots 1} = -h^n_{2\dots 2}. \tag{A.44}$$

The proof of this fact is known to the author, but it is tedious, and we do not demonstrate it here. Discuss the corollaries of this property.

The identities (A.33) following from (A.32) have been used to prove that $N(k) \equiv 1$. The determinant $N(k)$ is an infinite sum of the products of the functions G. It seems to be unexpected that the identity (A.35) can be split into an infinite sequence of the identities, each of which contain a finite num of products:

$$
\begin{aligned}
&G_1(k)G_2(k) \equiv 1, \\
&G_{121}(k)G_2(k) - G_{21}(k)G_{12}(k) + G_1(k)G_{212}(k) \equiv 0, \\
&G_{12121}(k)G_2(k) - G_{2121}(k)G_{12}(k) + G_{121}(k)G_{212}(k) - G_{21}(k)G_{1212}(k) + \\
&+G_1(k)G_{21212}(k) \equiv 0, \\
&\dots
\end{aligned}
$$

i.e., for any even $n \neq 0$

$$\sum_{m=0}^{n} (-1)^m G^{n-m}_{\dots 1}(k) G^{m}_{\dots 2}(k) \equiv 0. \tag{A.45}$$

The proof is elementary, but tedious. It is based on the identities (A.44).

The relations (A.45) seem to be unexpeted, but there exists one more proof of them based on the Liouville's theorem (see [5]).

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