New analytical results and numerical algorithms for quarter-plane diffraction. Part I: Modified Smyshlyaev's formulae based on the embedding formulae

Andrey V. Shanin[1]

Department of Physics, Moscow State University, Leninskie Gory, Moscow, Russia, 119992, e-mail: andrey_shanin@mail.ru

Abstract

We present new results related to quarter-plane diffraction problem. The Smyshlyaev's framework of separation of radial variable is taken as the base. Part I describes the application of the embedding methods to the quarter-plane problem. As the result, we obtain the embedding formula in the 3D space and the modified Smyshlyaev's formula for the ν-plane. The main feature of the modified Smyshlyaev's formula is that it uses the edge Green's functions instead of the usual Green's function on a sphere with a cut. The benefits of this are discussed.

1 Introduction

We are sure that the main avenue for numerical solving the conical problems (and a quarter-plane problem among them) is the separation of the radial variable and studying the Laplace-Beltrami problem on the unit sphere. Recently, such approach has been developed by the group of Smyshlyaev et. al. [2, 15]. In the current work we present the results for improving this approach. Our improvements result in several analytical relations, which also give significant gain in numerical calculations.

The approach by Smyshlyaev is as follows. On the *first step* the problem is formulated in the polar coordinates, and the radial variable is separated from the spherical ones. As the result, one obtains the formula for the diffraction coefficient of the conical problem:

$$f(\omega, \omega_0) = \frac{\mathrm{i}}{\pi} \int_\gamma e^{-\mathrm{i}\pi\nu} g(\omega, \omega_0, \nu)\nu \, d\nu \tag{1}$$

(see [2]). Here ω and ω_0 are the directions of incidence and scattering, ν is the separation constant of the spherical problem, g is the Green's function of the spherical problem.

On the *second step* the spherical Green's function g should be found. In the most general case this function can be computed by solving boundary integral equations for the spherical problem. Alternatively, the function g can

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be expressed trough the eigenfunctions Φ_j and eigenvalues ν_j of the spherical problem as follows:

$$g(\omega, \omega_0, \nu) = \sum_j \frac{\Phi_j(\omega)\Phi_j(\omega_0)}{\nu^2 - \nu_j^2}. \tag{2}$$

In some particular cases (we should note that the flat cone belongs to this set) the eigenfunctions can be found analytically by separating variables in the sphero-conal coordinates (see, e.g. Kraus and Levine [8] for the flat cone). This approach takes into account that the flat cone is a degenerate case of an elliptic cone.

In the current work we are going to modify this procedure as follows. In Part I we construct the modified Smyshlyaev's formulae to use instead of (1). These formulae are based on the embedding procedure. Instead of using the Green's function $g(\omega, \omega_0, \nu)$ we shall obtain the integral representation involving the *spherical edge Green's functions* $v^{1,2}(\omega, \nu)$, $w^{1,2}(\omega, \nu)$. Such a function can be treated as a limiting case of the Green's function g as the source location approaches the edge of the scatterer. This function depends on less variables (on 3 rather than on 5), so it is more convenient for numerical tabulation.

In Part II we develop a new method for computing the functions $v^{1,2}$ and $w^{1,2}$. Instead of using the integral equations or constructing the series like (2) we propose a completely novel method of the *coordinate equations*, which can be treated as a generalization of separation of variables. This new method enables us to find the edge Green's functions $v^{1,2}$, $w^{1,2}$ without series. It can be applied to a wide class of problems, e.g. to a trihedral cone problem.

Current work has been stimulated by the recent progress achieved for the 2D diffraction problems. Namely, the coordinate equations method was proposed by the author for the problem of diffraction by several ideal strips lying in one plane [14]. The method has some connections to the ideas developed by Latta [9] and Williams [18] for a single strip problem. Embedding formulae have been introduced initially by Williams, and then have been studied by Martin and Wickham [10] and by the group of Porter [3, 4, 5].

Formally, the current work does not contain the approximate formula for the quarter-plane diffraction coefficient. The reasons for submitting it are the following:
(i) We are sure that there are needed some more analytical steps before a reasonable formula will become available for this problem. The situation seems to be close to the one well known for matrix factorization where numerical attempts look questionable just because they do not reproduce the necessary analytical structure of the unknown function. We expect that the next step will be made within the Wiener-Hopf framework. The ideas of the coordinate equations treated as the Fuchsian equations in a 2D complex space could be very helpful for this. Actually, some excessive but not very successful studies in Wiener-Hopf generalizations for the 3D problems are in the background of the current work.
(ii) The results achieved seem to be quite important. The embedding formulae themselves should be a starting point of studying of any diffraction problem

with piecewise-straight boundaries. They reduce the problem to seeking the physically clear values, namely the directivities for the edge Green functions. The coordinate equations form the analytical method applicable to the conical problem that cannot be treated by traditional techniques. The examples are the problems of diffraction by a trihedral cone and by a set of flat complanar cones with a common vertex.
(iii) The analytical results obtained here lead to significant numerical advances for the Smyshlyaev's scheme. The modified Smyshlyaev's formulae extend the zone of fast convergence of the integral in the ν-plane, and the coordinate equations lead to very effective numerical algorithms for finding the edge Green functions.

Part I is organized as follows.

In Section 2 the scalar (Dirichlet and Neumann) problems of diffraction by a flat cone are formulated.

In Section 3 the *embedding formulae* are constructed. The embedding formula expresses the diffraction coefficient in terms of the edge Green's functions. We use the method developed in [6].

In Section 4 we introduce the spherical edge Green functions $v^{1,2}$ and $w^{1,2}$ and derive the modified Smyshlyaev's formulae.

In Section 5 we study the connections between the scalar Dirichlet / Neumann diffraction problem and the electromagnetic case. We use the Debye potentials technique for this.

In Section 6 we consider the numerical advantages of the new formulae being compared to the old ones.

The main results obtained in the paper are formulated as the Theorems.

2 Basic relations

2.1 Problem formulation

Instead of solving the electromagnetic diffraction problem from the very beginning we consider the scalar Dirichlet and Neumann diffraction problems. Later on (in Section 6) we shall write down the formulae based on the Debye potentials connecting electromagnetic case with the scalar one.

Let the Helmholtz equation

$$\Delta u + k_0^2 u = 0. \tag{3}$$

be valid in the 3D space (x, y, z). The time dependence of all variables has the form $e^{-\mathrm{i}\Omega t}$ and it is omitted henceforth.

The screen occupies the quarter-plane $z = 0, x > 0, y > 0$. The lines $(x > 0, y = 0, z = 0)$ and $(x = 0, y > 0, z = 0)$ will be named the *edges* of the scatterer and denoted by the symbols Λ_1 and Λ_2 (see Figure 1). The angle between the edges is equal to $\pi/2$.

We study two problems having the same geometry, but different boundary conditions, namely the Dirichlet and Neumann problems. Whenever it will be

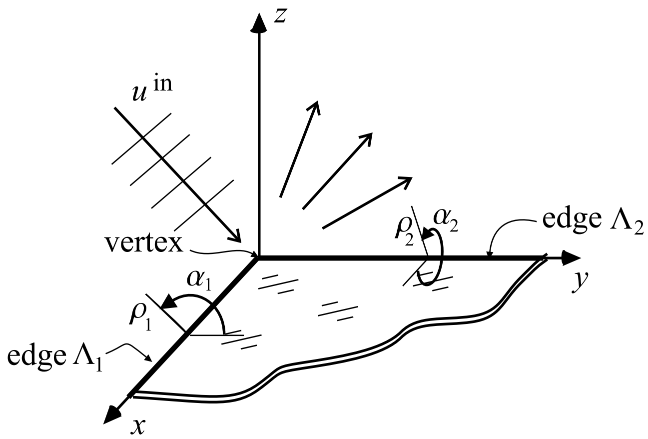

Figure 1: Geometry of the problem

necessary, we shall denote the values related to the Dirichlet problem by the index "D", and the values related to the Neumann problem by the index "N". The Dirichlet and Neumann boundary conditions fulfilled on both sides of the screen have the form:

$$u^D = 0, \qquad \frac{\partial}{\partial z} u^N = 0. \tag{4}$$

Let the incident field has the form of a plane wave

$$u^{\text{in}} = e^{-\mathrm{i}(k_x x + k_y y + k_z z)}, \tag{5}$$

where $k_x^2 + k_y^2 + k_z^2 = k_0^2$.

Besides the governing equation and boundary conditions, the radiation, edge and vertex conditions should be imposed to make a proper problem formulation. The radiation condition is not easy to formulate in this case, but its physical meaning is clear: there should not be components of the field coming from infinity except u^{in}. A usual procedure for the conical problems is to consider the point source problem instead of the plane wave incidence. However, here we shall avoid to discuss this matter.

The edge condition follows from the theory of diffraction by an ideal half-plane. The edge is source-free if the field near the edge behaves like

$$u^D \sim \rho_{1,2}^{1/2} \sin \frac{\alpha_{1,2}}{2}, \tag{6}$$

$$u^N \sim \text{const} + \rho_{1,2}^{1/2} \cos \frac{\alpha_{1,2}}{2}, \tag{7}$$

where ρ_j and α_j are the local cylindrical coordinates near the edge Λ_j.

The vertex conditions can be formulated in the form

$$u = O(1), \quad \nabla u = o(r^{-1/2}) \quad \text{as} \quad r \to 0, \tag{8}$$

where r is the distance from the vertex of the cone.

We shall assume below that the theorem of uniqueness is valid for the problem of diffraction by a cone, i.e. if a field satisfies the Helmhotz equation, Dirichlet boundary conditions, radiation, edge and vertex conditions, then it is identically equal to zero.

2.2 Diffraction coefficient

The total field u has a complicated structure. Roughly speaking, it consists of the incident field, reflected plane wave, two cylindrical waves radiated by the edges and the spherical wave radiated by the vertex. The boundary regions between the fields having different structure are called the penumbral zones; the field oscillates rapidly there.

Consider only the spherical component of the field. This component has the form

$$u^{D,N}(\omega, r) = 2\pi \frac{e^{\mathrm{i}k_0 r}}{k_0 r} f^{D,N}(\omega) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}). \tag{9}$$

Here r is the distance from the vertex; ω is the point on the unit sphere marking the direction of scattering, and $f^{D,N}(\omega)$ is the *diffraction coefficient*. We prefer to indicate explicitly the dependence of the diffraction coefficient on the direction ω_0 from which the incident plane wave is coming:

$$f^{D,N} = f^{D,N}(\omega; \omega_0).$$

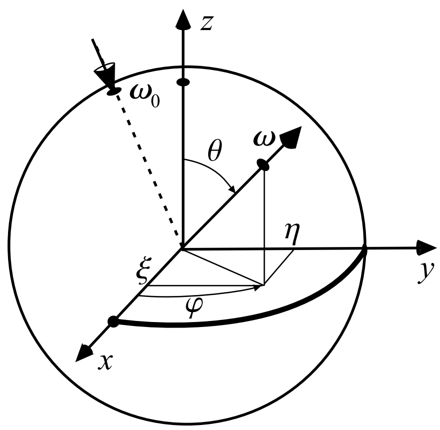

Figure 2: Coordinates on a sphere

Introduce the coordinates for the points ω and ω_0 of the unit sphere (see Figure 2). We shall use two different pairs of the coordinates. First, usual spherical coordinates (θ, φ) can be taken. The positive z-axis corresponds to the direction of $\theta = 0$, and the positive x-axis corresponds to $\theta = \pi/2$, $\varphi = 0$.

Besides, we shall use the "Cartesian" coordinates (ξ, η) for ω and ω_0 when it will be convenient. Namely, let ξ and η be the Cartesian coordinates of the projection of the point ω on the (x, y)-plane. Generally, there are two points on a sphere having the same set (ξ, η), so below we use these coordinates either when it is clear to what hemisphere the point belongs or this is not important. The Cartesian coordinates are connected with the spherical ones via the relations

$$\xi = \sin\theta\cos\varphi, \quad \eta = \sin\theta\sin\varphi.$$

The coordinates ξ_0 and η_0 correspond to the point ω_0. Note that using these coordinates the incident wave can be written in the form

$$u^{\rm in} = e^{-{\rm i}k_0(\xi_0 x + \eta_0 y + \sqrt{1-\xi_0^2-\eta_0^2}z)}.$$

The diffraction coefficients $f^{D,N}(\omega;\omega_0)$ are the main functions to be determined within the current research.

3 Embedding formulae

3.1 Edge Green's functions in the 3D space

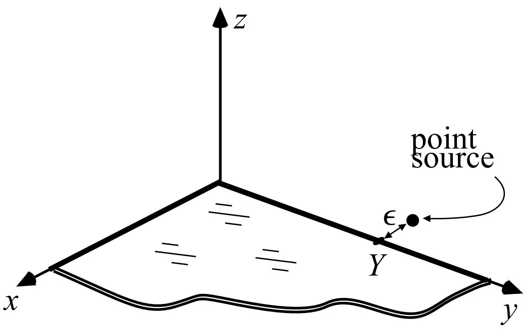

Figure 3: To the definition of the edge Green's function G_y

Let us introduce some definitions that will be used below. Consider the Dirichlet problem. Introduce the *edge Green's function* G_y^D for this problem having the prescribed oversingular (unphysical) asymptotics at the edge as follows. Consider the point source problem for the same flat cone. Let the source be located at the point near the edge Λ_2 having the coordinates $z = 0$, $x = -\epsilon$, $y = Y$, where ϵ tends to 0 (see Figure 3). To obtain the finite limit of the field, we should take the strength of the source depending on ϵ as $\sim \epsilon^{-1/2}$, namely we consider the inhomogeneous Helmholtz equation

$$\left(\Delta + k_0^2\right)\hat{G}_y^D(x, y, z; Y, \epsilon) = \sqrt{\frac{\pi}{\epsilon}}\,\delta(x + \epsilon, y - Y, z)$$

to formulate the point-source problem.

Solve this problem taking into account boundary, radiation, vertex, and edge conditions and take the limit

$$G_y^D(x,y,z;Y) = \lim_{\epsilon\to 0} \hat{G}_y^D(x,y,z;Y,\epsilon).$$

$G_y^D(x,y,z;Y)$ is one of the edge Green's functions for our problem. The index y indicates that the source is located at the y-axis.

Since the edge condition physically means the absence of the sources at the edge, and the function G_y^D, conversely, does possess the source at the edge, it should violate the edge condition. A detailed local study of the edge behaviour of G_y^D shows that the following property is valid. If the integral

$$I(x,y,z) = \int_0^\infty h(Y) G_y^D(x,y,z;Y)dY,$$

which is a convolution of a smooth enough density function h with G_y^D is constructed, then the edge asymptotics for I near the edge Λ_2 has the form

$$I(\rho_2,\alpha_2,y) = -\frac{h(y)}{\sqrt{\pi}}\rho_2^{-1/2}\sin\frac{\alpha_2}{2} + O(\rho_2^{1/2}\sin\frac{\alpha_2}{2})$$

in the local cylindrical coordinates.

Analogously, introduce another edge Green function $G_x^D(x,y,z;X)$ for the point source located near the edge Λ_1. By symmetry,

$$G_x^D(x,y,z;X) = G_y^D(y,x,z;X).$$

Introduce the directivities f_y^D and f_x^D of the edge Green's functions as the coefficients of the following asymptotic expansions:

$$G_y^D(\omega,r;Y) = 2\pi\frac{e^{\mathrm{i}k_0 r}}{k_0 r} f_y^D(\omega;Y) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}), \tag{10}$$

$$G_x^D(\omega,r;X) = 2\pi\frac{e^{\mathrm{i}k_0 r}}{k_0 r} f_x^D(\omega;X) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}). \tag{11}$$

Obviously,

$$f_x^D(\xi,\eta;X) = f_y^D(\eta,\xi;X). \tag{12}$$

Finally, define $C_G^D(x;Y)$ as the coefficient of the edge asymptotics of the edge Green function G_y^D (i.e. with the source at Λ_2), which is observed near the edge Λ_1:

$$G_y^D(\rho_1,\alpha_1,x;Y) = \frac{2C_G(x;Y)}{\sqrt{\pi}}\rho_1^{1/2}\sin\frac{\alpha_1}{2} + O(\rho_1^{3/2}).$$

Note that a similar coefficient can be defined by taking the source at the edge Λ_1 (i.e. by studying the edge Green function G_x^D) and the observation point at the edge Λ_2. However, due to the reciprocity principle the source and the observation point can be interchanged, so it will be the same coefficient. Moreover, due to symmetry

$$C_G(x;y) = C_G(y;x).$$

3.2 Derivation of the embedding formulae

Let us prove the following theorem for the Dirichlet problem.

Theorem 1 *The following integral representations (the so-called embedding formulae in 3D space) are valid for the diffraction coefficient $f^D(\omega, \omega_0)$:*

$$f^D(\omega, \omega_0) = \frac{4\pi^2 \mathrm{i}}{k_0^2(\xi + \xi_0)} \int_0^\infty f_y^D(\omega; Y)\, f_y^D(\omega_0; Y)\, dY, \tag{13}$$

$$f^D(\omega, \omega_0) = \frac{4\pi^2 \mathrm{i}}{k_0^2(\eta + \eta_0)} \int_0^\infty f_x^D(\omega; X)\, f_x^D(\omega_0; X)\, dX, \tag{14}$$

$$f^D(\omega, \omega_0) = \frac{4\pi^2}{k_0^3(\xi + \xi_0)(\eta + \eta_0)} \times$$
$$\iint_0^\infty \left[f_x^D(\omega; X)\, f_y^D(\omega_0; Y) + f_x^D(\omega_0; X)\, f_y^D(\omega; Y) \right] C_G(X; Y)\, dX\, dY. \tag{15}$$

Proof

To derive the embedding formula (13) we use the technique developed in [6]. Apply the operator

$$H_x = \frac{\partial}{\partial x} + \mathrm{i}k_x$$

to the total field $u^D(x, y, z)$. The function $H_x[u^D]$ has the following properties:
(i) it satisfies the Helmholtz equation, since the Laplacian is invariant with respect to the translations;
(ii) it satisfies the Dirichlet boundary conditions on the surface of the cone;
(iii) it satisfies the radiation conditions since $H_x[u^{\mathrm{in}}] \equiv 0$.

However, the function $H_x[u^D]$ does not satisfy the edge condition at Λ_2 and the vertex conditions, since the differentiation leads to higher exponents in the asymptotics. Let us "correct" the edge and vertex behaviour of the function.

Let the edge asymptotics of the plane-wave incidence solution u^D have the form:

$$u^D(\rho_1, \alpha_1, x) = \frac{2C_x(x)}{\sqrt{\pi}} \rho_1^{1/2} \sin\frac{\alpha_1}{2} + o(\rho_1^{1/2}) \tag{16}$$

for the edge Λ_1, and

$$u^D(\rho_2, \alpha_2, y) = \frac{2C_y(y)}{\sqrt{\pi}} \rho_2^{1/2} \sin\frac{\alpha_2}{2} + o(\rho_2^{1/2}) \tag{17}$$

for the edge Λ_2. Here $C_x(x)$ and $C_y(y)$ are unknown functions. Consider the combination

$$u^*(x, y, z) = H_x[u^D(x, y, z)] - \int_0^\infty C_y(Y)\, G_y^D(x, y, z; Y)\, dY \tag{18}$$

This function obeys the Helmholtz equation, boundary conditions, edge and radiation condition. Moreover, the detailed study of the vertex behaviour shows that this function satisfies the vertex condition. Due to the theorem of uniqueness we conclude that

$$u^* \equiv 0$$

and

$$H_x[u^D(x,y,z)] = \int_0^\infty C_y(Y)\, G_y^D(x,y,z;Y)\, dY. \tag{19}$$

This is the embedding formula in a *weak form.*

The coefficient C_y remains unknown in formula (19). To express it in terms of G perform the argument based on the reciprocity principle as follows. Let "1" be the point having the spherical coordinates (ω_0, R), and let "2" be the point having the Cartesian coordinates $(x = -\epsilon, y = Y, z = 0)$, i.e. "2" is the source location for $\hat{G}_y^D(x,y,x;Y,\epsilon)$. Let G_{12} be the value of the Green's function with the unit source originating at "1" and observation point located at "2". Analogously, let G_{21} be the Green's function of the same problem with the source located at "2" and the observation point located at "1". We assume that both Green's functions are found in consideration of the boundary, edge, vertex and radiation conditions formulated for the initial diffraction problem.

The source located far enough from the vertex (i.e., for $k_0 R \gg 1$) produces a wave whose structure near the vertex is asymptotically close to the incident plane wave. The amplitude of this wave wave at the vertex is approximately equal to $-e^{\mathrm{i}k_0 R}/(4\pi R)$. Using the definition of C_y one can write $G_{12}(\epsilon)$ in the form

$$G_{12} = -\frac{e^{\mathrm{i}k_0 R}\epsilon^{1/2}}{2R\pi^{3/2}} C_y(Y) + O(\epsilon^{3/2}) + O(e^{\mathrm{i}k_0 R}R^{-2}). \tag{20}$$

Due to the reciprocity theorem, the locations of the source and the observation point can be interchanged, thus giving $G_{12} = G_{21}$. The value G_{21} is naturally connected with the function $G_y^D(x,y,z;Y)$, namely

$$G_{21} = \frac{\epsilon^{1/2}}{\pi^{1/2}} \hat{G}_y(\omega_0, R; Y, \epsilon) + O(\epsilon^{3/2}).$$

Using (10), we find that

$$G_{21} = 2\frac{(\pi\epsilon)^{1/2} e^{\mathrm{i}k_0 R}}{k_0 R} f_y^D(\omega_0; Y) + O(\epsilon^{3/2}) + O(e^{\mathrm{i}k_0 R}R^{-2}). \tag{21}$$

Comparing (20) with (21) and taking the limits $\epsilon \to 0$, $R \to \infty$, we conclude that

$$C_y(Y) = -\frac{(2\pi)^2}{k_0} f_y^D(\omega_0; Y) \tag{22}$$

Substitute (22) into (19). Note that the operator H_x transforms the diffraction coefficient as follows:

$$f(\omega, \omega_0) \xrightarrow{H_x} \mathrm{i}k_0(\xi + \xi_0) f(\omega, \omega_0)$$

Finally, transform (19) into (13).

Similarly, the operator

$$H_y = \frac{\partial}{\partial y} + \mathrm{i}k_0\eta_0$$

can be used giving (14).

To derive (15) consider the operator

$$H_{xy} = \left(\frac{\partial}{\partial x} + \mathrm{i}k_0\xi_0\right)\left(\frac{\partial}{\partial y} + \mathrm{i}k_0\eta_0\right). \tag{23}$$

Apply this operator to the field u^D. Using the method introduced above and based on the theorem of uniqueness, one can obtain the following representation for f^D:

$$-k_0^2(\xi+\xi_0)(\eta+\eta_0)f^D(\omega,\omega_0) =$$

$$-\mathrm{i}4\pi^2\left(\xi_0\int\limits_0^\infty f_x^D(\omega;X)\,f_x^D(\omega_0;X)\,dX + \eta_0\int\limits_0^\infty f_y^D(\omega;Y)\,f_y^D(\omega_0;Y)\,dY\right) -$$

$$\frac{4\pi^2}{k_0}\left(\int\limits_0^\infty f_x^D(\omega;X)\,\frac{\partial}{\partial X}f_x^D(\omega_0;X)\,dX + \int\limits_0^\infty f_y^D(\omega;Y)\,\frac{\partial}{\partial Y}f_y^D(\omega_0;Y)\,dY\right) \tag{24}$$

Transform the second term of the r.h.s. Consider the function

$$\frac{\partial}{\partial Y}G_y^D(x,y,z;Y).$$

Add the expression $\partial G_y^D/\partial y$ to $\partial G_y^D/\partial Y$, and then compensate the singularities appearing at the edge Λ_1. As the result, obtain the formula:

$$\frac{\partial}{\partial Y}G_y^D(x,y,z;Y) = -\frac{\partial}{\partial y}G_y^D(x,y,z;Y) + \int\limits_0^\infty C_G(X;Y)\,G_x^D(x,y,z;X)\,dX.$$

This relation can be converted into the relation for the directivities:

$$\frac{\partial}{\partial Y}f_y^D(\omega_0;Y) = -\mathrm{i}k_0\eta_0 f_y^D(\omega_0;Y) + \int\limits_0^\infty C_G(X;Y)\,f_x^D(\omega_0;X)\,dX. \tag{25}$$

An analogous formula is valid for the derivative of f_x:

$$\frac{\partial}{\partial X}f_x^D(\omega_0;X) = -\mathrm{i}k_0\eta_0 f_x^D(\omega_0;X) + \int\limits_0^\infty C_G(Y;X)\,f_y^D(\omega_0;Y)\,dY. \tag{26}$$

Note that the coefficient C_G in (26) is the same as in (25) due to the symmetry and the reciprocity property. Finally, one can rewrite (24) in the form (15).
□

3.3 Embedding formulae for Neumann problem

Here we briefly show the results relating to the embedding formulae in the Neumann case. All calculations are analogous to the ones for the Dirichlet case.

Introduce the edge Green functions for the Neumann problem. The inhomogeneous equations for the approximations of these Green functions are as follows.

$$\left(\Delta + k_0^2\right) \hat{G}_x^N(x, y, z; X, \epsilon) = \frac{1}{2}\sqrt{\frac{\pi}{\epsilon}}\,[\delta(x-X, y-\epsilon, z-0) - \delta(x-X, y-\epsilon, z+0),$$

$$\left(\Delta + k_0^2\right) \hat{G}_y^N(x, y, z; Y, \epsilon) = \frac{1}{2}\sqrt{\frac{\pi}{\epsilon}}\,[\delta(x-\epsilon, y-Y, z+0) - \delta(x-\epsilon, y-Y, z-0).$$

These equations are solved taking into account the radiation, edge and vertex conditions and the limits

$$G_x^N(x, y, z; X) = \lim_{\epsilon\to 0} \hat{G}_x^N(x, y, z; X, \epsilon), \qquad G_y^N(x, y, z; Y) = \lim_{\epsilon\to 0} \hat{G}_y^N(x, y, z; Y, \epsilon). \tag{27}$$

We introduce the directivities of the edge Green functions analogously to (10) and (11):

$$G_{x,y}^N(\omega, r; X) = 2\pi \frac{e^{\mathrm{i}k_0 r}}{k_0 r} f_{x,y}^N(\omega; X) + O(e^{\mathrm{i}k_0 r}(k_0 r)^{-2}), \tag{28}$$

where X indicates the position of the source either on the edge Λ_1 or on the edge Λ_2.

We also define the coefficient E_G whose role is analogous to the coefficient C_G introduced for the Dirichlet case, i.e. we take the source at the edge Λ_2 and the observation point at the edge Λ_1:

$$G_y^N(\rho_1, \alpha_1, x; Y) = \frac{2E_G(x;Y)}{\sqrt{\pi}} \rho_1^{1/2} \cos\frac{\alpha_1}{2} + O(\rho_1^{3/2}).$$

Using the notations introduced here, we can formulate the following theorem:

Theorem 2 *The following integral representations are valid for the diffraction coefficient $f^N(\omega, \omega_0)$:*

$$f^N(\omega, \omega_0) = -\frac{4\pi^2\mathrm{i}}{k_0^2(\xi+\xi_0)} \int_0^\infty f_y^N(\omega; Y)\, f_y^N(\omega_0; Y)\, dY, \tag{29}$$

$$f^N(\omega, \omega_0) = -\frac{4\pi^2\mathrm{i}}{k_0^2(\eta+\eta_0)} \int_0^\infty f_x^N(\omega; X)\, f_x^N(\omega_0; X)\, dX, \tag{30}$$

$$f^N(\omega, \omega_0) = \frac{4\pi^2}{k_0^3(\xi+\xi_0)(\eta+\eta_0)} \times$$
$$\iint_0^\infty \left[f_x^N(\omega; X)\, f_y^N(\omega_0; Y) + f_x^N(\omega_0; X)\, f_y^N(\omega; Y)\right] E_G(X;Y)\, dX\, dY. \tag{31}$$

4 Modified Smyshlyaev's formulae

In this section we again consider in details the Dirichlet case, and then show briefly the analogous results related to the Neumann case.

4.1 Edge Green's functions on a sphere

Let us make some preliminary steps for the subsequent consideration. The embedding formulae (13), (14) and (15) will be used below to modify the representation (1). A natural way for this is to separate the radial variable and to study the spherical problem for each value of the separation constant. That is why we need an analog of the edge Green's function introduced for the problem on a sphere.

Define the Laplace-Beltrami operator in the spherical coordinates:

$$\tilde{\Delta} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

One can formulate an eigenvalue Dirichlet problem on a sphere as follows. Let the Laplace-Beltrami equation

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) v(\omega, \nu) = 0 \tag{32}$$

be valid on a sphere with the cut S corresponding to the cross-section of the flat cone (i.e. S is the line $\theta = \pi/2$, $0 < \varphi < \pi/2$). Let the Dirichlet boundary condition $v = 0$ be valid on S

The edge conditions are formulated for the *edges* L_1 and L_2 of the spherical problem, i.e. for the ends of S, which are the cross-sections of Λ_1 and Λ_2, respectively. Introduce the local spherical coordinates $\zeta_{1,2}$, $\phi_{1,2}$ near the edges as it is shown in Figure 4. One should demand that the solution near the edges grows no faster than $\zeta_{1,2}^{1/2}\sin(\phi_{1,2}/2)$.

The functions satisfying all these conditions exist for a discrete set of real values of ν (eigenvalues of the problem). These values form the *spectrum* of the problem and will be denoted by ν_j. Corresponding eigenfunctions will be denoted by $\Phi_j(\omega)$. Consider only the eigenfunctions having a specific symmetry with respect to the line $\theta = \pi/2$, namely

$$\Phi_j(\theta, \varphi) = \Phi_j(\pi - \theta, \varphi) \tag{33}$$

and corresponding eigenvalues. One can show that all $\nu_j > 1/2$ (see [12]).

We assume that the eigenfunctions are normalized as follows:

$$\iint \Phi_m(\omega)\Phi_n(\omega)\, d\omega = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

Here the integration is performed over the whole sphere.

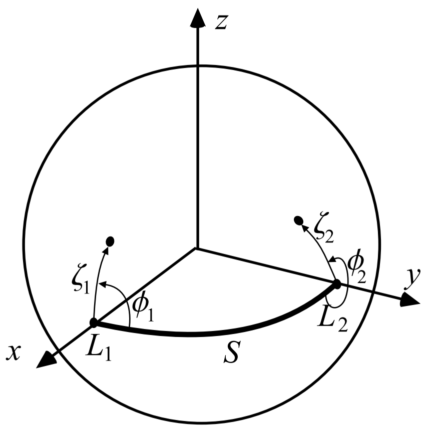

Figure 4: Local spherical coordinates near the edges

Define the *spherical edge Green's function* $v^1(\omega, \nu)$ for values of ν not belonging to the spectrum as follows. First, define the function $\hat{v}^1(\omega, \nu; \kappa)$ as the solution of the spherical problem with a point source located at the point $\omega_\kappa = (\theta = \pi/2, \varphi = -\kappa)$ close to the edge L_1 i.e. of the inhomogeneous Laplace-Beltrami equation

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) \hat{v}^1(\omega, \nu, \kappa) = \frac{\pi^{1/2}}{\kappa^{3/2}} \delta(\zeta_1 - \kappa, \phi_1 - \pi), \tag{34}$$

boundary condition at the cut and the edge conditions. The location of the source (i.e., the point ω_κ) is shown in Figure 5. Take the limit

$$v^1(\omega, \nu) = \lim_{\kappa \to 0} \hat{v}^1(\omega, \nu, \kappa).$$

Function v^1 can be represented through the eigenfunctions of the spherical problem. Namely, let the edge asymptotics of the eigenfunction Φ_j near L_1 and L_2, respectively, have the form:

$$\Phi_j(\zeta_1, \phi_1) = \frac{2C_j}{\sqrt{\pi}} \zeta_1^{1/2} \sin\frac{\phi_1}{2} + O(\zeta_1^{3/2}), \tag{35}$$

$$\Phi_j(\zeta_2, \phi_2) = \frac{2\tilde{C}_j}{\sqrt{\pi}} \zeta_2^{1/2} \sin\frac{\phi_2}{2} + O(\zeta_2^{3/2}). \tag{36}$$

Here C_j and $\tilde{C}_j$ are some constants.

Using orthogonality and completeness of the eigenfunctions, one can obtain the relation

$$v^1(\omega, \nu) = 2 \sum_j \frac{C_j \Phi_j(\omega)}{\nu^2 - \nu_j^2}. \tag{37}$$

Analogously, taking the source at the point $\tilde{\omega}_\kappa$ with the coordinates $\theta = \pi/2$, $\varphi = \pi/2 + \kappa$, one can introduce the edge Green's function $v^2(\omega, \nu)$. Due

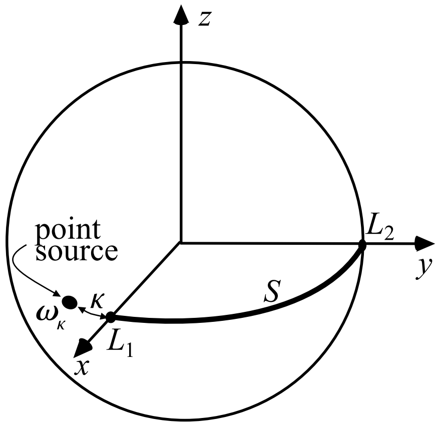

Figure 5: To the definition of edge Green's function v^1

to symmetry, $v^2(\theta, \varphi, \nu) = v^1(\theta, \pi/2 - \varphi, \nu)$. Using the asymptotics (36) this function can be written in the form

$$v^2(\omega, \nu) = 2 \sum_j \frac{\tilde{C}_j \Phi_j(\omega)}{\nu^2 - \nu_j^2}. \tag{38}$$

Introduce also the coefficient $C_2^1(\nu)$ describing the asymptotics of v^1 near the edge L_2:

$$v^1(\zeta_2, \phi_2, \nu) = \frac{2C_2^1(\nu)}{\sqrt{\pi}} \zeta_2^{1/2} \sin \frac{\phi_2}{2} + O(\zeta_2^{3/2}).$$

Note that

$$C_2^1(\nu) = \lim_{\kappa \to 0} \frac{\sqrt{\pi}}{2\sqrt{\kappa}} v^1(\theta = \pi/2, \varphi = \pi/2 + \kappa) = 2 \sum_j \frac{C_j \tilde{C}_j}{\nu^2 - \nu_j^2}.$$

4.2 Modified Smyshlyaev's formulae for the Dirichlet case

Let us prove the following theorem.

Theorem 3 *The following formulae are valid:*

$$f^D(\omega, \omega_0) = \frac{1}{4\pi \mathrm{i}(\eta + \eta_0)} \int_\gamma e^{-\mathrm{i}\pi\nu} [v^1(\omega_0, \nu) v^1(\omega, \nu+1) + v^1(\omega, \nu) v^1(\omega_0, \nu+1)]\, d\nu, \tag{39}$$

$$f^D(\omega, \omega_0) = \frac{1}{4\pi \mathrm{i}(\xi + \xi_0)} \int_\gamma e^{-\mathrm{i}\pi\nu} [v^2(\omega_0, \nu) v^2(\omega, \nu+1) + v^2(\omega, \nu) v^2(\omega_0, \nu+1)]\, d\nu, \tag{40}$$

$$f^D(\omega, \omega_0) = \frac{\mathrm{i}}{8\pi(\xi + \xi_0)(\eta + \eta_0)} \int_\Gamma \frac{e^{-\mathrm{i}\pi\nu}}{\nu} C_2^1(\nu) [B(\omega, \omega_0, \nu) + B(\omega_0, \omega, \nu)]\, d\nu, \tag{41}$$

where

$$B(\omega, \omega_0, \nu) = (v^1(\omega, \nu+1) - v^1(\omega, \nu-1))(v^2(\omega_0, \nu+1) - v^2(\omega_0, \nu-1)),$$

γ and Γ are the contours of integration shown in the Figure 6 and Figure 7. Contour Γ consists of the infinite loop and two small loops.

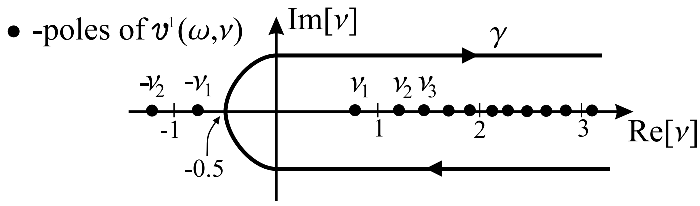

Figure 6: Contour of integration for formula (39)

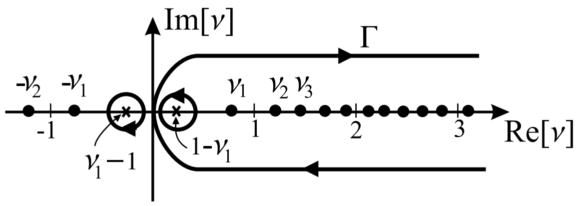

Figure 7: Contour of integration for formula (41)

Proof

Use the procedure similar to the one described in [15], [2]. Rewrite the Laplacian in the form admitting the separation of variables:

$$\Delta = r^{-2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + r^{-2} \tilde{\Delta}.$$

Rewrite the function $G_x^D(x, y, z; X)$ in the spherical coordinates, i.e. as $G_x^D(\omega, r; X)$. Construct the representation of this function through the eigenfunctions Φ_j of the spherical problem. Return to the definition of the edge Green's function, namely to the limiting procedure described above. Consider the point source located at the distance ϵ from the edge and having the strength equal to $(\pi/\epsilon)^{1/2}$. Following [2], write $\hat{G}_x^D(\omega, r; X, \epsilon)$ in the form:

$$\hat{G}_x^D(\omega, r; X, \epsilon) = \sqrt{\frac{\pi}{\epsilon}} \sum_j g_j(r, X) \Phi_j(\omega) \Phi_j(\omega_\kappa), \qquad (42)$$

where κ is connected with ϵ via the relation

$$\kappa = \frac{\epsilon}{X},$$

and $g_j(r,X)$ is the solution of the differential equation

$$\left[\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+k_0^2r^2-\nu_j^2+\frac{1}{4}\right]g_j(r,X)=\delta(r-X). \tag{43}$$

The solution of (43) obeying the radiation and vertex conditions is as follows [2]:

$$g_j(r,X)=\frac{\pi}{2\mathrm{i}}(rX)^{-1/2}J_{\nu_j}(k_0r_<)H^{(1)}_{\nu_j}(k_0r_>), \tag{44}$$

where $r_<\equiv\min\{r,X\}$, $r_>\equiv\max\{r,X\}$; J and H are the Bessel and Hankel functions.

Due to symmetry, the summation in (42) is performed only over the symmetrical eigenfunction, i.e. over those Φ_j for which $\Phi_j(\theta,\varphi)=\Phi_j(\pi-\theta,\varphi)$.

Using (35), take the limit of (42) as $\kappa\to 0$:

$$G_x(\omega,r;X)=\frac{\pi}{\mathrm{i}\sqrt{r}X}\sum_j C_j\Phi_j(\omega)J_{\nu_j}(k_0r_<)H^{(1)}_{\nu_j}(k_0r_>). \tag{45}$$

Construct the directivity $f_x(\omega)$. Take $r\gg X$ and use the asymptotics [1]

$$H^{(1)}_{\nu_j}(k_0r)=\sqrt{\frac{2}{\pi k_0r}}\exp\left\{\mathrm{i}k_0r-\mathrm{i}\frac{\pi}{2}\nu-\mathrm{i}\frac{\pi}{4}\right\}+O(e^{\mathrm{i}k_0r}(k_0r)^{-3/2}).$$

As the result, obtain the representation

$$f_x(\omega;X)=\sqrt{\frac{k_0}{2\pi}}\frac{e^{-\mathrm{i}3\pi/4}}{X}\sum_j C_j\Phi_j(\omega)J_{\nu_j}(k_0X)e^{-\mathrm{i}\pi\nu_j/2}. \tag{46}$$

Substitute (46) into (14). The result is the following:

$$f(\omega,\omega_0)=-\frac{2\pi}{k_0(\eta+\eta_0)}\sum_{m,n}C_mC_n\Phi_m(\omega)\Phi_n(\omega_0)I_{m,n}e^{-\mathrm{i}\pi(\nu_m+\nu_n)/2}, \tag{47}$$

where

$$I_{m,n}=\int_0^\infty\frac{J_{\nu_m}(k_0X)J_{\nu_n}(k_0X)}{X^2}\,dX.$$

Note that the integral is convergent for all m and n. This integral can be calculated using the standard formulae [1]:

$$I_{m,n}=-\frac{4k_0}{\pi}\frac{\cos\frac{\pi}{2}(\nu_m-\nu_n)}{((\nu_m+1)^2-\nu_n^2)((\nu_m-1)^2-\nu_n^2)}. \tag{48}$$

Finally, obtain

$$f(\omega,\omega_0)=\frac{4}{\eta+\eta_0}\sum_{m,n}\frac{C_mC_n\Phi_m(\omega)\Phi_n(\omega_0)}{((\nu_m+1)^2-\nu_n^2)((\nu_m-1)^2-\nu_n^2)}(e^{-\mathrm{i}\pi\nu_m}+e^{-\mathrm{i}\pi\nu_n}). \tag{49}$$

By the direct check based on the residue calculation, and taking into account (37), the identity (39) can be established. Analogously, formula (40) can be obtained from (13).

Finally, consider formula (15). Using the same technique and performing rather tedious calculations, we obtain the representation (41). The key relation for this formula is the following:

$$\iint_0^\infty J_\mu(t) J_\lambda(t) J_\beta(\min(t,z)) H_\beta^{(1)}(\max(t,z)) \frac{dt\,dz}{tz} =$$

$$= -\frac{2}{\pi^2}\left[\frac{e^{i\pi(\lambda+\mu-2\beta)}/2}{(\mu^2-\beta^2)(\lambda^2-\beta^2)} + \frac{e^{i\pi(\mu-\lambda)}/2}{(\lambda^2-\mu^2)(\lambda^2-\beta^2)} + \frac{e^{i\pi(\lambda-\mu)}/2}{(\mu^2-\beta^2)(\mu^2-\lambda^2)}\right].$$

The contour Γ consists of a big loop and two small ones encircling the pole $1-\nu_1$ in the positive direction and ν_1-1 in the negative direction (see Figure 7). Here we use the empirical fact that ν_1 is the only eigenvalue of the problem satisfying $1/2 < \nu < 1$.

□

Note that the embedding formulae formula (39), (40) and (41) have the form similar to (1), i.e. (12) of [2]. However, the integrands in (39), (40) and (41) are not equal to the integrand in (12) of [2]. This can be shown by studying the singularities of both integrands.

4.3 Modified Smyshlyaev's formulae for the Neumann case

Consider the Laplace-Beltrami problem on the sphere with the cut S on which the Neumann boundary conditions are given.

Define the spherical edge Green functions $w^{1,2}(\omega,\nu)$ for this problem. For this, formulate the inhomogeneous Laplace-Beltrami equations for the approximations $\hat{w}^{1,2}(\omega,\nu,\kappa)$:

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right)\hat{w}^1(\omega,\nu,\kappa) = \frac{\pi^{1/2}}{2\kappa^{3/2}}\,\delta(\zeta_1-\kappa)[\delta(\phi_1-0)-\delta(\phi_1+0)], \qquad (50)$$

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right)\hat{w}^2(\omega,\nu,\kappa) = \frac{\pi^{1/2}}{2\kappa^{3/2}}\,\delta(\zeta_2-\kappa)[\delta(\phi_2-0)-\delta(\phi_2+0)]. \qquad (51)$$

Solve these equations taking into account boundary and edge conditions and take the limits:

$$w^{1,2}(\omega,\nu) = \lim_{\kappa\to 0}\hat{w}^{1,2}(\omega,\nu,\kappa) \qquad (52)$$

Define the coefficient $E_2^1(\nu)$ as the factor at the main term of the asymptotics of w^1 observed near the edge L_2, i.e. using the asymptotics

$$w^1(\zeta_2,\phi_2,\nu) = \frac{2E_2^1(\nu)}{\sqrt{\pi}}\zeta_2^{1/2}\cos\frac{\phi_2}{2} + O(\zeta_2^{3/2}). \qquad (53)$$

Formulate the following theorem containing the modified Smyshlyaev's formulae for the Neumann case:

Theorem 4 *The following formulae are valid:*

$$f^N(\omega,\omega_0) = -\frac{1}{4\pi\mathrm{i}(\eta+\eta_0)}\times$$
$$\int_\gamma e^{-\mathrm{i}\pi\nu}[w^1(\omega_0,\nu)w^1(\omega,\nu+1)+w^1(\omega,\nu)w^1(\omega_0,\nu+1)]\,d\nu, \tag{54}$$
$$f^N(\omega,\omega_0) = -\frac{1}{4\pi\mathrm{i}(\xi+\xi_0)}\times$$
$$\int_\gamma e^{-\mathrm{i}\pi\nu}[w^2(\omega_0,\nu)w^2(\omega,\nu+1)+w^2(\omega,\nu)w^2(\omega_0,\nu+1)]\,d\nu, \tag{55}$$
$$f^N(\omega,\omega_0) = \frac{\mathrm{i}}{8\pi(\xi+\xi_0)(\eta+\eta_0)}\int_\Gamma \frac{e^{-\mathrm{i}\pi\nu}}{\nu}E_2^1(\nu)[D(\omega,\omega_0,\nu)+D(\omega_0,\omega,\nu)]\,d\nu, \tag{56}$$

where

$$D(\omega,\omega_0,\nu) = (w^1(\omega,\nu+1)-w^1(\omega,\nu-1))(w^2(\omega_0,\nu+1)-w^2(\omega_0,\nu-1)),$$

γ and Γ are the contours of integration shown in the Figure 6 and Figure 7, where now ν_1 is the first eigenvalue of the Neumann problem related to the eigenfunction having the symmetrical property $\Phi(\theta,\varphi)=-\Phi(\pi-\theta,\varphi)$.

The proof is analogous to the one proposed for the Dirichlet case.

5 Debye potentials and electromagnetic case

Now we are going to establish the link between the model scalar (acoustic) problem studied above and the actual electromagnetic diffraction problem. For this we are going to use the results of applying of the Debye potentials formalism.

The contents of this section is mainly taken from the work [16]. Author is grateful to Professor V.P.Smyshlyaev who consulted the author regarding the Debye potentials and helped to apply the formulae obtained in [16] to the problem under consideration.

We consider a problem of diffraction of a plane electromagnetic wave see Figure 8. A plane electromagnetic wave is approaching the scatterer from the direction ω_0. The polarization of the wave is indicated by two unit vectors e and h orthogonal to each other and to ω_0:

$$\left\{\begin{array}{c}\mathrm{E}^{\mathrm{in}}\\ \mathrm{H}^{\mathrm{in}}\end{array}\right\} = \left\{\begin{array}{c}\mathrm{e}\\ \mathrm{h}\end{array}\right\}\exp\{-\mathrm{i}k_0(z\cos\theta_0+x\cos\theta_0\cos\varphi_0+y\cos\theta_0\sin\phi_0)\}. \tag{57}$$

The formula for the diffracted electromagnetic wave has the form ((6) from [16]):

$$\left\{\begin{array}{c}\mathrm{E}\\ \mathrm{H}\end{array}\right\} = -2\pi\frac{e^{\mathrm{i}k_0r}}{k_0r}\left[\nabla_\omega\left\{\begin{array}{c}F_D(\omega,\omega_0)\\ F_N(\omega,\omega_0)\end{array}\right\}\mp\nabla_\omega\left\{\begin{array}{c}F_N(\omega,\omega_0)\\ F_D(\omega,\omega_0)\end{array}\right\}\times\omega\right]+O(e^{\mathrm{i}k_0r}(k_0r)^{-2}), \tag{58}$$

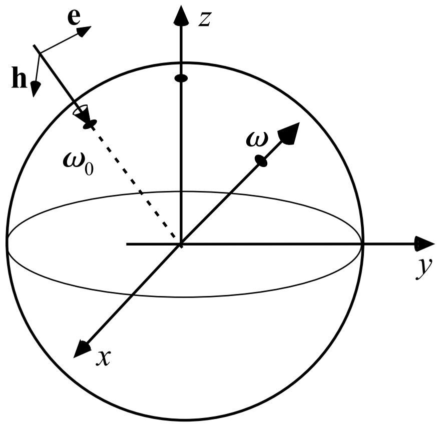

Figure 8: Incident wave in electromagnetic case

where ω is the direction of scattering, ∇_ω stands for the gradient operator with respect to the spherical variable, and the functions $F_{D,N}$ are defined below.

According to the definition (11) of [16],

$$F_{D,N}(\omega,\omega_0) = \frac{\mathrm{i}}{\pi}\int_\gamma \frac{e^{-\mathrm{i}\nu\pi}}{\nu^2-1/4} g_{D,N}(\omega,\omega_0,\nu)\,\nu\,d\nu. \tag{59}$$

Here $g_{D,N}$ are some special scalar Green's function on the sphere with the cuts bearing Dirichlet or Neumann boundary conditions. These Green's functions are defined by the inhomogeneous Laplace-Beltrami equations:

$$\left(\tilde{\Delta}_\omega + \nu^2 - \frac{1}{4}\right)\left\{\begin{array}{c} g_D(\omega,\omega_0,\nu) \\ g_N(\omega,\omega_0,\nu)\end{array}\right\} = \left\{\begin{array}{c} (\mathrm{h}\cdot\nabla_{\omega_0}) \\ (\mathrm{e}\cdot\nabla_{\omega_0})\end{array}\right\}\delta(\omega-\omega_0). \tag{60}$$

Compare this representation with (7) from [17] obtained for the Dirichlet / Neumann diffraction coefficient studied above:

$$f^{D,N}(\omega,\omega_0) = \frac{\mathrm{i}}{\pi}\int_\gamma e^{-\mathrm{i}\nu\pi} g^{D,N}(\omega,\omega_0,\nu)\,\nu\,d\nu, \tag{61}$$

where $g^{D,N}$ is the spherical Green's function of the Dirichlet / Neumann problem satisfying the inhomogeneous equation

$$\left(\tilde{\Delta}_\omega + \nu^2 - \frac{1}{4}\right) g^{D,N}(\omega,\omega_0,\nu) = \delta(\omega-\omega_0), \tag{62}$$

Comparing the definitions of $f^{D,N}$ and $F_{D,N}$ we conclude that the following identities are valid:

$$\tilde{\Delta}_\omega F_D = -(\mathrm{h}\cdot\nabla_{\omega_0}) f^D, \tag{63}$$
$$\tilde{\Delta}_\omega F_N = -(\mathrm{e}\cdot\nabla_{\omega_0}) f^N. \tag{64}$$

Consider the embedding formula (41). Obviously, it can be rewritten for the functions $F_{D,N}$ as follows:

$$F_D(\omega,\omega_0) = \frac{\mathrm{i}}{8\pi(\xi+\xi_0)(\eta+\eta_0)}\times$$

$$\int_\Gamma \frac{e^{-\mathrm{i}\pi\nu}}{\nu} C^1_2(\nu)(\mathrm{h}\cdot\nabla_{\omega_0})[B'(\omega,\omega_0,\nu)+B''(\omega,\omega_0,\nu)]\,d\nu, \qquad (65)$$

$$F_N(\omega,\omega_0) = \times$$

$$\frac{\mathrm{i}}{8\pi(\xi+\xi_0)(\eta+\eta_0)} \int_\Gamma \frac{e^{-\mathrm{i}\pi\nu}}{\nu} E^1_2(\nu)(\mathrm{e}\cdot\nabla_{\omega_0})[D'(\omega,\omega_0,\nu)+D''(\omega,\omega_0,\nu)]\,d\nu, \qquad (66)$$

where

$$B'(\omega,\omega_0,\nu) = \left(\frac{v^1(\omega,\nu+1)}{(\nu+1)^1-1/4} - \frac{v^1(\omega,\nu-1)}{(\nu-1)^1-1/4}\right)(v^2(\omega_0,\nu+1)-v^2(\omega_0,\nu-1)),$$

$$B''(\omega,\omega_0,\nu) = \left(\frac{v^2(\omega,\nu+1)}{(\nu+1)^1-1/4} - \frac{v^2(\omega,\nu-1)}{(\nu-1)^1-1/4}\right)(v^1(\omega_0,\nu+1)-v^1(\omega_0,\nu-1)),$$

$$D'(\omega,\omega_0,\nu) = \left(\frac{w^1(\omega,\nu+1)}{(\nu+1)^1-1/4} - \frac{w^1(\omega,\nu-1)}{(\nu-1)^1-1/4}\right)(w^2(\omega_0,\nu+1)-w^2(\omega_0,\nu-1)),$$

$$D''(\omega,\omega_0,\nu) = \left(\frac{w^2(\omega,\nu+1)}{(\nu+1)^1-1/4} - \frac{w^2(\omega,\nu-1)}{(\nu-1)^1-1/4}\right)(w^1(\omega_0,\nu+1)-w^1(\omega_0,\nu-1)).$$

Thus, the values standing in the formula (57) for the electromagnetic diffraction coefficients are expressed through the spherical Green functions $v^{1,2}$ and $w^{1,2}$.

6 Discussion of the properties of the modified Smyshlyaev's formulae

Compare the properties of the new formulae (39), (40), (41) with that of (1). There are two features making the embedding formulae preferable.
(i) The functions $v^{1,2}(\omega,\nu)$ depend on three scalar variables, while $g(\omega,\omega_0,\nu)$ depend on five ones. Therefore functions $v^{1,2}$ require less computational efforts, if their tabulation if necessary.
(ii) The integrals (39), (40) and (41) are better than (1) from the point of view of convergence.

The last statement should be explained. The integral (1) is convergent only in the sense of distributions. However, the authors of [2] discuss the possibility of transforming the contour of integration in (1) in such a way that the integrand decays exponentially along the contour. Obviously, the exponentially decaying integrals are much more attractive for the numerical analysis comparatively to the divergent ones.

In application to our case, the main result of [2] is that such a transformation is available only for the points ω and ω_0 satisfying the inequalities

$$\arccos \xi + \arccos \xi_0 > \pi \tag{67}$$

and

$$\arccos \eta + \arccos \eta_0 > \pi. \tag{68}$$

Let us study the growth of the integrands of (39), (40) and (41). Following [2] and [7], we obtain the estimations:

$$\begin{aligned} v^1(\omega, \nu) &\sim \exp\{-|\mathrm{Im}\,\nu| \arccos \xi\} |\nu|^{-1/2}, \\ v^2(\omega, \nu) &\sim \exp\{-|\mathrm{Im}\,\nu| \arccos \eta\} |\nu|^{-1/2}, \\ C_2^1(\nu) &\sim \exp\{-|\mathrm{Im}\,\nu| \pi/2\} \end{aligned}$$

The integrand of (39) can be estimated as

$$|\nu|^{-1} \exp\{-\mathrm{i}\pi\nu - |\mathrm{Im}\,\nu|(\arccos \xi + \arccos \xi_0)\}.$$

It means that the integral (39) is convergent for almost all ω, ω_0, and if the inequality (67) is valid, then this integral can be converted into the exponentially convergent one by using the contour γ' instead of γ as it is shown in Figure 9a. Note that the inequality (68) is not necessary in this case.

Similarly, the formula (40) can be considered.

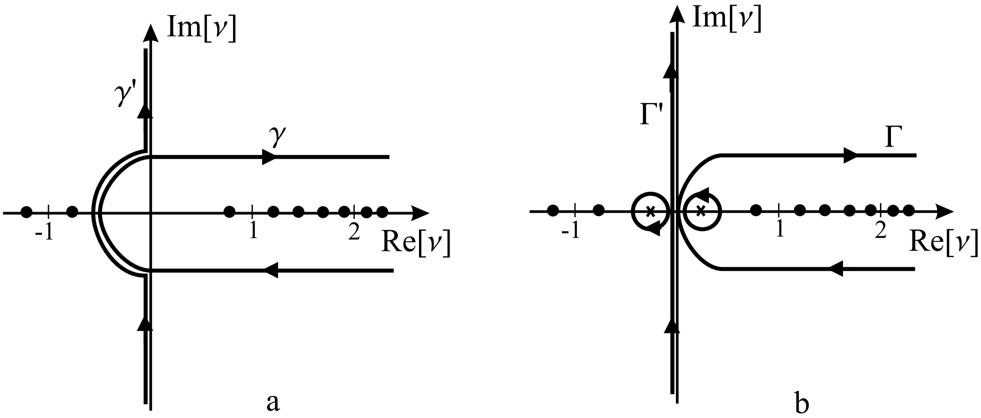

Figure 9: Deformation of the contours of integration

The integrand of (41) can be estimated as

$$|\nu|^{-2} \exp\{-\mathrm{i}\pi\nu\} \Big(\exp\{-|\mathrm{Im}\,\nu|(\arccos \xi + \arccos \eta_0 + \pi/2)\} +$$

$$\exp\{-|\mathrm{Im}\,\nu|(\arccos \xi_0 + \arccos \eta + \pi/2)\} \Big).$$

The integral is convergent for almost all ω and ω_0 (when the exponential factor oscillates). If

$$\arccos \xi + \arccos \eta_0 > \pi/2 \tag{69}$$

and

$$\arccos \xi_0 + \arccos \eta > \pi/2 \tag{70}$$

the open loop of the contour of integration Γ can be deformed into the contour Γ' shown in Fig. 9b, along which the convergence is exponential.

Consider the inequalities (67), (68), (69) and (70). When they stop to be valid it is reasonable to expect the singularities of the diffraction coefficient. All singularities of the diffraction coefficient can be found by simple physical consideration. The cylindrical wave diffracted by the edges of the quarter plane are due to the simple poles of f. These poles are located at the lines $\xi + \xi_0 = 0$ and $\eta + \eta_0 = 0$. The intersection of these lines correspond to the reflected plane wave. Beside these sets there can be "secondary" singularities corresponding to the rays diffracted first by one edge and then by another one (see Fig. 10). These rays result into the branch lines corresponding to the sets $\eta = \sqrt{1 - \xi_0^2}$ and $\xi = \sqrt{1 - \eta_0^2}$. The first singularity appears only if $\xi_0 > 0$, and the second one appears only if $\eta_0 > 0$.

According to this consideration, one can see that the embedding formulae enable one to "extract" the singularities (poles) out of the integral into the rational factor. The formulae (39) and (40) extract only one pole each, and the formula (41) extracts both poles. Only the secondary singularities remain in the integral term.

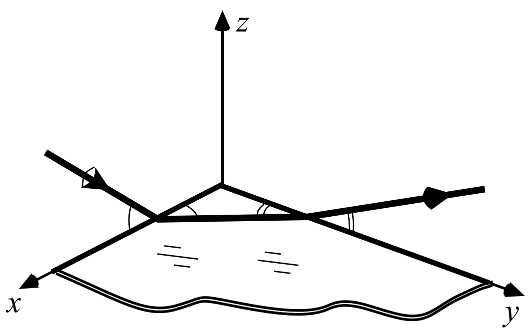

Figure 10: Rays corresponding to the secondary singularities

Physically, this can be explained as follows. The operator H_{xy} gives zero when it acts on the incident plane wave, reflected plane wave, and both scattered cylindrical waves.

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