New analytical results and numerical algorithms for quarter-plane diffraction. Part II: Coordinate equations for the edge Green's functions on the sphere

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Abstract

Here we describe a method for finding the edge Green's functions defined in Part I for the Laplace-Beltrami problem on a sphere with a Dirichlet / Neumann cut. This method can be considered as a generalization of the separation of variables, since it reduces the partial differential equation to a system of ordinary differential equation in some sense. We demonstrate also the numerical application of the new method.

1 Introduction

The edge Green's functions $v^{1,2}$ and $w^{1,2}$ are necessary to be determined for applying the modified Smyshlyaev's formulae (39)–(41) and (54)–(56) of Part I. We remind here how these functions are introduced. The Laplace-Beltrami equation is defined on the unit sphere with usual spherical coordinates (θ, φ). The equation is fulfilled everywhere on the sphere except the cut S, which is the line $\theta = \pi/2$, $0 < \varphi < \pi/2$, on which the Dirichlet / Neumann boundary conditions are fulfilled. The functions $v^{1,2}$ correspond to the Dirichlet problem, and the functions $w^{1,2}$ correspond to the Neumann problem.

The points L_1 and L_2 are the ends of the cut and they are called the edges of the problem. The edge Green's functions are constructed as the solutions of the point source problems with the sources located near the edges. Since we cannot simple place the source at the edge, we introduce a limiting procedure as follows. For each small finite κ we find the approximations $\hat{v}^{1,2}$ and $\hat{w}^{1,2}$ for $v^{1,2}$ and $w^{1,2}$, respectively, by means of the inhomogeneous equations:

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) \hat{v}^{1,2}(\omega, \nu, \kappa) = \frac{\pi^{1/2}}{\kappa^{3/2}} \delta(\zeta_{1,2} - \kappa, \phi_{1,2} - \pi), \tag{1}$$

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right) \hat{w}^{1,2}(\omega, \nu, \kappa) = \frac{\pi^{1/2}}{2\kappa^{3/2}} \delta(\zeta_{1,2} - \kappa)[\delta(\phi_{1,2} - 0) - \delta(\phi_{1,2} + 0)], \tag{2}$$

where

$$\tilde{\Delta} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2}.$$

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θ and φ are the usual spherical coordinates; ω is the notation for a point on the sphere; $\zeta_{1,2}$ and $\phi_{1,2}$ are the local spherical coordinates defined as it is shown in Figure 1

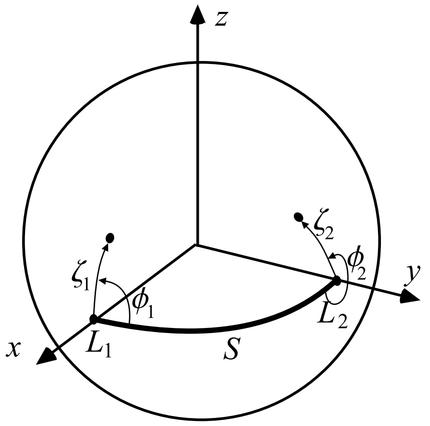

Figure 1: Local spherical coordinates near the edges

When solving the equations (1) and (2) we take into account the boundary conditions on S (the Dirichlet ones for $\hat{v}^{1,2}$ and the Neumann ones for $\hat{w}^{1,2}$) and the edge conditions. We also remind that the parameter ν should neither belong to the spectrum of the Dirichlet or the Neumann problem. The dependence of the strength of the sources on κ is chosen in such a way that there exist non-zero finite limits

$$v^{1,2}(\omega,\nu) = \lim_{\kappa\to 0} \hat{v}^{1,2}(\omega,\nu,\kappa), \qquad w^{1,2}(\omega,\nu) = \lim_{\kappa\to 0} \hat{w}^{1,2}(\omega,\nu,\kappa). \tag{3}$$

In other words, $v^{1,2}$ and $w^{1,2}$ are the limits of the "usual" Green's function $g^{D,N}(\omega,\omega_0,\nu)$ with the appropriate choice of the position and the amplitude of the sources.

There are two known ways for finding the edge Green's functions. Both of them are valid for the usual Green's functions. First, we can solve the boundary integral equation; and second, we can construct the representation for the edge Green's functions as the Fourier series of the eigenfunctions. Here we propose a completely new, third way for finding these functions. Within our technique, the Laplace-Beltrami equation is reduced to the *coordinate equations* system, which is a generalization of Fuchsian ordinary differential equation to the case of two variables.

Part II is organized as follows.

In Section 2 we demonstrate the main idea of generalization of the separation of variables method leading to the coordinate equations.

In Section 3 the coordinate equations are derived for the functions $v^{1,2}$, $w^{1,2}$ using the trick of oversingular combinations. As the result, the edge Green's functions become expressed as solutions for a set of ordinary differential equations of order 4. The coefficients of the coordinate equations contain 2 numerical parameters, which should be found numerically by using the boundary conditions imposed on the solutions.

In Section 4 we demonstrate the numerical results related to our theory. We concentrate on calculation of the edge Green's functions. A gradient method is applied to finding the unknown numerical of the coefficients of the coordinate equations. As we show, the method converges rapidly when the separation constant is not close to the resonant values.

In Appendix A we write down the explicit form of the coefficients of the coordinate equations for a right-angle flat cone.

2 The idea of the coordinate equations

Here we follow [2] to introduce the idea of the method. Consider the classical separation of variables for the sphere without cuts as an illustration. The problem can be solved using the spherical coordinates (θ, φ). Each eigenfunction can be represented as the product $u = M_1(\theta) M_2(\varphi)$. The functions M_1 and M_2 obey the equations

$$M_1'' - a(\theta) M_1' - b(\theta) M_1 = 0, \qquad M_2'' + \mu^2 M_2 = 0, \tag{4}$$

where μ is an integer, and

$$a(\theta) = -\frac{\cos\theta}{\sin\theta}, \qquad b(\theta) = \frac{1}{4} - \nu^2 + \frac{\mu^2}{\sin^2\theta}.$$

These equations can be rewritten in the form

$$\frac{\partial}{\partial\theta}\mathbf{U} = \mathbf{X}\mathbf{U}, \qquad \frac{\partial}{\partial\varphi}\mathbf{U} = \mathbf{Y}\mathbf{U}, \tag{5}$$

where

$$\mathbf{U} = \begin{pmatrix} M_1(\theta)M_2(\varphi) \\ M_1'(\theta)M_2(\varphi) \\ M_1(\theta)M_2'(\varphi) \\ M_1'(\theta)M_2'(\varphi) \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b(\theta) & a(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b(\theta) & a(\theta) \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu^2 & 0 & 0 & 0 \\ 0 & -\mu^2 & 0 & 0 \end{pmatrix}.$$

As one can see, the matrices $\mathbf{X}$ and $\mathbf{Y}$ have a very strict form. The idea proposed in [2] is as follows. The equations (5) (they will be called the *coordinate equations*) will be used instead of (4). The structure of the equations (5) will be allowed to be not so strict, as in classical separation of variables. The vector $\mathbf{U}$ can have an arbitrary dimension (normally, the more complicated the scatterer is, the higher dimension should appear). The first component of the vector is the solution that should be found, all other components are not necessarily the derivatives of the solution, but are any appropriate functions. The matrices $\mathbf{X}$ and $\mathbf{Y}$ can also have arbitrary structure.

In our case we shall construct the coordinate equations of dimension 4, whose first two components are the functions v^1 and v^2. These equation will be constructed for any ν not belonging to the spectrum. Obviously, such equations cannot be equivalent to any separation of variables, since the last one deals with the values of ν, conversely, belonging to the spectrum.

We should mention here that there is no general way to derive the coordinate equations from the Laplace-Beltrami equation for an arbitrary problem on a sphere. However, for our problem and some other cases the coordinate equations can be constructed using the uniqueness of the solution for non-resonant ν. Derivation of the coordinate equations is the main subject of the current paper.

Obviously, the coordinate equations can be useful only if the elements of the matrices $\mathbf{X}$ and $\mathbf{Y}$ are rather simple functions of the coordinates. As we show below, in our case it is possible to construct the coordinate equations with the elements who are trigonometric functions of the coordinates. It is also clear that the coordinate equations can be written for arbitrary coordinate system, i.e., for example, if the coordinate equations for the coordinates (θ, φ) are known, then one can transform them into the equations for the coordinates (ξ, η) or any others.

Let us explain why we say that the coordinate equations can be called a generalization of ordinary differential equation to the case of two independent variables. Formally, the equations (5) are in partial derivatives, but they have the properties peculiar to the ordinary differential equations (ODEs). The solution of a partial differential equation is specified by its values on a contour. Conversely, the solution of the coordinate equations is specified by the value of the vector $\mathbf{U}$ at any single point. For example, let the vector $\mathbf{U}$ be known at the point ω' (see Figure 2). Let us try to fund the vector $\mathbf{U}$ at any other point ω, which is the observation point now. Connect ω and ω' by a line on the sphere. Introduce the coordinate l along this line. Note that we can construct the restriction of the coordinate equation to this line. This restriction has the form of an ordinary differential equation

$$\frac{d}{dl}\mathbf{U} = \mathbf{Z}\mathbf{U},$$

where $\mathbf{Z}$ is a linear combination of $\mathbf{X}$ and $\mathbf{Y}$ at each point. This equation can be solved with the values $\mathbf{U}(\omega')$ taken as the initial conditions. So, the values $\mathbf{U}(\omega)$ for any ω are defined by the values of $\mathbf{U}$ at just one point ω'. Such behaviour is peculiar to ordinary differential equations rather than to partial ones.

Moreover, as we show below, our coordinate equations for the sphere with a cut have the following property: if a boundary condition is valid at only one point of the cut, then it is valid on the whole cut. The coordinate equations method is a way to reduce the partial differential problem to a set of ordinary differential equations, i.e. it can be considered as a generalization of the

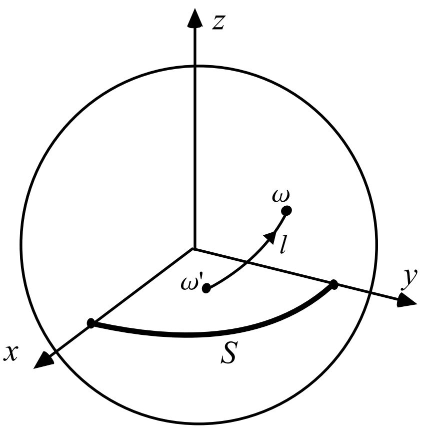

Figure 2: The reference point and the observation point

separation of variables.

In the next section we shall derive the coordinate equations for the vector

$$\mathbf{U} = (v^1, v^2, w^1, w^2)^T. \tag{6}$$

For this we shall use a trick based on the uniqueness of the edge Green's function for each ν not belonging to the spectrum. However, the possibility to write down the coordinate equations with rather simple coefficients (they are trigonometric functions of the coordinates) is deeply connected with the mathematical nature of the problem. Let us explain this point using an example.

Consider the case of one independent variable k. Let $u(k)$ be some function. Let us ask ourselves when one can write an ordinary differential equations with simple (say, rational) coefficients for u. The properties of the solutions of ordinary differential equations with rational coefficients are well known. There should exist a basis of several linearly independent solutions and each solution should be a linear combination of these basis solutions. Construct an analytical continuation of u on its Riemann surface. Even in the case of infinite number of branches the values on all branches should be linear combinations of the values on a finite number of branches. This is a very important property. Also we should demand a certain behaviour at the branch points and at infinity.

One can show that, conversely, if u on all its branches can be expressed as linear combinations of u at finite number of branches and the behaviour at the branch points and infinity is "good" in some sense, then an ordinary differential equation with rational coefficients can be constructed for u.

The situation with two independent variables (θ, φ) is quite similar. We should study the continuation of $\mathbf{U}$ onto its Riemann manifold. Again, $\mathbf{U}$ can have infinite number of branches, but its values on all them should be linear combinations of its values on a finite set of branches. A detailed study shows that the unknown function (6) does possess this property. However, the analysis of the analytical continuation of $\mathbf{U}$ is beyond the scope of the current paper.

3 Derivation of the coordinate equations

3.1 Properties of the unknown functions

Consider the problem on a sphere with the cut S. Construct the vector $\mathbf{U}$ for this problem. The dimension of this vector is equal to 4. We chose the functions $v^1(\omega, \nu)$ and $v^2(\omega, \nu)$ as the first two components of the vector $\mathbf{U}$. We remind that these functions are the edge Green's functions for the Dirichlet problem. The last two components are the functions $w^1(\omega, \nu)$ and $w^2(\omega, \nu)$, which are the edge Green's functions for the Neumann problem.

easy to show that the edge asymptotics of the edge Green's functions have the form:

$$v^m(\phi_n, \zeta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}} \zeta_n^{-1/2} \sin\frac{\phi_n}{2} + \frac{2C_n^m}{\sqrt{\pi}} \zeta_n^{1/2} \sin\frac{\phi_n}{2} + O(\zeta_n^{3/2}) \qquad (7)$$

and

$$w^m(\phi_n, \zeta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}} \zeta_n^{-1/2} \cos\frac{\phi_n}{2} + \frac{2E_n^m}{\sqrt{\pi}} \zeta_n^{1/2} \cos\frac{\phi_n}{2} + O(\zeta_n^{3/2}), \qquad (8)$$

where $m, n = 1, 2$; δ is the Kronecker's delta; C_n^m and E_n^m are some unknown coefficients depending on ν. Note that for $m = 1$, $n = 2$ the value of $C_n^m = C_2^1$ coincides with $C_2^1(\nu)$ from formula (41), Part I.

Due to the obvious symmetry

$$C_2^1 = C_1^2, \quad E_2^1 = E_1^2, \quad C_1^1 = C_2^2, \quad E_1^1 = E_2^2. \qquad (9)$$

Here and below we omit the argument ν of the functions $v^{1,2}$, $w^{1,2}$ and of the coefficients C_n^m and E_n^m. We assume that the non-resonant case is considered, i.e. that ν belongs neither to the spectrum of the Dirichlet nor of the Neumann problem. This means that if the field satisfies the homogeneous Laplace-Beltrami equation , boundary conditions of the Dirichlet or Neumann type on S and the Meixner's edge conditions (i.e. it grows at the edges no faster than $\zeta_n^{1/2}$), then the field is identically equal to zero.

3.2 Derivation of the coordinate equations for the edge Green's functions

Let us prove the following theorem.

Theorem 1 *Let the vector* $\mathbf{U}$ *be defined as (6) for any value of* ν *not belonging to the spectrum of the Dirichlet or Neumann problem. This vector obeys the coordinate equations of the form (5) with the coefficients* $\mathbf{X}$, $\mathbf{Y}$, *whose explicit form is given by the relations (35).*

PROOF

Call v an *oversingular function* if it satisfies the homogeneous Laplace-Beltrami equation, boundary conditions either of the Dirichlet or of the Neumann type on S and behaves at the edges like

$$v(\phi_n, \zeta_n) = \frac{C_n}{\sqrt{\pi}} \zeta_n^{-1/2} \sin \frac{\phi_n}{2} + O(\zeta_n^{1/2})$$

in the Dirichlet case or

$$v(\phi_n, \zeta_n) = \frac{E_n}{\sqrt{\pi}} \zeta_n^{-1/2} \cos \frac{\phi_n}{2} + O(\zeta_n^{1/2})$$

in the Neumann case. Note that the oversingular functions do not satisfy the Meixner's edge conditions, so they are not necessarily equal to zero.

In the non-resonant case (i.e. ν not belonging to the spectrum) it is clear that

$$v = -C_1 v^1 - C_2 v^2, \quad \text{or} \quad v = -E_1 w^1 - E_2 w^2. \tag{10}$$

(Note that, for example, the combination $u + C_1 v^1 + C_2 v^2$ in the Dirichlet case satisfies the Laplace-Beltrami equation, boundary and edge conditions, therefore this combination should be equal to zero.)

Derive the coordinate equation for the vector $\mathbf{U}$ of (6) as follows. Seek for the combinations of the derivatives of $v^{1,2}$ and $w^{1,2}$ that are oversingular functions. Note that generally the combination of the derivatives of $v^{1,2}$ and $w^{1,2}$ do not satisfy the conditions of the oversingular function since $v^{1,2}$ and $w^{1,2}$ are oversingular themselves, and therefore their derivatives normally contain the terms of $\zeta_{1,2}^{-3/2}$. So, only several specific combinations can be found, that form the basis of the oversingular differentiations of $v^{1,2}$ and $w^{1,2}$.

Introduce 3 differential operators T_1, T_2 and T_3 as follows. The operator T_3 is simply

$$T_3 = \frac{\partial}{\partial \varphi}, \tag{11}$$

where φ is the spherical coordinate used above. Two other operators are also differentiations with respect to the rotations, but the axes are chosen as the x and y directions, respectively. Thus,

$$T_1 = \frac{\partial}{\partial \phi_1}, \qquad T_2 = \frac{\partial}{\partial \phi_2}, \tag{12}$$

where ϕ_1 and ϕ_2 are considered as the global (rather than local) coordinates. The explicit form of T_1 and T_2 in the coordinates (θ, φ) is as follows:

$$\begin{aligned} T_1 &= -\sin\varphi \frac{\partial}{\partial \theta} - \cos\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial \varphi}, \\ T_2 &= \cos\varphi \frac{\partial}{\partial \theta} - \sin\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial \varphi}. \end{aligned}$$

Obviously, if some function v obeys the Laplace-Beltrami equation, then $T_j[v]$ also obeys the same equation. This follows from the fact that the Laplace-Beltrami operator is invariant with respect to any rotation of the sphere.

It is less obvious but easy to prove that if v obeys the Dirichlet boundary conditions on the sides of S, then $T_3[u]$ obeys the Dirichlet conditions, while $T_1[v]$ and $T_2[v]$ obey the Neumann condition on S. Conversely, if v obeys Neumann condition, then $T_3[v]$ obeys Neumann condition, while $T_1[v]$ and $T_2[v]$ obey the Dirichlet condition.

The first four oversingular combinations are the following:

$$T_1[v^1], \quad T_1[w^1], \quad T_2[v^2], \quad T_2[w^2].$$

The other four combinations are more complicated:

$$T_3[v^1] + T_2[w^1], \quad T_2[v^1] - T_3[w^1], \quad T_3[v^2] - T_1[w^2], \quad T_2[v^2] + T_3[w^2].$$

Studying the asymptotics of the oversingular combinations, we obtain eight representations of the form (10):

$$T_1[v^1] = C_2^1 w^2 + \frac{1}{2} w^1, \tag{13}$$

$$T_1[w^1] = E_2^1 v^2 - \frac{1}{2} v^1, \tag{14}$$

$$T_2[v^2] = C_1^2 w^1 + \frac{1}{2} w^2, \tag{15}$$

$$T_2[w^2] = E_1^2 v^1 - \frac{1}{2} v^2, \tag{16}$$

$$T_3[v^1] + T_2[w^1] = (C_1^1 + E_1^1) v^1 - C_2^1 v^2, \tag{17}$$

$$T_2[v^1] - T_3[w^1] = (C_1^1 + E_1^1) w^1 - E_2^1 w^2, \tag{18}$$

$$T_3[v^2] - T_1[w^2] = C_1^2 v^1 - (C_2^2 + E_2^2) v^2, \tag{19}$$

$$T_2[v^2] + T_3[w^2] = E_1^2 w^1 + (C_2^2 + E_2^2) w^2. \tag{20}$$

The system (13)–(20) consists of 8 equations and contains 8 independent derivatives of 4 functions $v^{1,2}$ and $w^{1,2}$ with respect to the coordinates (θ, φ). So, one can express these derivatives separately. The representation of the derivatives has the form of the equations (5) written for the vector $\mathbf{U}$ defined by (6) and the coefficients are defined by the formulae (35) given in Appendix A. These equations are the coordinate equations for the edge Green's functions.
$\square$

3.3 Some properties of the coordinate equations

Consider the equations (5), (35). They have the following properties.
(i) The solvability condition should be valid:

$$\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \varphi}\mathbf{U}\right) = \frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \theta}\mathbf{U}\right).$$

A sufficient condition for this is as follows:

$$\mathbf{XY} - \mathbf{YX} + \frac{\partial}{\partial \varphi}\mathbf{X} - \frac{\partial}{\partial \theta}\mathbf{Y} = 0. \tag{21}$$

One can check directly, that the matrices (35) obey this condition identically.
(ii) All components of the vector $\mathbf{U}$ should satisfy the homogeneous Laplace-Beltrami equation everywhere except the edges. Note that due to (5),

$$\tilde{\Delta}\mathbf{U} = \left[\mathbf{X}^2 + \frac{\cos\theta}{\sin\theta}\mathbf{X} + \frac{\partial}{\partial\theta}\mathbf{X} + \frac{1}{\sin^2\theta}\left(\mathbf{Y}^2 + \frac{\partial}{\partial\varphi}\mathbf{Y}\right)\right]\mathbf{U}. \tag{22}$$

A direct substitution of (35) into (22) shows that

$$\mathbf{X}^2 + \frac{\cos\theta}{\sin\theta}\mathbf{X} + \frac{\partial}{\partial\theta}\mathbf{X} + \frac{1}{\sin^2\theta}\left(\mathbf{Y}^2 + \frac{\partial}{\partial\varphi}\mathbf{Y}\right) = \left(\frac{1}{4} + (C_1^1 + E_1^1)^2 - (C_2^1)^2 - (E_2^1)^2\right)\mathbf{I}, \tag{23}$$

where $\mathbf{I}$ is the 4×4 identity matrix. Comparing (23) with

$$\left(\tilde{\Delta} + \nu^2 - \frac{1}{4}\right)\mathbf{U} = 0, \tag{24}$$

we conclude that (24) is fulfilled provided that

$$(C_1^1 + E_1^1)^2 = (C_2^1)^2 + (E_2^1)^2 - \nu^2. \tag{25}$$

The last relation makes it possible to express the combination $C_1^1 + E_1^1$ in terms of the parameters C_2^1 and E_2^1. It means that the coefficients of the equations (5), (35) contain only two unknown numerical parameters depending on ν, namely C_2^1 and E_2^1. We remind that these parameters are very important, because they stand in the modified Smyshlyaev's embedding formulae. In Section 4 we demonstrate the effective numerical algorithm for finding these parameters.
(iii) Consider the equations (5) at the cut S. Note that the boundary conditions on S have the form

$$v^1 = 0, \quad v^2 = 0, \quad \frac{\partial}{\partial\theta}w^1 = 0, \quad \frac{\partial}{\partial\theta}w^2 = 0. \tag{26}$$

Due to the form of the matrix $\mathbf{X}$ (see (35)) the last two conditions follow from the first two ones. Due to the form of the matrix $\mathbf{Y}$, if $v^1 = v^2 = 0$ at a single point of S, then the same conditions are valid on the whole cut S. I.e., it is necessary to check the boundary conditions (only two of them!) at a single point of S.
(iv) The coordinate equations have four singular points on the sphere, namely the points $(\xi = \pm 1, \eta = 0)$ and $(\xi = 0, \eta = \pm 1)$. One can perform the local analysis of the solutions near the singular points using the standard technique. Consider the edge L_1 (i.e., the point $(\xi = 1, \eta = 0)$) as an example. There are

four linearly independent solutions $\mathbf{U}_1 \dots \mathbf{U}_4$ of the coordinate equations. The following local expansions are valid for these solutions if $E_2^1 \neq 0$ or $C_2^1 \neq 0$:

$$\mathbf{U}_1 = \zeta^{-1/2} \begin{pmatrix} h_1^1(\xi,\phi)\,\sin\phi/2 \\ \zeta\, h_1^2(\xi,\phi)\,\sin\phi/2 \\ h_1^3(\xi,\phi)\,\cos\phi/2 \\ \zeta\, h_1^4(\xi,\phi)\,\cos\phi/2 \end{pmatrix}, \; \mathbf{U}_2 = \zeta^{-1/2} \begin{pmatrix} h_2^1(\xi,\phi)\,\cos\phi/2 \\ \zeta\, h_2^2(\xi,\phi)\,\cos\phi/2 \\ h_2^3(\xi,\phi)\,\sin\phi/2 \\ \zeta\, h_2^4(\xi,\phi)\,\sin\phi/2 \end{pmatrix}, \tag{27}$$

$$\mathbf{U}_3 = \begin{pmatrix} h_3^1(\xi,\phi) \\ h_3^2(\xi,\phi) \\ \zeta\sin\phi\, h_3^3(\xi,\phi) \\ \zeta\sin\phi\, h_3^4(\xi,\phi) \end{pmatrix}, \quad \mathbf{U}_4 = \begin{pmatrix} \zeta\sin\phi\, h_4^1(\xi,\phi) \\ \zeta\sin\phi\, h_4^2(\xi,\phi) \\ h_4^3(\xi,\phi) \\ h_4^4(\xi,\phi) \end{pmatrix}, \tag{28}$$

where $\zeta = \zeta_1$, $\phi = \phi_1$, the and functions h_n^m are analytic near L_1 and are even, i.e. obey the relation $h(\phi) = h(-\phi)$. Only the first solution satisfies the boundary and symmetry conditions.

Similar expansions can be obtained for the other three singular points. Note that for the singular points $(\xi = 0, \eta = -1)$ and $(\xi = -1, \eta = 0)$ only the expansion of the type $\mathbf{U}_3$ satisfy the boundary and symmetry conditions.

4 Numerical solution of the coordinate equations

The calculation of the diffraction coefficient can be performed as follows. One of the modified Smyshlyaev's formulae is chosen. A set of nodes is taken on the contour γ dense enough to provide the necessary accuracy. Then for each node (characterized by the value of ν) the spherical problem is solved numerically, i.e. the parameters C_2^1, E_2^1 and the functions $v^{1,2}$, $w^{1,2}$ are determined. Using this data the integration over γ is performed numerically using, say, the *trapezoid* formula.

Since the procedure associated with the integration over γ have been discussed extensively in [1], here we concentrate on solving the spherical problem for any particular value of ν. We assume that ν is located far enough from the spectrum. In practice, all points of the contour γ satisfy this condition.

We use the coordinate equations proposed in Section 4 to calculate the functions $v^{1,2}$. However, for this equations neither the values of the coefficients are known to us (we need to find $C_2^1(\nu)$ and $E_2^1(\nu)$), nor the "initial" conditions (i.e. the value of $\mathbf{U}$ at a single reference point). This data should be determined as the result of solving the eigenvalue problem for the ODE's. Below we formulate this problem and propose a method to solve it numerically.

Due to the symmetry with respect to the plane $\theta = \pi/2$, it is necessary to solve the coordinate equations only in one hemisphere, say for $\theta < \pi/2$.

The coordinate equations have 4 singular points, namely $L_1 \dots L_4$ (see Figure 3; note that L_1 and L_2 are the edges of the cut). These singular point break

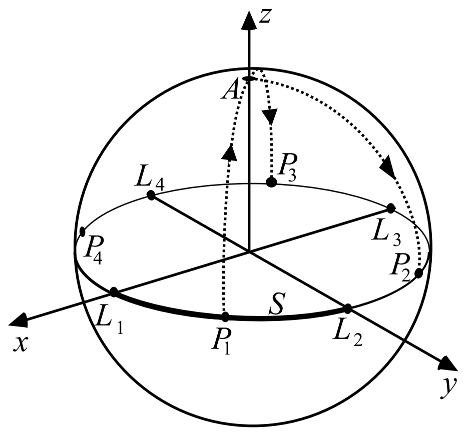

Figure 3: To the formulation of an eigenvalue problem on the sphere

the circle $\theta = \pi/2$ into 4 parts. One of these parts is S, on which the conditions

$$v^1 = v^2 = 0 \tag{29}$$

should be satisfied. On three other parts the conditions

$$w^1 = w^2 = 0 \tag{30}$$

should be valid due to the symmetrical properties of the functions w. As it follows from the first equation of (5) and the structure of the matrix $\mathbf{X}$, the conditions

$$\frac{\partial v^1}{\partial \theta} = \frac{\partial v^2}{\partial \theta} = 0$$

will follow from (30). Analogously to the previous consideration, the conditions (30) should be checked at just one point for each of three segments L_2L_3, L_3L_4 and L_4L_1.

Take four points $P_1 \ldots P_4$ on the circle $\theta = \pi/2$. The coordinate φ for these points is equal, respectively, to $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ (see Figure 3). The conditions (29) can be checked at the point P_1; the conditions (30) can be checked at the points P_2, P_3 and P_4. The detailed study of the edge behaviour of the coordinate equations shows that the conditions (29) and (30) guarantee the correct edge behaviour of the solution.

The value of $\mathbf{U}$ at the point P_1 should have the form

$$\mathbf{U}(P_1) = N\mathbf{U}_0, \qquad \mathbf{U}_0 = (0, 0, 1, -1)^T, \tag{31}$$

where N is a constant depending on ν. Let us get rid of N and solve the boundary problem using the condition

$$\mathbf{U}(P_1) = \mathbf{U}_0, \tag{32}$$

After this, N can be found using the main term of the edge asymptotics of $\mathbf{U}$. So, the initial condition is chosen and one should determine only the parameters C_2^1 and E_2^1.

Due to the geometrical symmetry and (32), if the conditions (30) are valid at P_2, then they are also valid at P_4. Therefore, it is necessary to check the conditions (30) only at the points P_2 and P_3.

Consider the boundary conditions at the point P_2. Due to the expansions (27), (28) and the initial conditions (32), it is necessary to check only one condition, e.g. $w^1(P_2) = 0$. The second condition $w^2 = 0$ will be valid automatically. Analogously, at the point P_3 it is necessary to check only one condition, e.g. $w^1(P_3) = 0$. We remind that this is true only if $E_2^1 \neq 0$ or $C_2^1 \neq 0$.

Thus, we have two independent unknown parameters E_2^1 and C_2^1, and two restrictions $w^1(P_2) = 0$ and $w^1(P_3) = 0$. Our aim is to find the values of the parameters providing the validity of the restrictions. The values of $w^1(P_2)$ and $w^1(P_3)$ can be chosen as the *discrepancies* of the problem:

$$D_1[C_2^1, E_2^1] = w^1(P_2), \qquad D_2[C_2^1, E_2^1] = w^1(P_3), \tag{33}$$

We propose a gradient procedure for finding the unknown parameters.

For each C_2^1 and E_2^1 define the numerical procedure for finding the discrepancies D_1 and D_2. Let the point A have the coordinate $\theta = 0$, i.e. A is the pole of the sphere. Solve the coordinate equation $\partial\mathbf{U}/\partial\theta = \mathbf{X}\mathbf{U}$ along the arc connecting P_1 and A using (32) as the initial conditions. As the result, we obtain the value of $\mathbf{U}$ at A. Second, solve the same equation along the arcs connecting A with P_2 and P_3 using the value $\mathbf{U}(A)$ as the initial conditions. As the result we obtain the values $\mathbf{U}(P_2)$ and $\mathbf{U}(P_3)$. Take the third component of both vectors as the discrepancies.

Take the initial values (i.e., the zero approximations) of C_2^1 and E_2^1. Denote them as $(C_2^1)_0$ and $(E_2^1)_0$, respectively. Perform *a step* of the gradient procedure as follows. Calculate the discrepancies D_1 and D_2 for given $(C_2^1)_0$ and $(E_2^1)_0$. Calculate the derivatives

$$\frac{\partial D_1}{\partial C_2^1}, \quad \frac{\partial D_1}{\partial E_2^1}, \quad \frac{\partial D_2}{\partial C_2^1}, \quad \frac{\partial D_2}{\partial E_2^1}$$

for given $(C_2^1)_0$ and $(E_2^1)_0$. This can be done by taking the values of C_2^1 and E_2^1 that differ a little from $(C_2^1)_0$ and $(E_2^1)_0$, calculating the discrepancies and performing the subtraction. Then calculate the corrections δC and δE by solving the linear system

$$\begin{aligned}
\delta C \,\frac{\partial D_1}{\partial C_2^1} + \delta E \,\frac{\partial D_1}{\partial E_2^1} &= -D_1[(C_2^1)_0, (E_2^1)_0], \\
\delta C \,\frac{\partial D_2}{\partial C_2^1} + \delta E \,\frac{\partial D_2}{\partial E_2^1} &= -D_2[(C_2^1)_0, (E_2^1)_0],
\end{aligned}$$

i.e. the corrections compensate the discrepancies in the linear approximation. Calculate the first approximations for C_2^1 and E_2^1 by the formulae

$$(C_2^1)_1 = (C_2^1)_0 + \delta C, \quad (E_2^1)_1 = (E_2^1)_0 + \delta E. \tag{34}$$

Analogously, calculate the second, third e.t.c. approximations until the discrepancies become small enough.

Finally, solve the equation for $\partial \mathbf{U}/\partial\varphi$ along the arc connecting the points P_1 and L_1. By comparing the edge asymptotic calculated numerically with that given by the relation (8) the parameter N can be determined.

The sketch of the algorithm described above is given in Figure 4.

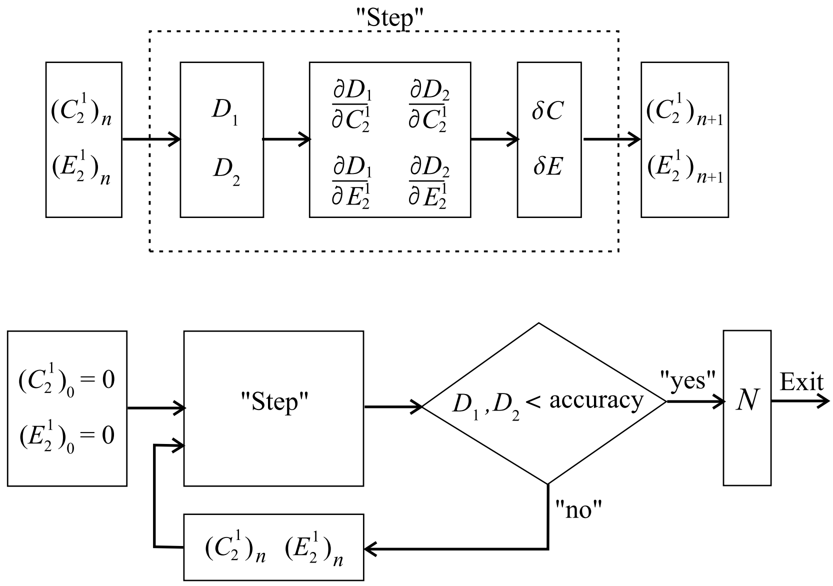

Figure 4: Block diagram of the gradient procedure

When the parameters C_2^1, E_2^1 and N are determined, it becomes possible to calculate the functions $v^{1,2}$ and $w^{1,2}$ on the whole hemisphere. For this, we solve the coordinate equations along the lines connecting A with $P_{1...4}$, tabulate these solutions dense enough and solve the coordinate equations along the lines $\theta = \text{const}$ using the tabulated values as the initial conditions. This procedure is schematically shown in Figure 5.

Below we demonstrate the results of the computations. We consider two different values of ν, namely

$$\nu = 0 \qquad \text{and} \qquad \nu = 20 + \mathrm{i}.$$

Tables 1 and 2 show step-by-step convergence of the gradient method for these two values of ν. The initial values of C_2^1 and E_2^1 are chosen equal to 0 in both cases. It is clear that as many as 5 steps are enough to achieve a reasonably high accuracy.

Figures 6 and 7 show the wave profiles of the functions v^1 and w^1 for $\nu = 20 + \mathrm{i}$. The graphs are presented in the spherical coordinates.

The number of steps can be reduced, if one notices that ν runs over the dense mesh lying on the contour γ. The neighboring nodes are close enough to

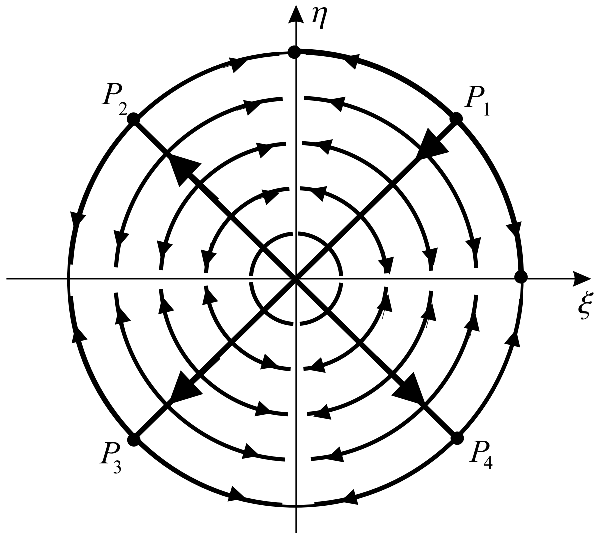

Figure 5: To the calculation of the edge Green's functions on the sphere

Table 1: Convergence of the gradient method for $\nu = 0$

n	$(C_2^1)_n$	$(E_2^1)_n$	$(C_1^1 + E_1^1)_n$	$\|D_1\|$	$\|D_2\|$
0	0	0	0	1	0
1	-0.450	0.450	0.637	0.587	1.159
2	-0.270	0.248	0.366	0.022	0.381
3	-0.191	0.293	0.349	0.005	0.018
4	-0.186	0.293	0.347	$7 \cdot 10^{-6}$	$1 \cdot 10^{-4}$
5	-0.186	0.293	0.347	$4 \cdot 10^{-10}$	$3 \cdot 10^{-9}$

each other, so one can use the values of C_2^1 and E_2^1 obtained on the previous step as the initial values $(C_2^1)_0$ and $(E_2^1)_0$ on the current step. This approach enables to reduce the number of iteration to 1–2 on each step.

The practical calculations show that the method works well when the value of ν is far enough from the spectrum of the problem. If, conversely, ν is real and more than $1/2$, the initial values of C_2^1 and E_2^1 should be very close to the exact values, otherwise the method diverges rapidly.

5 Concluding remarks

The coordinate equations enable one to reduce the partial differential equation on a sphere to a system of ordinary differential equations. The coordinate equations can be treated as the generalization of the separation of variables. Our method is not equivalent to the classical one based on the sphero-conal coordinates and the Lamè functions. The spherical edge Green's functions,

Table 2: Convergence of the gradient method for $\nu = 20 + \mathrm{i}$

n	$(C_2^1)_n$	$(E_2^1)_n$	$(C_1^1 + E_1^1)_n$	$\lvert D_1 \rvert$	$\lvert D_2 \rvert$
0	0	0	-20i	1	0
1	0.020+0.014i	0.558-0.512i	1.014-19.999i	0.158	0.252
2	0.015+0.017i	0.535-0.514i	1.014-20.000i	$1 \cdot 10^{-5}$	$5 \cdot 10^{-6}$
3	0.015+0.017i	0.535-0.514i	1.014-20.000i	$4 \cdot 10^{-11}$	$4 \cdot 10^{-11}$

which are the solutions of the coordinate equations, do not appear as the Fourier series. We are not aware of any method establishing the direct connection between the classical separation of variables in our case and the coordinate equations. The most important feature is that the new method is applicable to a wider class of the problems, for example to the case of several cuts on the equator or to the case of a sphere with a 1/8 triangular cut. The last one corresponds to diffraction by a cube vertex.

The new formulae are exact analytical results, i.e. no simplifying assumptions have been made for the problem. An effective numerical technique has been proposed to utilize this results in practical calculations. The coordinate equations lead to an eigenvalue problem for finding the parameters C_2^1 and E_2^1. A gradient method is applied to this problem.

We should mention one more feature of the coordinate equations, which we find important. Rewrite the coordinate equations for the coordinates (ξ, η). The coefficients of the equations will be expressed as the rational functions of ξ, η, $\sqrt{1-\xi^2}$, $\sqrt{1-\eta^2}$ and $\sqrt{1-\xi^2-\eta^2}$. The resulting equations can be treated as the generalization of the Fuchsian equations. Powerful methods of the analytical theory of differential equations can be applied to this problem, giving the information about the continuation of the field into the domain of complex ξ and η, about the Riemann manifold and the monodromy group of the solution etc.

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A The explicit form of the coordinate equations

The explicit expressions for the coefficients of the coordinate equations (5) are as follows:

$$\mathbf{X} = \begin{pmatrix} X_1^1 & X_2^1 & X_3^1 & X_4^1 \\ X_1^2 & X_2^2 & X_3^2 & X_4^2 \\ X_1^3 & X_2^3 & X_3^3 & X_4^3 \\ X_1^4 & X_2^4 & X_3^4 & X_4^4 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} Y_1^1 & Y_2^1 & Y_3^1 & Y_4^1 \\ Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 \\ Y_1^3 & Y_2^3 & Y_3^3 & Y_4^3 \\ Y_1^4 & Y_2^4 & Y_3^4 & Y_4^4 \end{pmatrix}, \tag{35}$$

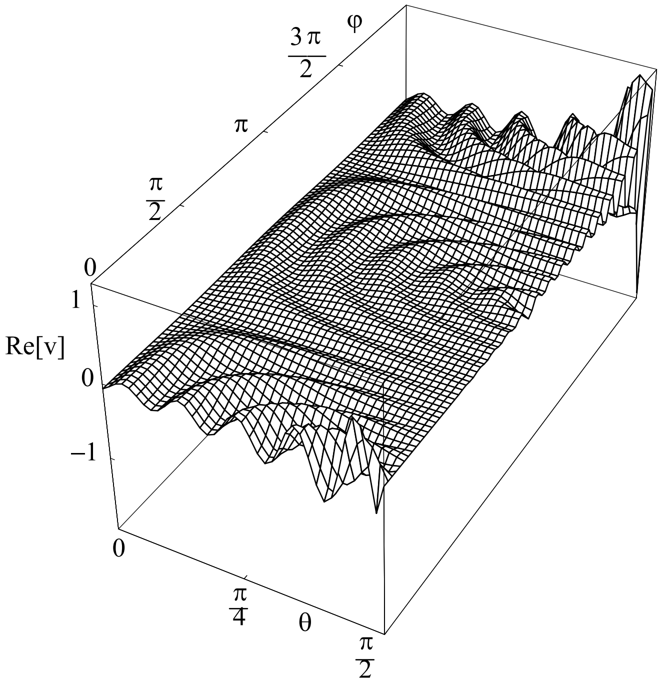

Figure 6: 3D graph of $\mathrm{Re}\,[v^1]$ for $\nu = 20 + \mathrm{i}$

where

$$\begin{aligned}
X_1^1 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{\cos\varphi - 2(C_1^1 + E_1^1)\sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)},\\
X_2^1 &= -\cos\varphi\,\cos\theta\,\sin\theta\,\frac{E_2^1\cos\varphi - C_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta},\\
X_3^1 &= \frac{(C_1^1 + E_1^1)\cos\varphi\,\cos^2\theta - \sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}\\
X_4^1 &= -\frac{E_2^1\cos\varphi\,\cos^2\theta + C_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}$$

$$\begin{aligned}
X_1^2 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta},\\
X_2^2 &= -\sin\varphi\,\cos\theta\,\sin\theta\,\frac{2(C_1^1 + E_1^1)\cos\varphi - \sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)},\\
X_3^2 &= \frac{C_2^1\cos\varphi + E_2^1\cos^2\theta\,\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}\\
X_4^2 &= \frac{\cos\varphi - 2(C_1^1 + E_1^1)\cos^2\theta\,\sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}$$

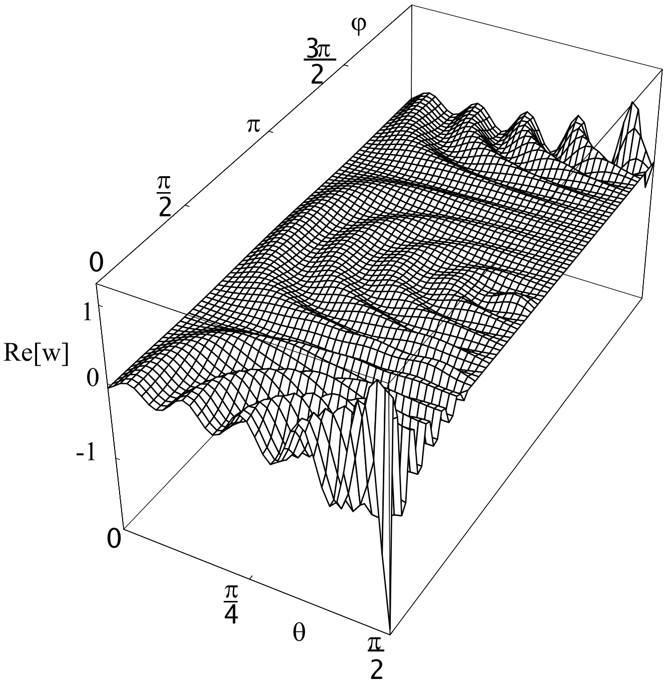

Figure 7: 3D graph of $\mathrm{Re}\,[w^1]$ for $\nu = 20 + \mathrm{i}$

$$
\begin{aligned}
X_1^3 &= \frac{2(C_1^1 + E_1^1)\cos\varphi\,\cos^2\theta + \sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}, \\
X_2^3 &= -\frac{C_2^1\cos\varphi\,\cos^2\theta + E_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}, \\
X_3^3 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{2(C_1^1 + E_1^1)\sin\varphi + \cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)} \\
X_4^3 &= \cos\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\cos\varphi - E_2^1\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
X_1^4 &= \frac{E_2^1\cos\varphi + C_2^1\cos^2\theta\,\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}, \\
X_2^4 &= -\frac{2(C_1^1 + E_1^1)\cos^2\theta\,\sin\varphi + \cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}, \\
X_3^4 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{C_2^1\sin\varphi - E_2^1\cos\varphi}{1-\sin^2\varphi\,\sin^2\theta} \\
X_4^4 &= \sin\varphi\,\cos\theta\,\sin\theta\,\frac{\sin\varphi + 2(C_1^1 + E_1^1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

$$
\begin{aligned}
Y^1_1 &= \sin\varphi\,\sin^2\theta\,\frac{2(C^1_1+E^1_1)\sin\varphi-\cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)},\\
Y^1_2 &= \sin\varphi\,\sin^2\theta\,\frac{E^1_2\cos\varphi-C^1_2\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta},\\
Y^1_3 &= -\cos\theta\,\sin\theta\,\frac{2(C^1_1+E^1_1)\sin\varphi+\cos\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}\\
Y^1_4 &= \cos\theta\,\sin\theta\,\frac{E^1_2\sin\varphi-C^1_2\cos\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
Y^2_1 &= \cos\varphi\,\sin^2\theta\,\frac{C^1_2\cos\varphi-E^1_2\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta},\\
Y^2_2 &= -\cos\varphi\,\sin^2\theta\,\frac{2(C^1_1+E^1_1)\cos\varphi-\sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)},\\
Y^2_3 &= \cos\theta\,\sin\theta\,\frac{E^1_2\cos\varphi-C^1_2\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta}\\
Y^2_4 &= -\cos\theta\,\sin\theta\,\frac{\sin\varphi+2(C^1_1+E^1_1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

$$
\begin{aligned}
Y^3_1 &= \cos\theta\,\sin\theta\,\frac{\cos\varphi-2(C^1_1+E^1_1)\sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)},\\
Y^3_2 &= -\cos\theta\,\sin\theta\,\frac{E^1_2\cos\varphi-C^1_2\sin\varphi}{1-\cos^2\varphi\,\sin^2\theta},\\
Y^3_3 &= -\sin\varphi\,\sin^2\theta\,\frac{\cos\varphi+2(C^1_1+E^1_1)\sin\varphi}{2(1-\cos^2\varphi\,\sin^2\theta)}\\
Y^3_4 &= \sin\varphi\,\sin^2\theta\,\frac{E^1_2\sin\varphi-C^1_2\cos\varphi}{1-\cos^2\varphi\,\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
Y^4_1 &= \cos\theta\,\sin\theta\,\frac{C^1_2\cos\varphi-E^1_2\sin\varphi}{1-\sin^2\varphi\,\sin^2\theta},\\
Y^4_2 &= -\cos\theta\,\sin\theta\,\frac{2(C^1_1+E^1_1)\cos\varphi-\sin\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)},\\
Y^4_3 &= \cos\varphi\,\sin^2\theta\,\frac{C^1_2\sin\varphi-E^1_2\cos\varphi}{1-\sin^2\varphi\,\sin^2\theta}\\
Y^4_4 &= \cos\varphi\,\sin^2\theta\,\frac{\sin\varphi+2(C^1_1+E^1_1)\cos\varphi}{2(1-\sin^2\varphi\,\sin^2\theta)}
\end{aligned}
$$

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