Diffraction of a high-frequency grazing wave by a grating with a complicated period

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January 3, 2013

Abstract

A 2D problem of propagation of a plane wave on a branched surface with a periodic set of branch points is studied. The periodic system of branch points plays the role of a diffraction grating. A period of the grating is composed of two branch points. The incident wave falls at a grazing angle with respect to the edge of the grating. The consideration is held in the parabolic approximation. The axis of propagation coincides with the edge of the grating.

Edge Green's functions of the problem are introduced. They are wave fields generated by point sources located near the branch points. An *embedding formula* is proven representing the unknown scattering coefficients of in terms of the directivities of the edge Green's functions. A *spectral equation* is derived for the directivities of the edge Green's functions. This equation is an ordinary differential equation whose coefficient is unknown. An *OE-equation* is proposed to find this coefficient.

1 Problem formulation for Helmholtz equation on a branched surface

It was shown in [1] that estimation of Q-factor of 2D rectangular resonators with windows demands solving of a family of the so-called Weinstein-class problems. They are 2D problems of diffraction of a plane wave by an infinite grating composed of branch points on a branched surface. These branch points are arranged in a line. Branched surfaces naturally emerge in resonator problems when the reflection principle is applied. Wave are assumed to have wavelength short comparatively to the distances between the branch points, and the incidence angle is assumed to be small (grazing).

The following problem is considered here. The branched surface consists of the "main" sheet and an infinite number of "auxiliary" sheets. The main sheet is the (x, y)-plane cut along the half-lines $y < 0$, $x = x_n$,

$$x_n = \begin{cases} (a+b)l, & n = 2l \\ (a+b)l + a, & n = 2l+1 \end{cases} \qquad l \in \mathbb{Z},$$

where a and b are the geometrical parameters of the problem. Auxiliary sheets are numbered by index n. Each of them is cut along the half-line $y < 0$, $x = x_n$. The branched surface is a result of attaching of the auxiliary sheets to the main sheet. The shores of the cut of the auxiliary sheet number n are attached to the shores of the cut of the main sheet going from the point $(x_n, 0)$. The right shore of the main sheet is attached to the left shore of the auxiliary sheet, and *vice versa*. Thus, the points $(x_n, 0)$ become the branch points of the surface. Each branch point has the second order.

The grating composed of the branch points has period equal to $a+b$ along the x-axis. The sketch of the surface is shown in Fig. 1.

The Helmholtz equation is valid on the branched surface:

$$\Delta u_{\mathrm{H}} + k^2 u_{\mathrm{H}} = 0. \tag{1}$$

The incident wave goes from infinity along the main sheet:

$$u_{\mathrm{Hin}} = \exp\{ik(x\cos\theta_{\mathrm{in}} - y\sin\theta_{\mathrm{in}})\}. \tag{2}$$

The incidence angle θ_{in} is small.

Time dependence of u_{H} is assumed to have form of $e^{-i\omega t}$, where ω is the circular frequency. The ratio ω/k is the wave velocity of the medium.

The total field should be continuous on the whole surface and obey Meixner's conditions (vertex conditions) near the branch points. The scattered wave should obey the radiation condition at infinity on each sheet. Mathematically, problem formulation is similar to that of, say, problem with a single branch point and two sheets, i.e. the classical Sommerfeld problem.

The presence of auxiliary sheets leads to the fact that the half-lines $y < 0$, $x = x_n$ on the main sheet can be considered as ideally absorbing screens.

The problem formulated here, being considered in terms of [1], corresponds for example to a high-frequency mode in the resonator shown in

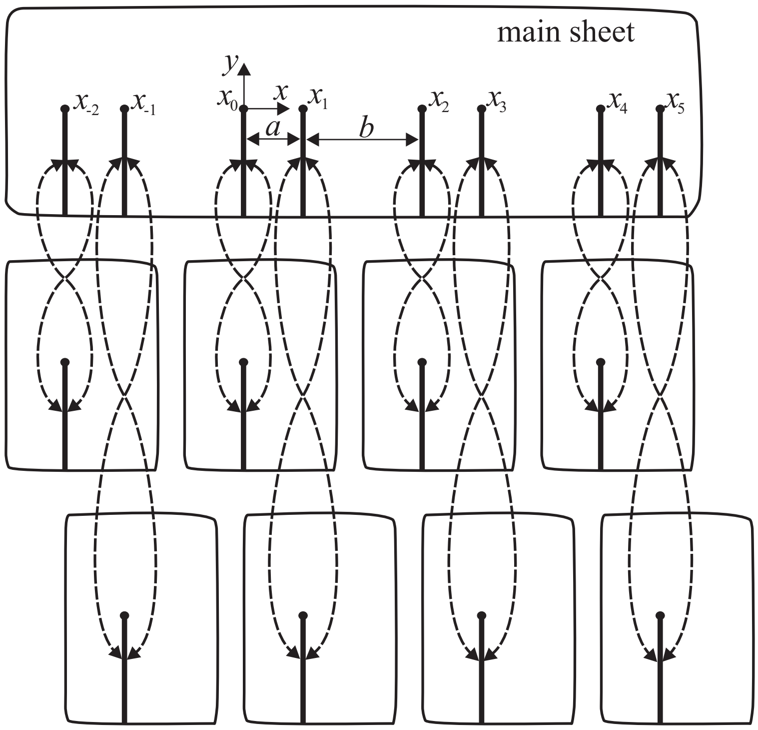

Figure 1: Structure of the branched surface

Fig. 2. The resonator is formed by a thin hard wall in the unbounded space. The wall has shape of a square with an angle taken off. The mode is formed by a a family of parallel rays. Parameter $a + b$ in this case is the total length of the ray $2\sqrt{2}D$. Parameter a is equal to $\sqrt{d}$.

2 Formulation of the problem in the parabolic approximation

The small value of $\theta_{\rm in}$ and the small value of the wavelength comparatively to a and b enable us to consider the problem in the parabolic approximation of diffraction theory. In our case this approximation means that we neglect the cylindrical edge waves diffracted at large angles, taking into account only the field scattered forward and having penumbral structure.

The positive x-direction is chosen as the main propagation direction, i.e.

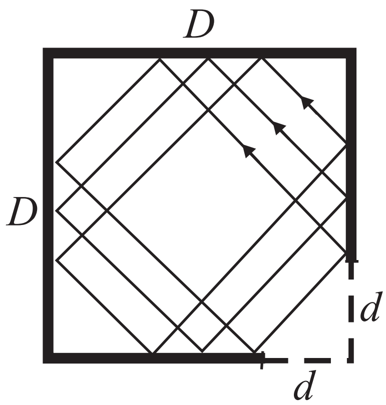

Figure 2: Resonator mode corresponding to the considered problem

the field is represented as

$$u_{\mathrm{H}} = \exp\{ikx\}u(x, y). \tag{3}$$

Function u depends on x slowly comparatively to u_{H}. Substituting (3) into (1) and neglecting the second derivative of u with respect to x (see [2]) obtain that u obeys approximately the parabolic equation

$$\left(\partial_y^2 + 2ki\partial_x\right) u = 0. \tag{4}$$

The main reason to study the parabolic equation instead of the Helmholtz one is simplification of description of wave processes. Namely, in any domain $x' < x < x''$ not containing branch points the values $u(x', y)$ can be considered as the initial data and the field in the whole domain can be computed by the formula

$$u(x, y) = \int_{-\infty}^{\infty} u(x', y')g(x - x', y - y')dy', \tag{5}$$

where g is the Green's function of the parabolic equation in unbounded plane:

$$g(x, y) = \begin{cases} k^{1/2}(2\pi x)^{-1/2} \exp\left\{iky^2/(2x) - i\pi/4\right\}, & x > 0, \\ 0, & x < 0. \end{cases} \tag{6}$$

As a consequence, the field on an auxiliary sheet is identically equal to zero everywhere to the left of the (only) branch point located on this sheet. This happens because due to the problem formulation there is no incident field on the auxiliary sheets and the scattered wave does not propagate from right to left in the parabolic approximation.

The parabolic equation can be considered on the branched surface built above, however it is possible to simplify the consideration by utilizing the absence of waves traveling from right to left. Consider the main sheet of the branched surface cut along the half-lines $y < 0$, $x = x_n$. Due to continuity of the field on the branched surface the field on the right shore of some cut coincides with the field on the left shore of a cut made on corresponding auxiliary sheet, i.e. this field is equal to zero:

$$u(x_n + 0, y < 0) = 0. \tag{7}$$

It is not necessary to formulate boundary conditions on left shores of the cuts on the main sheet, since the field is smoothly continued from these shores onto corresponding auxiliary sheets.

Below we consider field u only on the main sheet cut along the lines $y < 0$, $x = x_n$ on which the conditions (7) are fulfilled.

The total field u is a sum of the incident wave $u_{\rm in}$ and the scattered field $u_{\rm sc}$

$$u = u_{\rm in} + u_{\rm sc}. \tag{8}$$

The incident field in the parabolic approximation has form of

$$u_{\rm in}(x, y) = \exp\left\{-ikx\theta_{\rm in}^2/2 - ik\theta_{\rm in} y\right\}, \tag{9}$$

where $\theta_{\rm in}$ is the incident angle. This form guarantees that the incident wave obeys (4). The total field is continuous everywhere except the cuts. The scattered field obeys the boundary conditions following from (7):

$$u_{\rm sc}(x_m + 0, y) = -u_{\rm in}(x_m, y), \qquad y < 0. \tag{10}$$

Problem formulation should be supplied with the vertex conditions at the end points of the cuts and with the radiation condition. The vertex conditions state that the total field is limited near the end points. These conditions guarantee the absence of sources near the end points. Radiation condition states that there are no wave components coming from large positive or negative y except the incident wave. This condition will be taken into account when respective Fourier components will be considered.

Note that the validity of the parabolic approximation for the branched surface is violated near the branch points (say, several wavelengths from each branch point). Behavior of a solution of the parabolic equation near a branch point is rather complicated and it is very different from that of a solution of the Helmholtz equation. To see this, study function g as an example or study solution (78) from Appendix. For $x < x_m$ the field is continuous and smooth. The field at the point $(x_m, 0)$ is undetermined, and for $x > x_m$ it oscillates rapidly near the branch point. The end values $u(x_m, 0)$ play a huge role in what follows. In all cases we have in mind the values correctly defined by

$$u(x_m - 0, 0) \equiv \lim_{\epsilon \to 0} u(x_m - \epsilon, 0).$$

The formulation of the problem looks a bit strange because boundary conditions are imposed only on right shores of the cuts. We demonstrate in Appendix that in the simplest case of a single branch point such formulation leads to correct results.

A simpler problem belonging to the same class emerges if the grating has a single cut per period or if $a = b$ in our formulation. This problem was solved by L. A. Weinstein [3]. The most valuable and surprising result is that the coefficient of scattering into the main diffraction order tends to -1 as the incidence angle θ_{in} tends to zero. The solution was built by using the Wiener–Hopf method [4]. The problem considered here also can be treated by the Wiener–Hopf method, however in this case the problem becomes reduced to a matrix factorization problem whose solution is unknown.

The author proposed an alternative (with respect to Wiener–Hopf) method for the classical Weinstein's problem [5]. The method is based on the embedding formula and the spectral equation. In the current paper we generalize the proposed method to a new problem.

We should mention the papers [6, 7, 8], where the Weinstein–class problems are reduced to integral equations of the Wiener–Hopf type, and the papers [9, 10], where important technical issues related to the Weinstein–class problems are discussed.

3 Field representation according to the Floquet theory

The problem considered here has geometry periodic along the x-axis. Thus, it is possible to apply the Floquet theory. When a period is added to the

x-coordinate, the incident field is multiplied by the factor

$$\gamma \equiv \frac{u_{\rm in}(x+a+b)}{u_{\rm in}(x)} = \exp\left\{-ik(a+b)\,\theta_{\rm in}^2/2\right\}. \tag{11}$$

Obviously, the scattered field possesses the same property, i.e. it is multiplied by the same factor when a period is added to the x-coordinate.

The scattered field in the upper half-plane can be represented as a linear combination of the waves traveling in the positive direction of y or decaying there. Obviously, only waves obeying the Floquet property can participate in this linear combination. Thus the scattered field can be represented in the domain $y > 0$ as follows:

$$u_{\rm sc} = \sum_n R_n \exp\left\{-ikx\theta_n^2/2 + ik\,\theta_n y\right\}, \tag{12}$$

where θ_n is the propagation angle of the n-th mode:

$$\theta_n = \left(\theta_{\rm in}^2 + \frac{4\pi n}{k(a+b)}\right)^{1/2}, \quad n \in \mathbb{Z}. \tag{13}$$

Note that the numbering of the modes is chosen such that

$$\theta_0 = \theta_{\rm in}. \tag{14}$$

The branch of the square root in (13) is chosen in such a way that its value is real positive or imaginary positive. The first case corresponds to propagating waves (i.e. to the diffraction orders), and the second case corresponds to decaying components of the near field, i.e. to the inhomogeneous waves.

The main task of this paper is finding the scattering coefficients R_n in representation (12). We expect that for small $\theta_{\rm in}$ all coefficients except R_0 are small, so the reflection is close to the mirror one.

4 Edge Green's functions and their directivities

Define edge Green's functions $v_m(x, y)$ as follows. Consider the plane (x, y) with the cuts, i.e. consider the main sheet of the branched surface shown in Fig. 1. Functions v_m obey the following inhomogeneous parabolic equations:

$$\left(\partial_x + \frac{1}{2ik}\partial_y^2\right) v_m = \delta(x - x_m - 0)\delta(y). \tag{15}$$

Notation $x - x_m - 0$ means that for positive ϵ small enough we look for a solution of the problem having a source at the point $x_m + \epsilon$ (i.e. the source is located to the right of the point x_m), and after that we take the limit $\epsilon \to 0$. For any non-zero ϵ the solution should obey boundary condition (7), vertex conditions and the radiation condition.

It is easy to construct functions v_m explicitly. In the domain $x_m < x < x_{m+1}$ function v_m is represented as

$$v_m(x, y) = g(x - x_m, y). \tag{16}$$

This follows from the fact that function g defined in (6) obeys the following inhomogeneous parabolic equation:

$$\left(\partial_x + \frac{1}{2ik}\partial_y^2\right) g = \delta(x)\delta(y).$$

The inhomogeneous equation with a singular right-hand side is understood in a weak sense. The cut located at $x = x_m$ does not affect the field of the source located at $x = x_m + \epsilon$, thus the limiting procedure $\epsilon \to 0$ causes no difficulties.

Then, according to (5) and boundary condition (7), the field in the domain $x_{m+1} < x < x_{m+2}$ can be represented as

$$v_m(x, y) = \int_0^\infty g(x_{m+1} - x_m, y')g(x - x_{m+1}, y - y')dy', \tag{17}$$

the field in the domain $x_{m+2} < x < x_{m+3}$ can be represented as

$$v_m(x, y) =$$

$$\iint_0^\infty g(x_{m+1} - x_m, y')g(x_{m+2} - x_{m+1}, y'' - y')g(x - x_{m+2}, y - y'')dy'dy'', \tag{18}$$

etc. The formulae are explicit on each step, however the result is not practical, since the formulae contain nested integrals.

Obviously,

$$v_m(x, y) = 0 \quad x < x_m.$$

Due to the periodicity, all functions v_m can be obtained from v_0 and v_1 by x-coordinate shifts:

$$v_{m+2}(x, y) = v_m(x - (a + b), y).$$

Let us make an important note. The procedure of placing a point source near a branch point in the parabolic equation case is very different from the Helmholtz case. It is not true that function v_m is an asymptotic representation of the Helmholtz field produced by a unit strength source located near the branch point. In classical works by Leontovich and Fok dedicated to the parabolic equation it is mentioned that the parabolic approximation becomes invalid near the source. Thus, a correct procedure is matching of the far–field asymptotics of the Green's function of the Helmholtz equation with g. This is not what is done in this paper. Instead, function v_m is considered as an auxiliary function having sense only in the parabolic approximation.

The directivities $V_m(\theta)$ of the edge Green's functions $v_m(x, y)$ are introduced as the coefficients in the following asymptotic representations:

$$v_m(x, y) = g(x - x_m, y) V_m \left(\frac{y}{x - x_m} \right) + o(x^{-1/2}), \tag{19}$$

where the values y and $x - x_m$ are assumed to be asymptotically large, while their ratio remains constant. This representation is constructed by analogy with the Helmholtz equation case. The main term of the field v_m is represented as the Green's function g multiplied by a directivity depending only on the scattering angle. Obviously, directivities $V_m(\theta)$ are defined only for $\theta > 0$. Moreover, asymptotics (19) can be uniform only for θ bigger than some non-zero positive value.

Introduce also a directivity of some arbitrary parabolic field $w(x, y)$ with respect to the point X as coefficient $W^X(\theta)$ in the representation

$$w(x, y) = g(x - X, y) W^X \left(\frac{y}{x - X} \right) + o(x^{-1/2}),$$

if such representation is valid for the field. Transformation formula for the directivities of the same field computed with respect to different points is as follows:

$$W^{X'}(\theta) = \exp\{ik(X - X')\theta^2/2\}\, W^X(\theta). \tag{20}$$

Due to periodicity,

$$V_{m+2}(\theta) = V_m(\theta). \tag{21}$$

5 Some properties of the fields and their directivities in the parabolic approximation

As it is known, directivity of a field in the Helmholtz equation case is expressed through the Fourier transformation of the field taken on the scatterers of some other surfaces. The key point for building these representations is the Green's formula. We need similar representations for the parabolic equation, and the first step is constructing the Green's formula for the parabolic case.

Theorem 1 *Let functions $v(x, y)$ and $w(x, y)$ obey the following inhomogeneous equations in some domain Ω:*

$$\left(\partial_x + \frac{1}{2ik}\partial_y^2\right) v = f(x, y), \qquad \left(-\partial_x + \frac{1}{2ik}\partial_y^2\right) w = h(x, y). \tag{22}$$

Then the following equality is valid

$$\int_{\partial\Omega} [(\mathbf{v} \cdot \mathbf{n})w - (\mathbf{w} \cdot \mathbf{n})v] dl = 2ik \int_{\Omega} [fw - hv] ds. \tag{23}$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$, and vector flows $\mathbf{v}$, $\mathbf{w}$ are given by

$$\mathbf{v} = \begin{pmatrix} ikv \\ \partial_y v \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -ikw \\ \partial_y w \end{pmatrix}, \tag{24}$$

One can prove the theorem by applying Gauss–Ostrogradsky theorem.

Let us formulate several propositions related to directivities of wave fields and for the coefficients R_n.

Proposition 1 *Coefficients R_n from (12) are expressed by the formula*

$$R_n = \frac{\exp\{-ik\theta_n y_*\}}{a+b} \int_0^{a+b} u_{\mathrm{sc}}(x, y_*) \exp\left\{i\frac{k\theta_n^2}{2}x\right\} dx \tag{25}$$

for any $y_* > 0$.

The proof is elementary. Expand the (periodic) function

$$u_{\mathrm{sc}}(x, y_*) \exp\left\{i\frac{k\theta_0^2}{2}x\right\}$$

as Fourier series.

Proposition 2 *For large positive X, Y*

$$v_m(X,Y) = \sqrt{\frac{k}{2\pi X}} \exp\left\{ i\frac{kY^2}{2X} - i\frac{\pi}{4} \right\} \times$$

$$\frac{Y}{X} \int_{-\infty}^{\infty} v_m(x, y_*) \exp\left\{ ik \left(\frac{1}{2}\frac{Y^2}{X^2} x - \frac{Y}{X} y_* \right) \right\} dx + o(X^{-1/2}) \qquad (26)$$

for any fixed $y_* > 0$.

To prove this statement use (23) substituting v_m as v, and

$$w(x,y) = g(X - x, y - Y)$$

as w. Domain Ω is the set of points (x, y) with $y > y_*$.

For large X, Y and small x, y_* we can use the approximation

$$w(x, y_*) \approx \sqrt{\frac{k}{2\pi X}} \exp\left\{ i\frac{kY^2}{2X} - i\frac{\pi}{4} \right\} \exp\left\{ ik \left(\frac{1}{2}\frac{Y^2}{X^2} x - \frac{Y}{X} y_* \right) \right\}.$$

The possibility to consider x and y_* as small (comparatively to X and Y) follows from the fact that the edge Green's functions decay as $x \to \infty$ and are equal to zero for large negative x.

For using (23) it is necessary to transform the combination

$$\int_{-\infty}^{\infty} \partial_y v_m(x, y_*) \exp\left\{ ik \frac{1}{2}\frac{Y^2}{X^2} x \right\} dx.$$

Take into account the radiation condition, i.e. note that in the upper half-plane the field is represented as a linear combination of outgoing and decaying waves. Fourier transformation expands the field v_m in plane waves, and it becomes possible to express the vertical derivative for each plane wave:

$$\int_{-\infty}^{\infty} \partial_y v_m(x, y_*) \exp\left\{ \frac{ik}{2}\frac{Y^2}{X^2} x \right\} dx = ik\frac{Y}{X} \int_{-\infty}^{\infty} v_m(x, y_*) \exp\left\{ \frac{ik}{2}\frac{Y^2}{X^2} x \right\} dx.$$

Substitution of these representations into (23) gives (26).

Proposition 3 *Directivities $V_m(\theta)$ can be computed by the formula*

$$V_m(\theta) = \exp\left\{-i\frac{k\theta^2}{2}x_m\right\}\theta\int_{-\infty}^{\infty} v_m(x, y_*)\exp\left\{ik\left(\frac{\theta^2}{2}x - \theta y_*\right)\right\}dx \quad (27)$$

for any $y_* > 0$.

Expression (27) follows from (26) after taking into account that

$$g(X, Y) \approx \exp\left\{-\frac{ik}{2}\frac{Y^2}{X^2}x_m\right\}g(X - x_m, Y)$$

and

$$\theta = \frac{Y}{X - x_m} \approx \frac{Y}{X}.$$

Proposition 4 *The following representations are valid for $V_m(\theta)$:*

$$V_m(\theta) = 1 - \sum_{n=m+1}^{\infty}\exp\left\{ik\frac{\theta^2}{2}(x_n - x_m)\right\}\int_{-\infty}^{0} v_m(x_n - 0, y)e^{-ik\theta y}dy \quad (28)$$

This proposition can be proven by applying (23) to functions v_m and

$$w = \exp\left\{ik\left(\theta^2 x/2 - \theta y\right)\right\}$$

with $\theta > 0$ in the domain shown in Fig. 3. Lower horizontal segments of the boundary correspond to $y = -L$ for large L. The radiation condition guarantee that the integral over these segments tends to zero as $L \to \infty$.

Denote by c_m the edge values of the total field $u(x, y)$ (for the plane wave incidence) multiplied by a coefficient for convenience:

$$c_m \equiv u(x_m - 0, 0)\exp\left\{ikx_m\theta_{\text{in}}^2/2\right\}. \quad (29)$$

Due to Floquet condition,

$$c_{n+2} = c_n,$$

thus, to determine all c_m it is sufficient to find only c_0 and c_1.

Proposition 5 *The following representation is valid for $c_{0,1}$*

$$c_{1-m} = V_m(\theta_{\text{in}}). \quad (30)$$

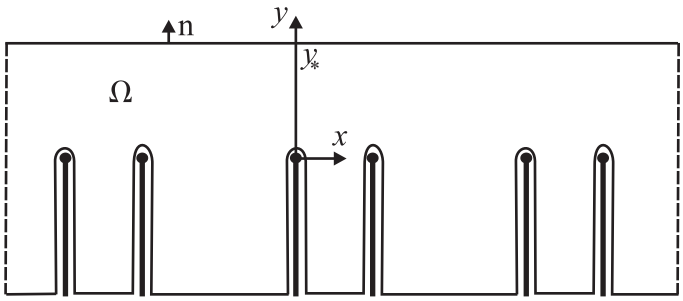

Figure 3: Domain Ω for derivation of (28)

Introduce $v(x, y; x', y')$ as the Green's function of the following problem. v should obey the inhomogeneous parabolic equation

$$\left(\partial_x + \frac{1}{2ik}\partial_y^2\right) v = \delta(x - x')\delta(y - y')$$

on a plane with the cuts $x = x_m$, $y < 0$. It should obey boundary conditions (7) on the right shores of the cuts. Moreover, the function (with respect to variables x,y) should obey the radiation condition and vertex conditions.

Obviously, the edge Green's functions can be considered as partial cases of v:

$$v_m(x, y) = v(x, y; x_m + 0, 0).$$

Directivities V_m can be defined as

$$V_m(\theta) = \lim_{x' \to \infty} \frac{v_m(x' + x_m, x'\theta)}{g(x', x'\theta)}. \tag{31}$$

Also it is obvious that the total field with plane incident wave (at least in some domain containing the origin) can be considered as a limiting case of the Green's function with a remote source:

$$u(x, y) = \lim_{x' \to \infty} \frac{v(x, y; -x', x'\theta_{\text{in}})}{g(x', -x'\theta_{\text{in}})} \tag{32}$$

and, therefore

$$c_m = \exp\{ikx_m\theta_{\text{in}}^2/2\} \lim_{x' \to \infty} \frac{v(x_m, y; -x', x'\theta_{\text{in}})}{g(x', -x'\theta_{\text{in}})}. \tag{33}$$

Note that the function

$$w(x, y) = v(a - x, y; a - x'', y'')$$

(again, considered as a function of x, y) obeys the second equation (22) with the right-hand side

$$h(x, y) = \delta(x - x'', y - y'')$$

and boundary conditions $w = 0$ set on the left shores of the cuts $x = x_m$, $y < 0$. Apply (23) to the functions $v(x, y; x', y')$ and $v(a - x, y; a - x'', y'')$. Take the whole cut plane as the domain Ω. The integral over the shores of the cuts is equal to zero due to the boundary conditions. The integral over the remote horizontal segments tends to zero due to the radiation condition. Thus, we get

$$v(x'', y''; x', y') = v(a - x', y'; a - x'', y'')$$

and therefore

$$v(x_{1-m}, 0; -x', y') = v_m(x' + a, y').$$

Substitute the last equation into (31) and (33), and take the limit $x' \to \infty$, $y' = \theta_{\mathrm{in}} x'$. As the result, get (30).

6 Embedding formula

Prove one more proposition.

Proposition 6 *Let $u(x, y)$ be a function obeying a homogeneous parabolic equation (4) everywhere except the cuts and boundary conditions (7) on the right shores of the cuts, and limited near the end points of the cuts. Then function $w(x, y) = \partial_y u$ obeys the inhomogeneous parabolic equation*

$$\left(\partial_x + \frac{1}{2ik} \partial_y^2 \right) w = \sum_{m=-\infty}^{\infty} u(x_m - 0, 0) \delta(x - x_m - 0) \delta(y) \qquad (34)$$

and boundary conditions (7).

Everywhere except the vicinity of the cuts w obeys the homogeneous parabolic equation since the operator ∂_y commutes with the parabolic operator. The boundary conditions can be checked directly. The proposition above states that the operator ∂_y generates monopole sources at the ends of the cuts. Let us check this.

Consider a narrow strip $x_m < x < x_m + \Delta$, $-\infty < y < \infty$. According to (5) the field in this strip can be written as

$$u(x,y) = \lim_{\epsilon\to 0} \int\limits_{\epsilon}^{\infty} u(x_m, y')g(x - x_m, y - y')dy'. \tag{35}$$

The integration is held over the positive half-axis, since the field $u(x_m+0, y')$ on the negative half-axis is equal to zero due to boundary condition (7). The integral converges near the end point $y = 0$ due to the vertex condition. Apply the operator ∂_y and integrate by parts:

$$w(x,y) = \lim_{\epsilon\to 0} \int\limits_{\epsilon}^{\infty} w(x_m, y')g(x - x_m, y - y')dy' +$$

$$\lim_{\epsilon\to 0} u(x_m, \epsilon)g(x - x_m, y - \epsilon).$$

Due to (5), the value of $w(x_m, \epsilon)$ in the first term tends to a finite limit as $\epsilon \to 0$ (this limit is equal to $w(x_m - 0, 0)$). In the second term $u(x_m, +0) = u(x_m - 0, 0)$. Thus,

$$w(x,y) = \int\limits_{0}^{\infty} w(x_m - 0, y')g(x - x_m, y - y')dy' + u(x_m - 0, 0)g(x - x_m, y).$$

The first term in the right-hand side corresponds a field without sources, and the second term corresponds to the field of a monopole source located at the point $(x_m + 0, 0)$ and having amplitude $u(x_m - 0, 0)$. A similar procedure can be performed with each other vertex.

The embedding formula links the scattering coefficients R_n with directivities of the edge Green's functions. This formula is given by the following theorem.

Theorem 2

$$R_m = \frac{1}{ik(a+b)\theta_m(\theta_m + \theta_{\rm in})} \sum_{n=0}^{1} \exp\left\{\frac{2\pi i m x_n}{a+b}\right\} V_{1-n}(\theta_{\rm in}) V_n(\theta_m). \tag{36}$$

PROOF. Apply the operator

$$\mathrm{H}[u](x,y) = (\partial_y + ik\theta_{\rm in})u.$$

to the field $u(x,y)$ (i.e. to the solution of the problem with plane incident wave). This operator nullifies the incident wave, keeps valid the parabolic equation in the internal points of the domain and keeps valid the boundary conditions. Obviously, the field $\mathrm{H}[u]$ obeys the radiation condition. According to Proposition 6 the field w has sources located at the edges $(x_m, 0)$ (i.e. at the end points of the scatterer). The amplitudes of these sources are equal to $u(x_m - 0, 0)$. According to uniqueness of the solution of the diffraction problem,

$$\mathrm{H}[u](x,y) = \sum_{n=-\infty}^{\infty} u(x_n - 0, 0) v_n(x,y) = \sum_{n=-\infty}^{\infty} \exp\{-ikx_n\theta_{\rm in}^2/2\} c_n v_n(x,y). \tag{37}$$

Multiply (37) by

$$\exp\{ikx\theta_m^2/2 - ik\theta_m y\}.$$

Fix an arbitrary positive value $y = y_*$ and integrate (37) along the segment $0 < x < (a+b)$, $y = y_*$. Take into account that H standing in the left nullifies the incident wave, and it is possible to compute the y-derivative for each Fourier component of the scattered field. Use (25) to transform the right-hand side:

$$ik(\theta_m + \theta_{\rm in})(a+b)R_m =$$

$$\sum_{n=-\infty}^{\infty} c_n \exp\{-ikx_n\theta_{\rm in}^2/2\} \int_0^{a+b} v_n(x, y_*) \exp\{ikx\theta_m^2/2 - ik\theta_m y_*\} dx \tag{38}$$

Due to periodicty

$$v_n(x, y_*) c_n \exp\{ik(x\theta_m^2 - x_n\theta_{\rm in}^2)/2\} = v_{n'}(x', y_*) c_{n'} \exp\{ik(x'\theta_m^2 - x_{n'}\theta_{\rm in}^2)/2\},$$

$$x' = x + (a+b)l, \quad n' = n + 2l, \quad l \in \mathbb{Z}.$$

The last formula enables one to replace summation and integration along a segment by integration along a line:

$$ik(\theta_m + \theta_{\rm in})(a+b)R_m =$$

$$\sum_{n=0}^{1} c_n \exp\{-ikx_n\theta_{\rm in}^2/2\} \int\limits_{-\infty}^{\infty} v_n(x,y_*) \exp\{ikx\theta_m^2/2 - ik\theta_m y_*\} dx. \qquad (39)$$

Applying formulae (27) and (30), obtain

$$ik(\theta_m + \theta_{\rm in})(a+b)R_m = \sum_{n=0}^{1} \exp\left\{ikx_n \frac{\theta_m^2 - \theta_{\rm in}^2}{2}\right\} \frac{V_{1-n}(\theta_{\rm in})V_n(\theta_m)}{\theta_m}. \qquad (40)$$

Finally, applying (13) obtain (36). $\square$

Thus, scattering coefficients R_n become expressed through the directivities of the edge Green's functions. Below our efforts are focused on computation of these directivities.

7 Spectral equation

Directivities $V_m(\theta)$ can be computed by solving an ordinary differential equation (ODE) called the spectral equation. To derive the spectral equation apply the operator

$$\mathrm{K}_n = (x - x_n)\partial_y - iky$$

to the edge Green's function $v_n(x,y)$. Note that operator K_n commutes with the parabolic operator $2ik\partial_x + \partial_y^2$, thus $\mathrm{K}_n[v_n]$ obeys the parabolic equation in all internal points of the cut plane. Operator K_n is analogous to the operator of differentiation with respect to the angle of rotation about the point $(x_n, 0)$ in the Helmholtz case. Further, $\mathrm{K}_n[v_n]$ obeys boundary conditions (7) and the radiation condition. Finally,

$$\mathrm{K}_n[g(x - x_n, y)] \equiv 0.$$

At the same time, $v_n(x,y) = g(x - x_n, y)$ for $x_n < x < x_{n+1}$. Thus the operator K_n nullifies the source of the field v_n. Analogously to Proposition 6 one can show that the function

$$w(x,y) = \mathrm{K}_n[v_n](x,y)$$

obeys the inhomogeneous parabolic equation

$$\left(\partial_x + \frac{1}{2ik}\partial_y^2\right) w = \sum_{m=n+1}^{\infty} (x_m - x_n)\, v_n(x_m - 0, 0)\, \delta(x - x_m - 0)\, \delta(y).$$

Due to uniqueness of the solution of the diffraction problem,

$$\mathrm{K}_n[v_n](x,y) = \sum_{m=n+1}^{\infty} (x_m - x_n) v_n(x_m - 0, 0) v_m(x,y). \tag{41}$$

Consider directivities of the fields in (41). It is easy to check that operator K_n acts on a directivity calculated with respect to the point x_n as differentiation with respect to θ. Taking into account (20) obtain

$$\partial_\theta V_n(\theta) = \sum_{m=n+1}^{\infty} (x_m - x_n) v_n(x_m - 0, 0) \exp\{ik(x_m - x_n)\theta^2/2\} V_m(\theta). \tag{42}$$

Note that $V_m(\theta) = V_{m+2}(\theta)$. Introduce a matrix of coefficients

$$\mathrm{C}(\theta) = \begin{pmatrix} C_{0,0} & C_{0,1} \\ C_{1,0} & C_{1,1} \end{pmatrix}, \tag{43}$$

where

$$C_{0,0}(\theta) = \sum_{l=1}^{\infty} (x_{2l} - x_0) v_0(x_{2l} - 0, 0) \exp\{ik(x_{2l} - x_0)\theta^2/2\}, \tag{44}$$

$$C_{1,0}(\theta) = \sum_{l=0}^{\infty} (x_{2l+1} - x_0) v_0(x_{2l+1} - 0, 0) \exp\{ik(x_{2l+1} - x_0)\theta^2/2\}, \tag{45}$$

$$C_{0,1}(\theta) = \sum_{l=1}^{\infty} (x_{2l} - x_1) v_1(x_{2l} - 0, 0) \exp\{ik(x_{2l} - x_1)\theta^2/2\}, \tag{46}$$

$$C_{1,1}(\theta) = \sum_{l=1}^{\infty} (x_{2l+1} - x_1) v_1(x_{2l+1} - 0, 0) \exp\{ik(x_{2l+1} - x_1)\theta^2/2\}. \tag{47}$$

Using the new notation obtain that the following theorem is valid:

Theorem 3 *Directivities $V_0(\theta)$, $V_1(\theta)$ obtain the following ordinary differential equation*

$$\frac{d}{d\theta}(V_0, V_1) = (V_0, V_1) \cdot \mathrm{C}(\theta), \tag{48}$$

where C *is given by (43)–(47).*

This equation will be called *the spectral equation.* The coefficient of this equation include unknown values $v_0(x_m - 0, 0)$, $v_1(x_m - 0, 0)$ (edge values of the edge Green's functions). Computation of the coefficient and the initial condition of the spectral equation is the subject of next two sections.

8 More general form of the spectral equation

Introduce the notation

$$\tilde{V}_{m,n}(\theta) = -\exp\{ik(x_m - x_n)\theta^2/2\} \int\limits_{-\infty}^{0} v_n(x_m - 0, y) \exp\{-ik\theta y\} dy, \quad m > n. \tag{49}$$

Define also

$$\tilde{V}_{n,n}(\theta) = 1, \qquad \tilde{V}_{m,n}(\theta) = 0, \quad m < n. \tag{50}$$

Return to (41). Multiply both sides by

$$-\exp\{ik(x_m - x_n)\theta^2/2 - ik\theta y\}$$

and integrate along the line $x = x_m - 0$, $-\infty < y < 0$. As the result, obtain

$$\partial_\theta \tilde{V}_{m,n}(\theta) = \sum_{l=n+1}^{m} (x_l - x_n) v_n(x_l - 0, 0) \exp\{ik(x_l - x_n)\theta^2/2\} \tilde{V}_{m,l}(\theta). \tag{51}$$

Introduce the values

$$a_{l,n}(\theta) = (x_l - x_n) v_n(x_l - 0, 0) \exp\{ik(x_l - x_n)\theta^2/2\}.$$

Equation (51) can be rewritten in the form

$$\partial_\theta \tilde{V}_{m,n}(\theta) = \sum_{l=n+1}^{m} \tilde{V}_{m,l}(\theta) a_{l,n}(\theta). \tag{52}$$

Equations (52) form an infinite system of equations for all integer m and n. Note that "unknown functions" $\tilde{V}_{m,n}$ and "coefficients" $a_{m,n}$ obey the periodicity condition

$$\tilde{V}_{m,n}(\theta) = \tilde{V}_{m+2,n+2}(\theta), \qquad a_{m,n}(\theta) = a_{m+2,n+2}(\theta)$$

Moreover, the right-hand side (52) has a convolution type. These two circumstances enable one to apply Fourier transform to the system (52). Namely, for any arbitrary p with $\mathrm{Im}[p] \geq 0$ multiply (52) by $\exp\{ikp(x_m - x_n)\}$, fix $n = 0$ or 1 and sum over all m. In the right-hand side use the identity

$$\exp\{ikp(x_m - x_n)\} = \exp\{ikp(x_m - x_l)\} \exp\{ikp(x_l - x_n)\}.$$

As the result, obtain

$$\partial_\theta \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} = \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} \begin{pmatrix} \bar{a}_{0,0} & \bar{a}_{0,1} \\ \bar{a}_{1,0} & \bar{a}_{1,1} \end{pmatrix}, \tag{53}$$

where the following values are introduced:

$$\bar{V}_{0,0}(\theta,p) = \sum_{l=0}^{\infty} \tilde{V}_{2l,0}(\theta) \exp\{ik(x_{2l}-x_0)p\}, \tag{54}$$

$$\bar{V}_{1,0}(\theta,p) = \sum_{l=0}^{\infty} \tilde{V}_{2l+1,0}(\theta) \exp\{ik(x_{2l+1}-x_0)p\}, \tag{55}$$

$$\bar{V}_{0,1}(\theta,p) = \sum_{l=1}^{\infty} \tilde{V}_{2l,1}(\theta) \exp\{ik(x_{2l}-x_1)p\}, \tag{56}$$

$$\bar{V}_{1,1}(\theta,p) = \sum_{l=0}^{\infty} \tilde{V}_{2l+1,1}(\theta) \exp\{ik(x_{2l+1}-x_1)p\}, \tag{57}$$

and

$$\bar{a}_{0,0}(\theta,p) = \sum_{l=1}^{\infty} a_{2l,0}(\theta) \exp\{ik(x_{2l}-x_0)p\}, \tag{58}$$

$$\bar{a}_{1,0}(\theta,p) = \sum_{l=0}^{\infty} a_{2l+1,0}(\theta) \exp\{ik(x_{2l+1}-x_0)p\}, \tag{59}$$

$$\bar{a}_{0,1}(\theta,p) = \sum_{l=1}^{\infty} a_{2l,1}(\theta) \exp\{ik(x_{2l}-x_1)p\}, \tag{60}$$

$$\bar{a}_{1,1}(\theta,p) = \sum_{l=1}^{\infty} a_{2l+1,1}(\theta) \exp\{ik(x_{2l+1}-x_1)p\}, \tag{61}$$

Note that according to (28),

$$V_0(\theta) = \bar{V}_{0,0}(\theta,0) + \bar{V}_{1,0}(\theta,0), \qquad V_1(\theta) = \bar{V}_{0,1}(\theta,0) + \bar{V}_{1,1}(\theta,0). \tag{62}$$

Comparing (58)–(61) with (44)–(47), obtain

$$\begin{pmatrix} \bar{a}_{0,0} & \bar{a}_{0,1} \\ \bar{a}_{1,0} & \bar{a}_{1,1} \end{pmatrix} (\theta,p) = \mathrm{C}(\sqrt{\theta^2+2p}) \tag{63}$$

As the result obtain that the values $\tilde{V}_{\alpha,\beta}(\theta, p)$ obey an ODE of the form

$$\partial_\theta \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} = \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} \cdot \mathrm{C}(\sqrt{\theta^2 + 2p}). \tag{64}$$

This is an auxiliary equation for building an OE-equation. The OE-equation is necessary for finding the coefficient of the spectral equation $\mathrm{C}(\theta)$. The spectral equation (48) can be obtained from (64) by taking $p = 0$.

9 OE–equation

Consider the equation

$$\partial_\theta \mathrm{X}(\theta) = \mathrm{X}(\theta) \cdot \mathrm{A}(\theta), \tag{65}$$

for an unknown matrix X of dimension 2×2. Matrix $\mathrm{A}(\theta)$ is the coefficient of this equation. This equation is solved along a contour γ in the complex plane of variable θ. Let t_1 and t_2 be the starting point and the ending point of this contour. Let the initial condition

$$\mathrm{X}(t_1) = \mathrm{I},$$

be set at the starting point of the contour. Here I is the unit 2×2 matrix. Introduce the notation

$$\mathrm{OE}_\gamma[A(\theta)] \equiv \mathrm{X}(t_2). \tag{66}$$

The notation OE goes from the concept of *the ordered exponential* widely accepted in quantum mechanics.

Theorem 4 *Function* $\mathrm{C}(\theta)$ *obeys the following equation*

$$\mathrm{OE}_\gamma[\mathrm{C}(\sqrt{\theta^2 + 2p})] = \mathrm{T}(p), \qquad \mathrm{Im}[p] > 0, \tag{67}$$

where

$$\mathrm{T}(p) = \begin{pmatrix} 1 & -\exp\{ikbp\} \\ -\exp\{ikap\} & 1 \end{pmatrix}, \tag{68}$$

contour γ *is the real axis passed in the negative direction (i.e. from* $+\infty$ *to* $-\infty$*). In the left-hand side of (67) variable* θ *is taken as an independent variable and* p *is taken as a parameter.*

Equation (67) will be called *the OE–equation.*

To prove this theorem it is sufficient to show that

$$\lim_{\theta\to\infty}\begin{pmatrix}\bar{V}_{0,0} & \bar{V}_{0,1}\\ \bar{V}_{1,0} & \bar{V}_{1,1}\end{pmatrix}(\theta,p)=\mathrm{I}, \tag{69}$$

$$\lim_{\theta\to-\infty}\begin{pmatrix}\bar{V}_{0,0} & \bar{V}_{0,1}\\ \bar{V}_{1,0} & \bar{V}_{1,1}\end{pmatrix}(\theta,p)=\mathrm{T}(p). \tag{70}$$

Note that $\tilde{V}_{n,n}=1$. Direct computations based on (16) show that

$$\lim_{\theta\to-\infty}\tilde{V}_{n+1,n}(\theta)=-1,\qquad \lim_{\theta\to\infty}\tilde{V}_{n+1,n}(\theta)=0.$$

The terms $\tilde{V}_{n,n}$ and $\tilde{V}_{n+1,n}$ tend to limiting values (69) and (70) provided that all other terms (i.e. $\tilde{V}_{m,n}(\theta)$, $m>n+1$) tend to zero as $|\theta|\to 0$. Let us prove the last statement.

Let us show that L^2 norm of the terms $\tilde{V}_{m,n}(\theta)$, $m>n+1$ is finite and grows with m no faster than algebraically. This will be enough to prove our statement for functions $\tilde{V}_{m,n}(\theta)$ smooth enough and for $\mathrm{Im}[p]>0$.

Consider as an example the terms $\tilde{V}_{m,0}(\theta)$, $m=2,3,\ldots$. According to Parseval's theorem, instead of norm of $\tilde{V}_{m,0}(\theta)$ one can consider the L^2-norm of the functions $v_0(x_m-0,y)$, $y<0$. Consider the propagation operators

$$\Pi_a[w](y)=\int\limits_{-\infty}^{\infty}w(y')g(a,y-y')dy',\qquad \Pi_b[w](y)=\int\limits_{-\infty}^{\infty}w(y')g(b,y-y')dy', \tag{71}$$

and the projectors

$$P_+[w](y)=w(y)h(y),\qquad P_-[w](y)=w(y)h(-y), \tag{72}$$

$$h(y)=\begin{cases}1, & y\geq 0,\\ 0, & y<0.\end{cases}$$

The key statement of the proof is that the operators Π_a,Π_b maintain the L^2-norm. This can be checked by studying these operators in the Fourier representation. Projectors P_+ and P_- do not increase the norm.

In the operator notation the following identities can be written

$$v_0(x_1-0,y)=\Pi_a\circ\delta(y),$$

$$v_0(x_2 - 0, y) = \Pi_b \circ P_+ \circ \Pi_a \circ \delta(y),$$
$$v_0(x_3 - 0, y) = \Pi_a \circ P_+ \circ \Pi_b \circ P_+ \circ \Pi_a \circ \delta(y),$$
$$v_0(x_{2l+1} - 0, y) = \Pi_a \circ (P_+ \circ \Pi_b \circ P_+ \circ \Pi_a)^l \circ \delta(y),$$
$$v_0(x_{2l+2} - 0, y) = \Pi_b \circ P_+ \circ \Pi_a \circ (P_+ \circ \Pi_b \circ P_+ \circ \Pi_a)^l \circ \delta(y).$$

For $l \geq 0$ introduce the functions

$$s_{2l+1}(y) \equiv (P_+ \circ \Pi_b \circ P_+ \circ \Pi_a)^l \circ \delta(y),$$
$$s_{2l+2}(y) \equiv P_+ \circ \Pi_a \circ (P_+ \circ \Pi_b \circ P_+ \circ \Pi_a)^l \circ \delta(y).$$

Obviously,

$$h(-y)\, v_0(x_{2l+1} - 0, y) = P_- \circ \Pi_a \circ s_{2l+1}(y),$$
$$h(-y)\, v_0(x_{2l+2} - 0, y) = P_- \circ \Pi_b \circ s_{2l+2}(y).$$

Let us show that for $m \geq 2$

$$s_m(y) = P_+ \circ g(x_{m-1}, y) + s'_m(y), \tag{73}$$

where s'_m is a function with finite L^2-norm. Proof this by induction. The base (i.e. the case $m = 2$) is obvious. Then,

$$s_{m+1}(y) = P_+ \circ \Pi \circ s_m(y) - P_+ \circ \Pi \circ P_- \circ g(x_{m-1}, y),$$

where Π stands for Π_a or Π_b. The first term in the right has finite norm due to the properties of the operators Π and P_+. The second term can be computed explicitly. Its norm is finite. Also, this reasoning gives estimation of growth for s'_m. Then,

$$h(-y)\, v_0(x_m - 0, y) = P_- \circ \Pi \circ s_m(y) + P_- \circ \Pi \circ P_+ \circ g(x_{m-1}, y).$$

The first term in the right has the norm not greater than $\|s_m(y)\|_2$. The second term can be computed, and its norm is finite. $\square$

Physically, the terms $\tilde{V}_{m,m}$ and $\tilde{V}_{m+1,m}$ are different from other ones. The difference is that all other terms are wave fields diffracted at least once, while the terms $\tilde{V}_{m,m}$ and $\tilde{V}_{m+1,m}$ contain rays traveling directly from the source to the observation point.

Let us make an important notice. Continue (69) to $p = 0$ by continuity. According to (62) this gives

$$V_0(+\infty) = 1, \qquad V_1(+\infty) = 1. \tag{74}$$

These identities can be taken as initial conditions for the spectral equation (48).

10 Concluding remarks

We propose a new approach to solving of problems of diffraction by a periodic system of branch points of the type described above. The following steps are performed within this approach.

1. OE-equation (67) with the right-hand side (68) is solved (for example, numerically) and matrix function $\mathrm{C}(\theta)$ is found.

2. Coefficient $\mathrm{C}(\theta)$ found on the previous step is substituted into the spectral equation (48). The spectral equation is solved with boundary condition (74). Functions $V_0(\theta)$, $V_1(\theta)$ are found.

3. Functions $V_0(\theta)$, $V_1(\theta)$ are substituted into the embedding formula (36), and the scattering coefficients R_n are found.

Further investigations related to the topic may include development of numerical methods to solve the OE-equation, studies of the asymptotical and analytical properties of the OE-equation.

The work is supported by the grants of Russian Federation Government 11.G34.31.0066, “Scientific schools” grant 2631.2012.2, RFBR grant 12-02-00114. The author is grateful to the participants of the seminar on Wave diffraction and propagation having place in PDMI RAS institute for lively discussion of the work and valuable notes carefully taken into account by the author.

Appendix. The simplest problem for the parabolic equation on a branched surface

Let the surface have structure shown in Fig. 4. The surface is composed of two sheets cut along the half-lines $y < 0$, $x = 0$. The shores of the cuts are attached to each other as it is shown in the figure. One of the sheets will be called main, and another one will be called auxiliary. Everywhere on the surface except the point $(0, 0)$ the parabolic equation (4) is valid. The incident field having form (9) falls from infinity along the main sheet. There is no incident field on the auxiliary sheet, therefore the total field is equal to zero for $x < 0$ on the auxiliary sheet. Thus, the field on the main and on the auxiliary sheet (u_{m} and u_{a}, respectively) are as follows for $x < 0$:

$$u_{\mathrm{m}} = u_{\mathrm{in}}, \qquad u_{\mathrm{a}} = 0. \tag{75}$$

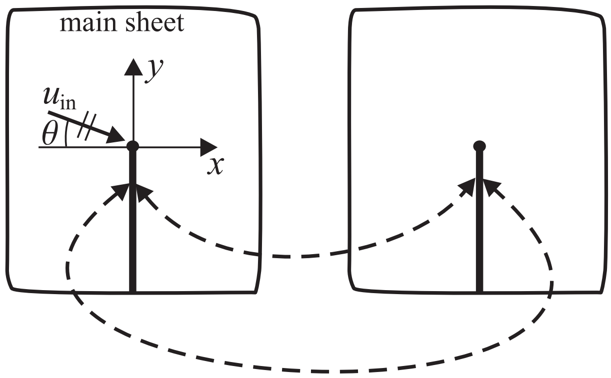

Figure 4: Two-sheet branched surface for the model problem

Compute the field to the right of the branch point, i.e. for $x > 0$. On the main sheet for $x = +0$ the field has form

$$u_{\text{m}}(+0, y) = \begin{cases} u_{\text{in}}(0, y), & y > 0, \\ 0, & y < 0 \end{cases} \tag{76}$$

This formula takes into account that for $y > 0$ the field is a continuation from the main sheet, and for $y < 0$ it is the continuation from the auxiliary sheet. The field on the auxiliary sheet on the line $x = +0$ has form

$$u_{\text{a}}(+0, y) = \begin{cases} 0, & y > 0, \\ u_{\text{in}}(0, y), & y < 0. \end{cases} \tag{77}$$

Now we can use formula (5) for finding the fields for $x > 0$:

$$u_{\text{m}}(x, y) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \exp\left\{-ik\left(\frac{\theta_{\text{in}}^2}{2}x + \theta_{\text{in}}y\right)\right\} F\left(-(y + \theta_{\text{in}}x)\sqrt{\frac{k}{2x}}\right), \tag{78}$$

$$u_{\text{a}}(x, y) = \exp\left\{-ik\left(\frac{\theta_{\text{in}}^2}{2}x + \theta_{\text{in}}y\right)\right\} \left[1 - \frac{e^{-i\pi/4}}{\sqrt{\pi}} F\left(-(y + \theta_{\text{in}}x)\sqrt{\frac{k}{2x}}\right)\right], \tag{79}$$

$$F(\xi) = \int_{\xi}^{\infty} e^{i\tau^2} d\tau.$$

The expressions obtained above are the standard representations for the wave field scattered at small angles.

The problem considered here was studied on a two-sheet surface, however the field on the main sheet can be found without solving the problem on the auxiliary sheet. For these, one can consider the field only on the main sheet cut along the half-line $y < 0$, $x = 0$ with condition (7) on the right shore.

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