Diffraction by a grating consisting of absorbing screens of different height. New equations

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Abstract

The problem of diffraction of a plane wave by a grating consisting of absolutely absorbing screens of different height is studied. It is supposed that the angle of incidence is small. The problem is considered in the parabolic approximation.

Edge Green functions are introduced. The embedding formula and the spectral equation for the edge Green functions are derived. The OE-equation for the coefficients of the spectral equation is constructed. The latter is solved numerically. The evolution equation describing the dependence of the edge Green functions on the geometrical parameter (the height of the screens) is proven. Using this equation, the asymptotics of the reflection coefficient is calculated for the principal mode.

1 Introduction

The problem of scattering of a waveguide mode by an open end of a waveguide with acoustically hard thin walls was studied by L. A. Weinstein [1]. The problem was solved by applying Wiener–Hopf method [2]. The coefficients of scattering of the incident mode (the mode moving towards the open end of the waveguide) into the reflected modes (the modes moving back from the open end) were obtained. This problem is of a significant practical value. It helps to describe oscillations in a Fabry–Perot resonator, which is considering as a part of a waveguide in this case. A wave process in such a resonator is represented as successive reflections by open ends of the waveguide.

The key result obtained by L. A. Weinstein is as follows. A high-frequency mode (i. e. a mode with wavelength much smaller than the width of the waveguide) close to the cut-off frequency (i. e. composed of partial waves traveling almost perpendicular to the axis of the waveguide) has reflection coefficient close to -1. This result is quite surprising because the open end of the waveguide has no reflective structures. The coefficient close to 0 would be rather expected due to radiation of the wave energy into the open space. The fact that the reflection coefficient is close to -1 explains high Q-factor of Fabry–Perot resonators in the absence of focusing elements.

The problem of high-frequency scattering close to the cut-off frequency is quite complicated since the problem cannot be solved in terms of the geometrical theory of diffraction: the field contains multiple penumbra components. Here we mean that a primary penumbral field being scattered by an edge generates a secondary penumbra, etc.

L. A. Weinstein noticed that by using the reflection principle one can reformulate the problem of diffraction by a waveguide outlet as a problem of scattering on a branched surface by a periodic diffraction grating formed by branch points. This formulation seems to be more convenient due to the possibility to analyze this problem using the formalism of the edge Green functions. One can use a parabolic approximation in the high-frequency case. In the parabolic approximation one can can study a grating composed of perfectly absorbing screens instead of the branched surface.

Calculation of the Q-factor of high-frequency modes in open rectangular resonators leads to the problems similar to the Weinstein's one [3]. Modes in such resonators can be represented as families of rays propagating along closed billiard trajectories. The losses of the modes are due to diffraction. One can associate periodic gratings having more complex periods with such problems. Unfortunately, these problems cannot be solved using the classic Wiener–Hopf method. Particularly the problem solved here can be reduced to a matrix Wiener–Hopf equation. Solution of this equation is unknown.

A new technique of solving Weinstein's problems has been proposed by the authors [4, 5]. This technique does not rely on the Wiener–Hopf method. The main aim of the current work is to develop this technique further and to construct efficient numerical and asymptotical methods based on it.

A physical problem that motivates the present study is the problem of radiation by an open end of a plane waveguide with walls of different height (see Fig. 1a)). A stationary acoustical problem is considered, the walls of the waveguide are supposed to be perfectly reflecting (acoustically hard) and infinitely thin. The incident waveguide mode is supposed to be close to its cut-off frequency, i. e. the partial Brillouin waves propagate almost perpendicularly to the axis of the waveguide. It is necessary to find the coefficients of scattering into the reflected modes.

In the language of Fabry–Perot resonators this problem is related to a res-

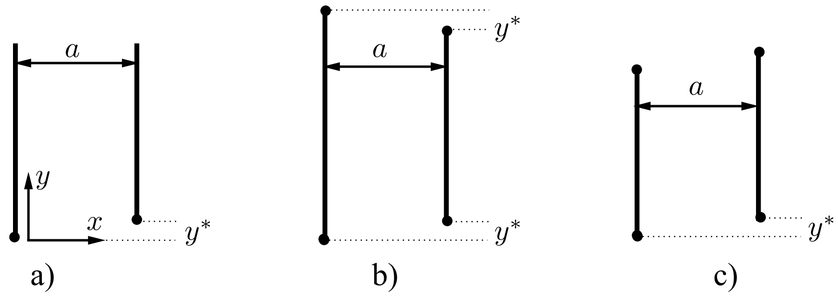

Fig. 1: Geometry of studying systems : a) Open waveguide, b),c) Fabry-Perot resonators

onator with parallel mirrors of different height or with staggered mirrors(see Fig. 1b), c)). This problem was studied in [1] using Fox–Li integral equations. The integral equations were reduced to a functional equation for the reflections coefficients and in the approximation of small angles some rough estimations of the mirror reflection coefficient have been made. L. A. Weinstein did not obtain any asymptotic expression for the scattering coefficients.

Using the reflection method [5] this problem can be reduced to the problem of scattering by a periodic grating of perfectly absorbing screens (see Fig. 2). In the high-frequency case the last problem can be studied in the parabolic approximation. The reflection coefficients for the initial problem correspond to the scattering coefficients for the grating.

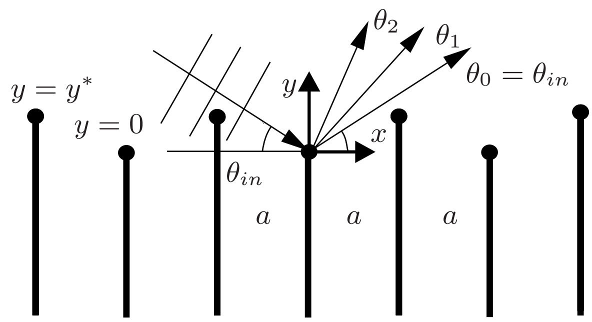

Fig. 2: Geometry of the periodic grating

The embedding formula is derived for the problem in the current paper. This formula expresses the scattering coefficients in terms of the directivities of the edge Green functions.

Then the spectral equation is derived, which is the ordinary differential equation for the directivities of the edge Green functions. An OE-equation for the coefficient of this differential equation is constructed. The OE-equation is solved numerically. This part of the paper is close to [5]. Then the evolution equation describing the dependence of the edge Green functions and of the coefficients of the spectral equation on the geometrical parameter y^* is derived. Asymptotic expressions for the directivities of the edge Green functions and asymptotics of the reflection coefficient are built by using the evolution equation.

We use the results obtained in [5] where it is possible.

2 Problem formulation

Consider Cartesian plane (x, y). Let the field variable u satisfy the parabolic diffraction equation:

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik} \frac{\partial^2}{\partial y^2} \right) u = 0. \tag{1}$$

Perfectly absorbing screens are placed along the lines $x = x_m$, $y < y_m$, $m \in \mathbb{Z}$,

$$x_m = am, \tag{2}$$

$$y_m = \left\{ \begin{array}{ll} 0 & m \text{ even}, \\ y^* & m \text{ odd}. \end{array} \right. \tag{3}$$

The field on the screens is discontinuous. Perfect absorption means that the following boundary conditions are satisfied on the right sides of the screens ($x = x_m + 0$, $y < y_m$):

$$u(x_m + 0, y) = 0. \tag{4}$$

Such boundary conditions can be imposed in the framework of the parabolic approximation since the governing equation (1) is of order one with respect to x.

Short wave case is considered:

$$ka \gg 1. \tag{5}$$

The total field is represented as

$$u = u_{\rm in} + u_{\rm sc},$$

where $u_{\rm in}$ is the incident plane wave

$$u_{\rm in}(x, y) = \exp \left\{ -ikx \frac{\theta_{\rm in}^2}{2} - iky\theta_{\rm in} \right\}, \tag{6}$$

and $u_{\rm sc}$ is the scattered field. Angle $\theta_{\rm in}$ is small:

$$\theta_{\rm in} \ll 1. \qquad (7)$$

Boundary conditions for $u_{\rm sc}$ are as follows:

$$u_{\rm sc}(x_m + 0, y) = -u_{\rm in}(x_m, y), \qquad y < y_m. \qquad (8)$$

Problem formulation should be supplemented with conditions at the ends of the screens and with radiation condition. Near the end points (x_m, y_m) the total field should be bounded. Boundedness of the field guarantees that there are no sources at the end points.

Due to the radiation condition the scattered field should not contain components coming from the area of large $|y|$ (this fact is taken into account when the Fourier expansions are constructed).

Geometrical period of the grating along the x axis is equal to $2a$. The incident wave has property:

$$u_{\rm in}(x + 2a, y) = \chi u_{\rm in}(x, y), \qquad \chi = \exp\{-ika\theta_{\rm in}^2\}.$$

Due to Floquet theory the scattered field should have the same property:

$$u_{\rm sc}(x + 2a, y) = \chi u_{\rm sc}(x, y). \qquad (9)$$

So, in the upper half-plane $(y > \max(0, y^*))$ the scattered field can be expanded as series:

$$u_{\rm sc} = \sum_{m=-\infty}^{\infty} R_m \exp\left\{-ikx\frac{\theta_m^2}{2} + iky\theta_m\right\}, \qquad (10)$$

$$\theta_m = \sqrt{\theta_{\rm in}^2 + \frac{2\pi m}{ka}}. \qquad (11)$$

The values of the square root are chosen positive real or positive imaginary in consistence with radiation condition. Positive real θ_m correspond to diffractions orders, while positive imaginary θ_m correspond to inhomogeneous waves existing only in the near field. Note that $\theta_0 = \theta_{\rm in}$. Our aim is to find scattering coefficients R_m.

The problem contains four geometrical parameters: k, a, y^*, $\theta_{\rm in}$. Due to the structure of the parabolic equation the problem depends (up to scaling) on two dimensionless parameters: $\theta_{\rm in}\sqrt{ka}$ and $y^*\sqrt{k/a}$. The first one indicates the domain of small angles by inequality

$$\theta_{\rm in}\sqrt{ka} \ll 1.$$

Note that this condition yields $R_0 \approx -1$ for the Weinstein's problem. Note also that the values $1/\sqrt{ka}$ and $\theta_1 - \theta_0$ are of the same order according to (11).

The second dimensionless parameter represents the ratio of the height mismatch y^* to the size of the first Fresnel zone $\sqrt{a/k}$. The size of the first Fresnel zone (not the wavelength!) is an important spatial scale in the y direction.

Thus

$$y^* \sqrt{k/a} \ll 1$$

corresponds to small difference of the screen heights.

3 Fox–Li integral equations and matrix Wiener–Hopf problem

Consider an arbitrary strip $X_1 < x < X_2$ without screens. It is known that parabolic equation (1) has an explicit solution in this strip:

$$u(x, y) = \int_{-\infty}^{\infty} u(X_1, y') g(x - X_1, y - y') dy', \tag{12}$$

where

$$g(x, y) = \begin{cases} k^{1/2} (2\pi x)^{-1/2} \exp\{iky^2/(2x) - i\pi/4\}, & x > 0, \\ 0, & x < 0. \end{cases} \tag{13}$$

is the Green function of the entire plane. Using this formula it is easy to obtain a system of integral equations for the scattered field u_{sc} on the half-lines $x = x_0(= 0)$, $y > y_0(= 0)$ и $x = x_1(= a)$, $y > y_1(= y^*)$:

$$u_{\text{sc}}(a, y) = \int_{0}^{\infty} u_{\text{sc}}(0, y') g(a, y - y') dy' - \int_{-\infty}^{0} u_{\text{in}}(0, y') g(a, y - y') dy', \qquad y > y^*, \tag{14}$$

$$\chi u_{\text{sc}}(0, y) = \int_{y^*}^{\infty} u_{\text{sc}}(a, y') g(a, y - y') dy' - \int_{-\infty}^{y^*} u_{\text{in}}(a, y') g(a, y - y') dy', \qquad y > 0. \tag{15}$$

Boundary conditions (8) are taken into account here. Also, the Floquet relation (9) is used in derivation of the second equation.

Equations (14) and (15) form the system of Fox–Li integral equations for the considered problem. Apparently, this system can be solved by iterations. Such a solution is not however helpful for finding the coefficients R_m due to slow convergence of the iterative series. Moreover, the iterative series does not enable one to establish asymptotic properties of the scattering coefficients. That is why we propose another method for solving this problem (the method of embedding formula, spectral equation and OE-equation).

A matrix Wiener–Hopf equation can be constructed from system (14), (15). Introduce Fourier transforms for the unknown functions:

$$\hat{U}_0(\xi) = \int_0^\infty u_{\rm sc}(0, y) e^{-i\xi y} dy, \tag{16}$$

$$\hat{U}_1(\xi) = \int_{y^*}^\infty u_{\rm sc}(a, y) e^{-i\xi(y-y^*)} dy. \tag{17}$$

Also, introduce auxiliary funcitons

$$\hat{W}_0(\xi) = \int_{-\infty}^0 u_{\rm sc}(-0, y) e^{-i\xi y} dy, \tag{18}$$

$$\hat{W}_1(\xi) = \int_{-\infty}^{y^*} u_{\rm sc}(a-0, y) e^{-i\xi(y-y^*)} dy. \tag{19}$$

Rewrite the system (14), (15) as follows:

$$\begin{pmatrix} \hat{W}_0 \\ \hat{W}_1 \end{pmatrix} + {\rm K} \begin{pmatrix} \hat{U}_0 \\ \hat{U}_1 \end{pmatrix} = {\rm r}, \tag{20}$$

where

$${\rm K}(\xi) = \begin{pmatrix} 1 & -\exp\{ika\theta_{\rm in}^2 - i\xi y^* - ia\xi^2/(2k)\} \\ -\exp\{i\xi y^* - ia\xi^2/(2k)\} & 1 \end{pmatrix}, \tag{21}$$

$${\rm r}(\xi) = \frac{-i}{\xi + k\theta_{\rm in}} \begin{pmatrix} \exp\{i\xi y^* - ia\xi^2/(2k)\} \\ \exp\{ika\theta_{\rm in}^2 - i\xi y^* - ia\xi^2/(2k)\} \end{pmatrix}. \tag{22}$$

Unknowns $\hat{U}_0$, $\hat{U}_1$ in (20) should be regular and decreasing functions in the lower half-plane of the complex variable ξ. Functions $\hat{W}_0$, $\hat{W}_1$ should be regular and decrease in the upper half-plane.

Wiener–Hopf problem derived here is a matrix one, and the authors are not aware of its solution.

4 Edge Green functions

The main idea of "embedding" is to reduce the initial problem to two (in the current case) simpler auxiliary problems. Introduce the edge Green functions v_0 and v_1 as solutions of the following inhomogeneous parabolic equations:

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right) v_0 = \delta\,(x - 0, y)\,, \tag{23}$$

and

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right) v_1 = \delta\,(x - a - 0, y - y^*) \tag{24}$$

on the plane with perfectly absorbing screens described earlier. One can see that the edge Green functions are generated by point sources. The sources are shown in Fig. 3. Arguments $x - 0$ и $x - a - 0$ mean that the sources are

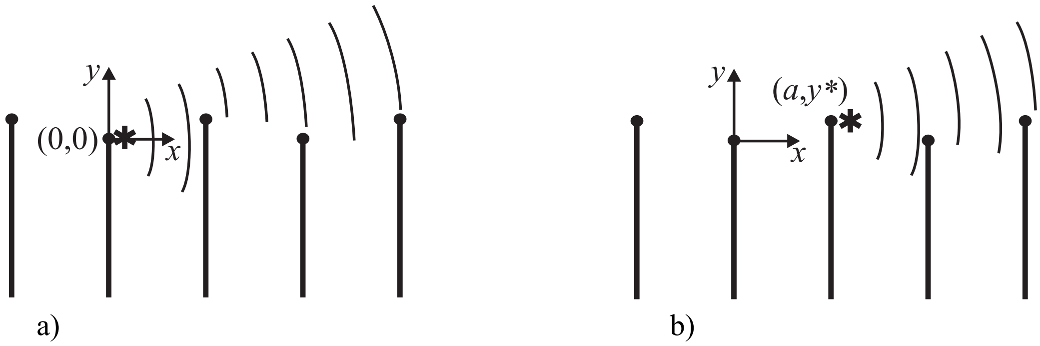

Fig. 3: Problems for the edge Green functions: case a) corresponds to v_0, case b) corresponds to v_1.

placed to the right of $(0,0)$ and (a, y^*). This lay-out guarantees that function $v_0(x,y)$ is equal to $g(x,y)$ in the area $0 < x < a$ and function $v_1(x,y)$ is equal to $g(x-a, y-y^*)$ in the area $a < x < 2a$. By using geometrical periodicity introduce edge Green functions v_n for any $n \in \mathbb{Z}$:

$$v_{m+2n}(x,y) = v_m(x - 2an, y), \qquad m = 0, 1.$$

Define the directivities V_m of the edge Green functions v_m as coefficients of the leading terms of the asymptotic expansions:

$$v_m(x,y) = g(x - x_m, y - y_m) V_m \left(\frac{y - y_m}{x - x_m}\right) + o\left((x - x_m)^{-1/2}\right), \quad m = 0, 1. \tag{25}$$

The parabolic equation and formula (12) enables one to construct a formal representation for $v_m(x, y)$. Obviously,

$$v_m(x, y) = 0 \qquad \text{for } x < x_m. \tag{26}$$

Then

$$v_m(x, y) = g(x - x_m, y - y_m) \qquad \text{for } x_m < x < x_{m+1}, \tag{27}$$

$$v_m(x, y) = \int_{y_n}^{\infty} v_m(x_n - 0, y')g(x - x_n, y - y')dy' \quad \text{for } x_n < x < x_{n+1}. \tag{28}$$

Thus, the expression for $v_m(x, y)$ in the strip $x_n < x < x_{n+1}$ contains $n - m$ nested integrals. However, such a solution is not useful for calculation of the directivities $V_m(\theta)$.

Introduce the values that play an important role below, namely the edge values of the edge Green functions:

$$z_{m,n} = \lim_{x \to x_n - 0} v_m(x_n, y_n) \qquad m < n. \tag{29}$$

The limiting procedure is necessary because solution of the parabolic equation obtained by (28) is generally discontinuous at the point (x_n, y_n). The left limit is taken due to the continuity of the field on the left side of the screen.

Prove an important formula that will be used in derivation of the spectral and evolution equation. Consider a more general problem. Let the ends of the absorbing screens have arbitrary coordinates (x_m, y_m) (i. e. we discard conditions (2), (3)). By analogy, introduce edge Green functions for this problem. Obviously, edge Green functions are still expressed by formulae (26), (27), (28). Consider a family of such problems indexed by parameter α. Let values y_m in this family be smooth enough functions of α. We denote the edge Green functions of this family by $v_m(\alpha, x, y)$ and the edge values of the edge Green functions by $z_{m,n}(\alpha)$ (by analogy with (29)).

Theorem 1 *The following formula is valid*

$$\frac{\partial v_m(\alpha, x, y)}{\partial \alpha} =$$

$$\sum_{n=m+1}^{\infty} \left(\frac{\partial y_m}{\partial \alpha} - \frac{\partial y_n}{\partial \alpha} \right) z_{m,n}(\alpha) v_n(\alpha, x, y) - \frac{\partial y_m}{\partial \alpha} \frac{\partial v_m(\alpha, x, y)}{\partial y}. \tag{30}$$

Proof Introduce for each x an integer parameter $n(x)$, such that $x_{n(x)} < x \leq x_{n(x)} + 1$. The theorem can be proven easily by induction with respect to $n(x) - m$. Formulae (27), (28) should be used.

Formula (27) works for $n(x) = m$, and this case is easy to verify (there is no summation in the right side of (30)). This is the base of induction. Make the inductive step. Suppose that the statement (30) is valid for $n(x) - m = l - 1$. Then for $n(x) - m = l$ we have

$$\frac{\partial v_m}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int\limits_{y_{m+l}}^{\infty} v_m(x_{m+l} - 0, y') g(x - x_{m+l}, y - y') dy' =$$

$$-\frac{\partial y_{m+l}}{\partial \alpha} v_m(x_{m+l} - 0, y_{m+l}) g(x - x_{m+l}, y - y_{m+l})$$

$$+ \int\limits_{y_{m+l}}^{\infty} \frac{\partial v_m(x_{m+l} - 0, y')}{\partial \alpha} g(x - x_{m+l}, y - y') dy'.$$

The derivative in the last integral can be transformed using (30) (here we use the induction conjecture). Taking into account that

$$\int\limits_{y_{m+l}}^{\infty} v_n(x_{m+l} - 0, y') g(x - x_{m+l}, y - y') dy' = v_n(x, y),$$

and also

$$\int\limits_{y_{m+l}}^{\infty} \frac{\partial v_m(\alpha, x_{m+l} - 0, y')}{\partial \alpha} g(x - x_{m+l}, y - y') dy' =$$

$$\left(-z_{m,m+l}(\alpha)\, v_{m+l}(\alpha, x, y) + \frac{\partial v_m(\alpha, x, y)}{\partial y} \right) \frac{\partial y_m}{\partial \alpha},$$

we obtain (30) for $n(x) - m = l$. $\square$

5 Embedding formula

An embedding formula links the scattering coefficients of the initial problem with the directivities of the edge Green functions. The formula is given by the following theorem:

Theorem 2 *For m corresponding to real positive θ_m the following identity is valid:*

$$R_m = \frac{\sum_{n=0}^{1} V_n(\theta_{\text{in}}) V_n(\theta_m) \exp\{ikx_n(\theta_m^2 - \theta_{\text{in}}^2)/2 - iky_n(\theta_m - \theta_{\text{in}})\}}{2ika\theta_m(\theta_{\text{in}} + \theta_m)}. \quad (31)$$

Sketch of the proof The proof of this theorem is very close to the proof of corresponding statement from [5]. Here we list only the main steps of the derivation of the embedding formula.

Apply differential operator

$$H \equiv \frac{\partial}{\partial y} + iky\theta_{\rm in} \qquad (32)$$

to the total field $u(x, y)$. It was shown in [5] that the field $w(x, y) = H[u](x, y)$ satisfies the parabolic equation, the boundary conditions on the screens, and the radiation condition, but it has sources with amplitudes $u_m(x_m, y_m)$ at the vertices (x_m, y_m). Thus, using uniqueness of the solution one can represent $w(x, y)$ as a superposition of the edge Green functions v_n:

$$H[u](x, y) = \sum_{n=-\infty}^{\infty} u(x_n - 0, y_n) v_n(x, y). \qquad (33)$$

The values of the field $u(x_m - 0, y_m)$ at the end points of the screens can be expressed through the edge Green functions with the help of the reciprocity theorem for the parabolic equation. The expression is as follows:

$$u(x_m - 0, y_m) = \exp\left\{-ikx_m\theta_{\rm in}^2/2 + iky_m\theta_{\rm in}\right\} V_m(\theta_{\rm in}). \qquad (34)$$

Substituting this expression into (33) and considering the directivities of the fields we obtain (31). □

(31) takes the simplest form for the case of mirror reflection, i. e. for R_0:

$$R_0 = \frac{(V_0(\theta_{\rm in}))^2 + (V_1(\theta_m))^2}{4ika\theta_{\rm in}^2}. \qquad (35)$$

Thus, in order to solve the original problem we need to find the directivities V_0, V_1.

6 Spectral equation

Theorem 3 *A row–vector comprised of directivities* $(V_0(\theta), V_1(\theta))$ *satisfies the ordinary differential equation*

$$\frac{d}{d\theta}(V_0, V_1) = (V_0, V_1)\Pi(\theta y^*)\mathrm{C}(\theta, y^*)\Pi^{-1}(\theta y^*), \qquad (36)$$

with the initial condition

$$V_0(+\infty) = 1, \quad V_1(+\infty) = 1, \qquad (37)$$

where

$$\Pi(\theta y^*) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\{-iky^*\theta\} \end{pmatrix}, \tag{38}$$

$$\mathrm{C}(\theta, y^*) = \begin{pmatrix} C_{0,0} & C_{0,1} \\ C_{1,0} & C_{1,1} \end{pmatrix}, \tag{39}$$

$$C_{0,0}(\theta, y^*) = \sum_{m=1}^{\infty} (x_{2m} - x_0) z_{0,2m} \exp\left\{ik(x_{2m} - x_0)\theta^2/2\right\}, \tag{40}$$

$$C_{1,0}(\theta, y^*) = \sum_{m=0}^{\infty} (x_{2m+1} - x_0) z_{0,2m+1} \exp\left\{ik(x_{2m+1} - x_0)\theta^2/2\right\}, \tag{41}$$

$$C_{0,1}(\theta, y^*) = \sum_{m=0}^{\infty} (x_{2m+2} - x_1) z_{1,2m+2} \exp\left\{ik(x_{2m+2} - x_1)\theta^2/2\right\}, \tag{42}$$

$$C_{1,1}(\theta, y^*) = \sum_{m=1}^{\infty} (x_{2m+1} - x_1) z_{1,2m+1} \exp\left\{ik(x_{2m+1} - x_1)\theta^2/2\right\}. \tag{43}$$

Proof Derivation of the spectral equation is very close to the derivation of the corresponding equation from [5]. Here we use (30) for this. Fix value of m as 0 or 1. Introduce a family of problems having the same values x_n as in (2) but different values y'_m:

$$y'_n = y_n + \alpha(x_n - x_m).$$

Due to the fact that operator

$$K_m = (x - x_m)\frac{\partial}{\partial y} - iky \tag{44}$$

is the symmetry operator of the parabolic equation in the classical sense, it is possible to construct solutions for all α if a solution for $\alpha = 0$ is known:

$$v_n(\alpha, x, y) = v_n(x, y - \alpha(x - x_m)) \exp\{ik\alpha(y - y_m) + ik\alpha^2(x - x_m)/2\}.$$

Besides, it follows from the last equality that

$$K_m[v_m] = -\frac{\partial v_m}{\partial \alpha}.$$

Apply formula (30) to the left-hand side of the last equality and put $\alpha = 0$. It yields

$$K_m[v_m](x, y) = \sum_{n=m+1}^{\infty} (x_n - x_m) z_{m,n} v_n(x, y).$$

Consider the directivities of the fields in the right and in the left. As the result, obtain (36). □

In [5] it was shown that the spectral equation can be formulated in a more general way. A similar statement is valid in our case.

Theorem 4 *The following ordinary differential equation is valid:*

$$\frac{\partial}{\partial\theta}\begin{pmatrix}\bar{V}_{0,0} & \bar{V}_{0,1}\\ \bar{V}_{1,0} & \bar{V}_{1,1}\end{pmatrix} = \begin{pmatrix}\bar{V}_{0,0} & \bar{V}_{0,1}\\ \bar{V}_{1,0} & \bar{V}_{1,1}\end{pmatrix}\Pi(\theta y^*)\mathrm{C}(\sqrt{\theta^2+2p})\Pi^{-1}(\theta y^*), \tag{45}$$

where

$$\bar{V}_{0,0}(\theta,p) = \sum_{l=0}^{\infty}\tilde{V}_{2l,0}(\theta)\exp\left\{ik(x_{2l}-x_0)p\right\}, \tag{46}$$

$$\bar{V}_{1,0}(\theta,p) = \sum_{l=0}^{\infty}\tilde{V}_{2l+1,0}(\theta)\exp\left\{ik(x_{2l+1}-x_0)p\right\}, \tag{47}$$

$$\bar{V}_{0,1}(\theta,p) = \sum_{l=1}^{\infty}\tilde{V}_{2l,1}(\theta)\exp\left\{ik(x_{2l}-x_1)p\right\}, \tag{48}$$

$$\bar{V}_{1,1}(\theta,p) = \sum_{l=0}^{\infty}\tilde{V}_{2l+1,1}(\theta)\exp\left\{ik(x_{2l+1}-x_1)p\right\}, \tag{49}$$

$$\begin{gathered}\tilde{V}_{m,n}(\theta) = \\ -\exp\{ik(x_m-x_n)\theta^2/2\}\int\limits_{-\infty}^{y_m} v_n(x_m-0,y)\exp\{-ik\theta(y-y_n)\}dy, \qquad m>n, \\ \tilde{V}_{m,m}(\theta)\equiv 1, \\ \tilde{V}_{m,n}(\theta)\equiv 0, \qquad m<n,\end{gathered} \tag{50}$$

and p has an arbitrary complex value with a nonnegative imaginary part.

Values $\bar{V}_{0,0}(\theta,p)$ are directly connected with the directivities:

$$V_0(\theta) = \bar{V}_{0,0}(\theta,0)+\bar{V}_{1,0}(\theta,0), \quad V_1(\theta) = \bar{V}_{0,1}(\theta,0)+\bar{V}_{1,1}(\theta,0). \tag{51}$$

These relations are close to (28) from [5]. Then,

$$V_m(\theta) = 1-\sum_{n=m+1}^{\infty}\exp\left\{ik\frac{\theta^2}{2}(x_n-x_m)\right\}\int\limits_{-\infty}^{y_n} v_m(x_n-0,y)\exp\{ik\theta(y_m-y)\}dy. \tag{52}$$

The values introduced here can be interpreted as follows. Formula (52) expresses the directivities of the edge Green functions through the integrals of the field on the surface of the screens. The first term that does not contain an integral (the unity) represents a contribution from the point source. Contributions of different screens can be treated separately. These contributions are closely related to the directivities on the branched surface introduced in [5]. Parameter p has no physical meaning, it is the variable of the discrete Fourier transform with respect to the number of the screen. Spectral equation (36) follows from (45) when $p = 0$.

Thus when matrix $\mathrm{C}(\theta, y^*)$ is constructed it will be possible to determine V_0, V_1 as a solution of the spectral equation (36) with the initial condition (37).

Formulae (40), (41), (42), (43) cannot be used for calculation of the coefficient $\mathrm{C}(\theta)$ due to slow convergence of the series there.

7 OE–equation

Introduce a notation for a solution of a matrix ordinary differential equation. Consider an equation of the first order:

$$\frac{d}{d\tau}\mathrm{X}(\tau) = \mathrm{X}(\tau)\mathrm{K}(\tau). \tag{53}$$

Solve it along contour h with start point τ_1 and end point τ_2. Also set the following initial condition:

$$\mathrm{X}(\tau_1) = \mathrm{I}.$$

By definition

$$\mathrm{OE}_h\,[\mathrm{K}(\tau)\,d\tau] \equiv \mathrm{X}(\tau_2). \tag{54}$$

Theorem 5 *Function* $\mathrm{C}(\theta, y^*)$ *satisfies the following equation:*

$$\mathrm{OE}_\gamma[\Pi(\theta y^*)\,\mathrm{C}(\sqrt{\theta^2 + 2p})\,\Pi^{-1}(\theta y^*)d\theta] = \mathrm{T}(p) \tag{55}$$

for any p *(*$\mathrm{Im}[p] \geq 0$*),*

$$\mathrm{T}(p) = \begin{pmatrix} 1 & -\exp\{ikap\} \\ -\exp\{ikap\} & 1 \end{pmatrix}, \tag{56}$$

contour γ *is the real axis passed in the negative direction.* θ *in the left-hand side is an independent variable, while* p *is a parameter.*

The proof of this theorem is similar to the proof of the corresponding statement in [5].

8 Evolution equation of the first kind

Introduce matrix

$$\mathrm{D}(\theta, y^*) = \begin{pmatrix} D_{0,0} & D_{0,1} \\ D_{1,0} & D_{1,1} \end{pmatrix}, \tag{57}$$

$$D_{0,0}(\theta, y^*) = \sum_{m=1}^{\infty} z_{0,2m} \exp\left\{ik(x_{2m} - x_0)\theta^2/2\right\}, \tag{58}$$

$$D_{1,0}(\theta, y^*) = \sum_{m=0}^{\infty} z_{0,2m+1} \exp\left\{ik(x_{2m+1} - x_0)\theta^2/2\right\}, \tag{59}$$

$$D_{0,1}(\theta, y^*) = \sum_{m=0}^{\infty} z_{1,2m+2} \exp\left\{ik(x_{2m+2} - x_1)\theta^2/2\right\}, \tag{60}$$

$$D_{1,1}(\theta, y^*) = \sum_{m=1}^{\infty} z_{1,2m+1} \exp\left\{ik(x_{2m+1} - x_1)\theta^2/2\right\}. \tag{61}$$

Obviously,

$$\mathrm{C} = \frac{1}{ik\theta}\frac{\partial \mathrm{D}}{\partial \theta}. \tag{62}$$

Theorem 6 *Row–vector comprised of directivities* $(V_0(\theta), V_1(\theta))$ *satisfies the following differential equation with respect to* y^**:*

$$\frac{d}{dy^*}(V_0, V_1) = (V_0, V_1)\Pi(\theta y^*)\tilde{\mathrm{D}}(\theta, y^*)\Pi^{-1}(\theta y^*), \tag{63}$$

where

$$\tilde{\mathrm{D}}(\theta, y^*) = \begin{pmatrix} 0 & D_{0,1} \\ -D_{1,0} & 0 \end{pmatrix}. \tag{64}$$

Proof To prove the theorem use the formula (30). Take y^* as α. This yields

$$\frac{\partial v_m(x,y)}{\partial y^*} = -\sum_{n=m+1} (\psi_n - \psi_m) z_{m,n} v_n(x,y) - \psi_m \frac{\partial v_m(x,y)}{\partial y},$$

where $\psi_n = 1$ for odd n and $\psi_n = 0$ for even n. Particularly,

$$\frac{\partial v_0(x,y)}{\partial y^*} = -\sum_{l=0}^{\infty} z_{0,2l+1}\, v_{2l+1}(x,y), \tag{65}$$

$$\frac{\partial v_1(x,y)}{\partial y^*} = \sum_{l=0}^{\infty} z_{1,2l+2}\, v_{2l+2}(x,y) - \frac{\partial v_1(x,y)}{\partial y}. \tag{66}$$

Consider the directivities in the previous equations:

$$\frac{\partial V_0}{\partial y^*} = -V_1 \exp\{-ik\theta y^*\} \sum_{l=0}^{\infty} z_{0,2l+1} \exp\left\{ik(x_{2l+1} - x_0)\frac{\theta^2}{2}\right\}, \quad (67)$$

$$\frac{\partial V_1}{\partial y^*} = V_0 \exp\{ik\theta y^*\} \sum_{l=0}^{\infty} z_{1,2l+2} \exp\left\{ik(x_{2l+2} - x_1)\frac{\theta^2}{2}\right\}. \quad (68)$$

The last pair of equations can be rewritten in terms of (63). □

Evolution equation can be written in more general form for values $\bar{V}_{m,n}(\theta, p)$ (as for the spectral equation).

Theorem 7 *The following ordinary differential equation is valid :*

$$\frac{\partial}{\partial y^*}\begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} = \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} \Pi(\theta y^*)\tilde{\mathrm{D}}(\sqrt{\theta^2 + 2p})\Pi^{-1}(\theta y^*). \quad (69)$$

Proof To prove this statement return to (65)–(66). Multiply the left-hand side and the right-hand side by

$$-\exp\{ik(x_m - x_n)\theta^2/2 - ik\theta(y - y_n)\}, \qquad n = 0, 1$$

and integrate them along the line $x = x_m - 0, -\infty < y < y_m$. This yields

$$\frac{\partial}{\partial y^*}\tilde{V}_{m,0}(\theta) = -\sum_{l=0}^{\infty} z_{0,2l+1} \exp\{ik(x_{2l+1} - x_0)\theta^2/2 - ik\theta y^*\}\tilde{V}_{m,2l+1}, \quad (70)$$

$$\frac{\partial}{\partial y^*}\tilde{V}_{m,1}(\theta) = \sum_{l=0}^{\infty} z_{1,2l+2} \exp\{ik(x_{2l+2} - x_1)\theta^2/2 + ik\theta y^*\}\tilde{V}_{m,2l+2}. \quad (71)$$

Introduce the values

$$a_{l,n}(\theta) = z_{n,l} \exp\{ik(x_l - x_n)\theta^2/2\}. \quad (72)$$

Equations (70)–(71) can be rewritten in the following way:

$$\frac{\partial}{\partial y^*}\tilde{V}_{m,0}(\theta) = -\sum_{l=0}^{\infty} a_{2l+1,0}\tilde{V}_{m,2l+1}(\theta) \exp\{-ik\theta y^*\}, \quad (73)$$

$$\frac{\partial}{\partial y^*}\tilde{V}_{m,1}(\theta) = \sum_{l=0}^{\infty} a_{2l+2,1}\tilde{V}_{m,2l+2}(\theta) \exp\{ik\theta y^*\}. \quad (74)$$

It is easy to notice that functions $\tilde{V}_{m,n}$ and coefficients $a_{l,n}$ satisfy the periodicity conditions

$$\tilde{V}_{m,n}(\theta) = \tilde{V}_{m+2,n+2}(\theta), \qquad a_{m,n}(\theta) = a_{m+2,n+2}(\theta).$$

Moreover, the right-hand sides of the equations are of a convolution nature. This fact enables one to apply the discrete Fourier transform to equations (73) and (74). Let us multiply (73) and (74) by $\exp\{ik(x_m - x_n)p\}$ for arbitrary p with $\mathrm{Im}[p] \geq 0$ and make a summation over all m. The result is as follows

$$\frac{\partial}{\partial y^*}\begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} = \begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} \Pi(\theta y^*) \begin{pmatrix} 0 & \bar{a}_{0,1} \\ -\bar{a}_{1,0} & 0 \end{pmatrix} \Pi^{-1}(\theta y^*), \tag{75}$$

where

$$\bar{a}_{1,0}(\theta, p) = \sum_{l=0}^{\infty} a_{2l+1,0}(\theta) \exp\{ik(x_{2l+1} - x_0)p\}, \tag{76}$$

$$\bar{a}_{0,1}(\theta, p) = \sum_{l=0}^{\infty} a_{2l+2,1}(\theta) \exp\{ik(x_{2l+2} - x_1)p\}. \tag{77}$$

Besides, by comparing (59)–(60) with (76)–(77) we get the identity

$$\begin{pmatrix} 0 & \bar{a}_{0,1} \\ -\bar{a}_{1,0} & 0 \end{pmatrix} (\theta, p) = \tilde{\mathrm{D}}(\sqrt{\theta^2 + 2p}). \tag{78}$$

□

Equation (63) also follows from this result.

9 Evolution equation of the second kind

Equation (79) will be called the evolution equation of the second kind.

Theorem 8 *Matrix* $\mathrm{D}(\theta, y^*)$ *satisfies the following equation:*

$$\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{D}}{\partial y^*}\right) = -k^2 y^* \theta \left[\tilde{\mathrm{D}}, \Xi\right] + \left[\frac{\partial \mathrm{D}}{\partial \theta}, \tilde{\mathrm{D}}\right], \tag{79}$$

where

$$\Xi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{80}$$

Proof To prove the theorem consider equations (45) and (69) written for matrix $(\bar{V}_{m,n})$ with $p = 0$ and use the obvious property

$$\frac{\partial}{\partial \theta}\frac{\partial}{\partial y^*}\begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix} = \frac{\partial}{\partial y^*}\frac{\partial}{\partial \theta}\begin{pmatrix} \bar{V}_{0,0} & \bar{V}_{0,1} \\ \bar{V}_{1,0} & \bar{V}_{1,1} \end{pmatrix}. \tag{81}$$

After a simple algebra obtain matrix equation

$$\frac{1}{ik\theta}\frac{\partial}{\partial y^*}\left(\Pi\frac{\partial \mathrm{D}}{\partial\theta}\Pi^{-1}\right) = \frac{\partial}{\partial\theta}\left(\Pi\tilde{\mathrm{D}}\Pi^{-1}\right) + \frac{1}{ik\theta}\Pi\left[\frac{\partial \mathrm{D}}{\partial\theta},\tilde{\mathrm{D}}\right]\Pi^{-1}. \tag{82}$$

Multiply the equation by Π^{-1} on the left and by Π on the right. Also, it is taken into account that

$$\Pi^{-1}\frac{\partial\Pi}{\partial y^*} = -ik\theta\Xi, \qquad \frac{\partial\Pi^{-1}}{\partial y^*}\Pi = ik\theta\Xi. \tag{83}$$

Thereby obtain (79). □

Evolution equation of the second kind can be integrated at $y^* = 0$. The result is

$$\frac{\partial \mathrm{D}(\theta,0)}{\partial y^*} = \begin{pmatrix} (D_{0,1})^2 & 0 \\ 0 & -(D_{0,1})^2 \end{pmatrix}. \tag{84}$$

The value of $D_{0,1}(\theta,0)$ can be found with the help of (60):

$$D_{0,1}(\theta,0) = \sum_{m=0}^{\infty} z_{0,2m+1}\exp\left\{ik(x_{2m+1})\theta^2/2\right\}, \tag{85}$$

The values $z_{0,n}$ at $y^* = 0$ are represented by the following nested integrals:

$$z_{0,n} = \tag{86}$$

$$\int_0^\infty \ldots \int_0^\infty g(a,y_1)\, g(a,y_2-y_1)\ldots g(a,y_n-y_{n-1})\, g(x-x_n,y-y_n)\, dy_1\ldots dy_n.$$

Integrals of the class of (86) are calculated explicitly in [6]:

$$D_{0,1}(\theta,0) = \sqrt{\frac{k}{2\pi a i}}\sum_{n=1}^{\infty}\left(\frac{e^{inka\theta^2/2}}{n^{3/2}} - \frac{e^{inka\theta^2}}{(2n)^{3/2}}\right). \tag{87}$$

It follows from (84) that

$$\frac{\partial\tilde{\mathrm{D}}}{\partial y^*}(\theta,0) = 0. \tag{88}$$

Therefore matrix $\tilde{\mathrm{D}}$ obeys the following asymptotics for $y^*\sqrt{k/a} \ll 1$:

$$\tilde{\mathrm{D}}(\theta,y^*) = \tilde{\mathrm{D}}(\theta,0) + O\left(\left(y^*\sqrt{k/a}\right)^2\right), \tag{89}$$

where

$$\tilde{\mathrm{D}}(\theta,0) = \begin{pmatrix} 0 & D_{0,1}(\theta,0) \\ -D_{0,1}(\theta,0) & 0 \end{pmatrix}. \tag{90}$$

10 Asymptotic estimation of the coefficient R_0

Introduce the following notation:

$$\eta = y^* \sqrt{k/a}. \tag{91}$$

Consider again the evolution equation of the first kind (63). Search the solution as an expansion in a small parameter η:

$$V_m(\theta, y^*) = V^{(0)}(\theta) + \eta V_m^{(1)}(\theta) + \eta^2 V_m^{(2)}(\theta) + ..., \qquad m = 0, 1, \tag{92}$$

where $V^{(0)}$ is a solution of the classical Weinstein problem ($y^* = 0$), which can be calculated explicitly and which has the following form [4]:

$$V^0 = \exp\left\{ -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\mathrm{erfc}(\theta\sqrt{-in/2ka})}{n} \right\}, \tag{93}$$

where erfc(z) is the complementary error function:

$$\mathrm{erfc(z)} = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\tau^2} d\tau.$$

Substitute (89) and (92) into (63) and expand in η. In the first approximation we get:

$$V_m = V^0 - (-1)^m D_{0,1}(\theta, 0) V^0(\theta)\eta + ... \tag{94}$$

One can notice (by substituting (94) into embedding formula (31)) that the term having first order in η does not contribute to the scattering coefficient R_0. This result is quite obvious from the geometrical point of view.

In order to compute the quadratic term (with respect to η) it is necessary to use relation (88). Due to (63),

$$\frac{\partial^2}{\partial(y^*)^2}(V_0, V_1) = (V_0, V_1)\Pi \left(\tilde{\mathrm{D}}^2 + \frac{\partial \tilde{\mathrm{D}}}{\partial y^*} + ik\theta[\tilde{\mathrm{D}}, \Xi] \right) \Pi^{-1}. \tag{95}$$

We are interested in the value of the second derivative at $y^* = 0$, therefore $\Pi = \mathrm{I}$. Also

$$\tilde{\mathrm{D}} = [\mathrm{D}, \Xi].$$

Using (95) and (88), we get the following expression in the second approximation

$$V_m = V^0 - (-1)^m D_{0,1}(\theta, 0) V^0(\theta)\eta - \frac{a}{k}\frac{\eta^2}{2} \left((D_{0,1}(\theta, 0))^2 - ik\theta D_{0,1}(\theta, 0) \right) V^0(\theta) + ... \tag{96}$$

Eventually by substituting last relation into the embedding formula we obtain the following expression for the reflection coefficient R_0:

$$R_0 = \frac{(V^0(\theta))^2 \, (1 + ia\theta D_{0,1}(\theta, 0)\eta^2)}{2ika\theta^2} + o(\eta^2). \tag{97}$$

For small angles θ formula (97) takes the following form:

$$R_0 = -1 - \zeta\left(\frac{1}{2}\right)\sqrt{\frac{ka}{\pi}}(1-i)\theta - \zeta\left(\frac{3}{2}\right)\sqrt{\frac{ka}{\pi}}(1+i)2^{-5/2}(2\sqrt{2}-1)\theta\eta^2 + o(\eta^2). \tag{98}$$

It follows from (98) that a high-frequency mode close to the cut-off in a waveguide with staggered walls has is reflected from the open end of the waveguide with the coefficient close to -1 provided that $y^* \sqrt{k/a} \ll 1$. This follows from the fact that the term containing η^2 is proportional to θ. This fact is similar to the Weinstein's result.

The authors suppose that the coefficient $R_0(\theta, y^*)$ tends to -1 for arbitrary y^* as $\theta \to 0$. However, this statement cannot be proved by the technique developed here.

11 Numerical results

Here we determine the coefficient $\mathrm{C}\left(\sqrt{\theta^2 + 2p}\right)$ by direct solving of the OE-equation (55).

Introduce new variables

$$\tau = ka\theta^2 + \beta, \quad \beta = 2kap.$$

OE-equation changes as follows:

$$\mathrm{OE}_{\gamma'(\beta)}\left[\frac{\mathrm{B}(\tau, \beta)d\tau}{2\sqrt{ka(\tau - \beta)}}\right] = \mathrm{T}'(\beta), \quad \mathrm{Im}[\beta] \geq 0, \tag{99}$$

where

$$\mathrm{B}(\tau, \beta) = \Pi\left(\sqrt{\frac{\tau - \beta}{ka}}\right) \mathrm{C}'(\tau)\, \Pi^{-1}\left(\sqrt{\frac{\tau - \beta}{ka}}\right),$$

$$\mathrm{T}'(\beta) = \begin{pmatrix} 1 & -\exp\{i\beta/2\} \\ -\exp\{i\beta/2\} & 1 \end{pmatrix}, \tag{100}$$

$$\mathrm{C}'(\tau) = \mathrm{C}\left(\sqrt{\tau/(ka)}\right).$$

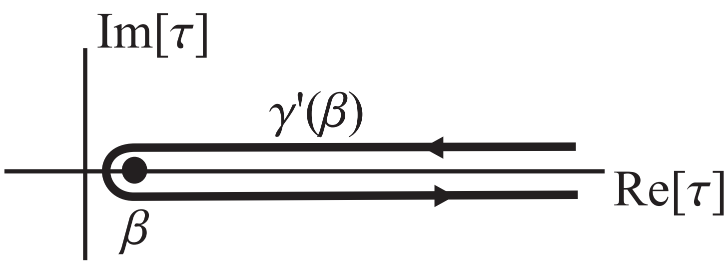

Fig. 4: *Contour* $\gamma'(\beta)$.

and τ is going along contour γ' shown in Fig. 4.

It is necessary to determine $\mathrm{C}'(\tau)$ on the positive half of the real axis in order to calculate reflection coefficients R_m. However, matrix $\mathrm{C}'(\tau)$ transforms as follows:

$$\mathrm{C}'(\tau + 2\pi n) = \Pi_n \mathrm{C}'(\tau) \Pi_n^{-1}, \tag{101}$$

where

$$\Pi_n = \begin{pmatrix} 1 & 0 \\ 0 & \exp\{-i\pi n\} \end{pmatrix}, \tag{102}$$

therefore it is sufficient to solve the OE-equation only on the segment $(0, 2\pi)$. OE-equation is solved along contours Γ_1 and Γ_2 shown in Fig. 5. We solve the

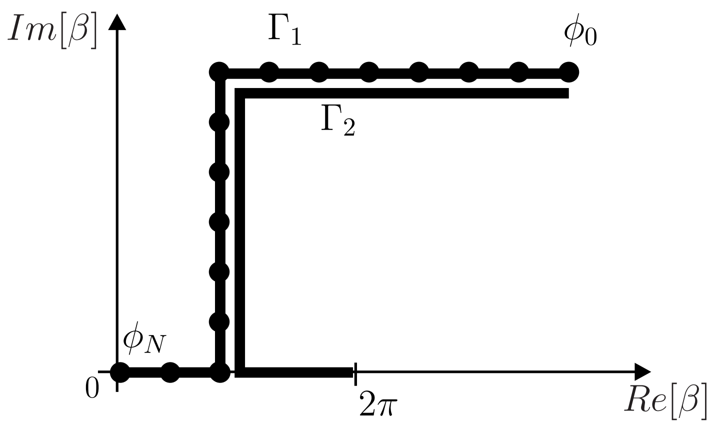

Fig. 5: *Контуры* Γ_1 *и* Γ_2.

OE-equation using a gradient procedure. Both contours start at the point ϕ_0 having large imaginary part. Therefore we set

$$\mathrm{C}'(\phi_0) = 0.$$

Contour Γ_1 is divided into small segments by nodes ϕ_n, $n = 0, 1, 2, ..., N$. Let $\phi_N = \epsilon$, where ϵ is a small positive value. (We choose a nonzero ϵ because the OE-equation has singularities at the points $\beta = \pi n$, $n \in \mathbb{Z}$.) Then we find C step by step at the points $\phi_1, \phi_2, \phi_3, ...\phi_N$. To do this we have to solve the OE-equation (99) with parameter $\beta = \phi_n$. We suppose that $\mathrm{C}'(\tau)$ is known at the nodes $\phi_1, \phi_2, \phi_3, ...\phi_{n-1}$. The coefficient at the node ϕ_n is found by the gradient procedure. The starting value for gradient procedure is chosen as follows

$$\mathrm{C}'(\tau)(\phi_0) = \mathrm{C}'(\tau)(\phi_{\mathrm{n}-1}), \tag{103}$$

i. e. we choose the value of the previous point as the starting approximation. The residual is calculated:

$$\mathrm{OE}_\gamma \left[\frac{\mathrm{B}(\tau, \phi_n)}{2\sqrt{ka(\tau - \phi_n)}} \right] - \mathrm{T}(\phi_n) \tag{104}$$

Using the gradient procedure we make the residual small enough. Then we move to the next step and so on.

The same procedure is repeated on contour Γ_2. Coefficient $\mathrm{C}'(\tau)$ becomes known on the segment $(0, 2\pi)$. Then the spectral equation (36) is solved along contours $\Gamma_{1,2}$, $\Gamma_{1,2} + 2\pi$, $\Gamma_{1,2} + 4\pi$ etc. Directivities V_0, V_1 become determined. Finally by applying the embedding formula (31) we find the scattering coefficients R_m.

To verify the validity of the numerical results we use direct calculation based on formulae similar to (28). Numerical results are shown in Fig. 6 and Fig. 7. Fig. 6 displays the dependence of the scattering coefficient R_0 on $p = ak\theta_{in}^2$ for

$$y^*/\sqrt{(a/k)} = 1/4.$$

Fig. 7 displays the same dependence for

$$y^*/\sqrt{(a/k)} = 1/3.$$

Solid lines correspond to calculations performed using the spectral equation method. The results of the direct computations are plotted by dotted lines. One can see that the agreement is reasonable. The embedding formula is not checked, since it is assumed to be a well-established result in the diffraction theory (e. g. see [7]).

12 Conclusion

The method of spectral equation and embedding formula [5] is applied in the present paper to a problem of diffraction by two-height grating. Some

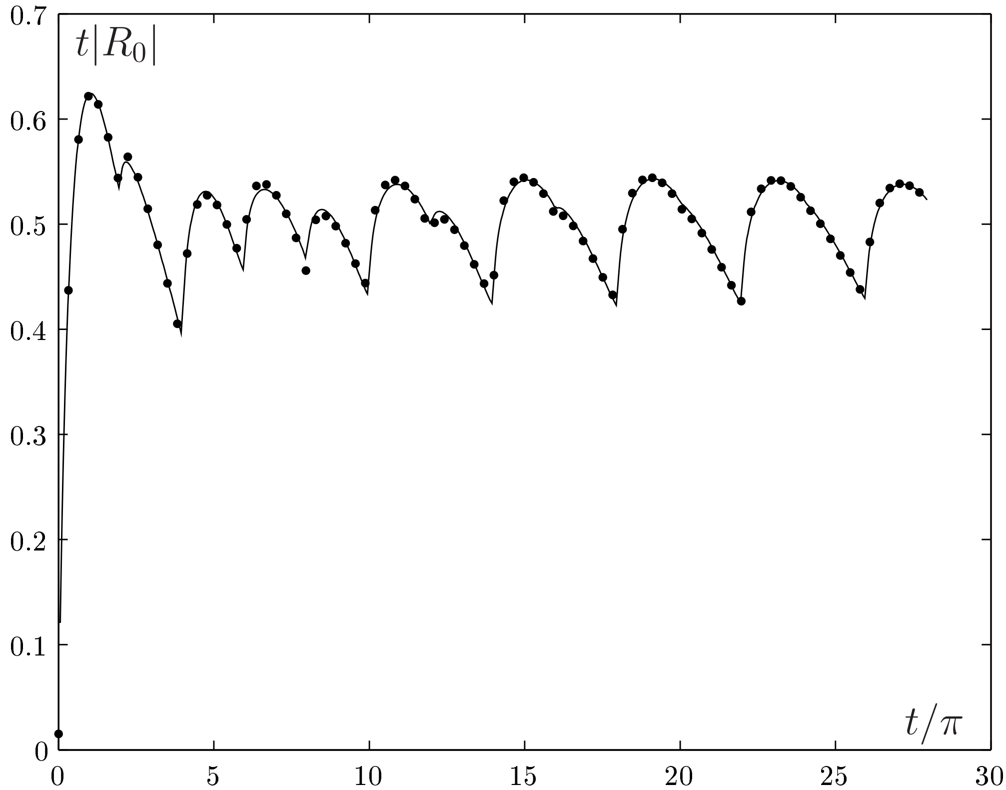

Fig. 6: *Dependence* $p|R_0(p)|$ *on* p *for* $y^*/\sqrt{(a/k)} = 1/4$. *The solid line corresponds to spectral equation method and the dotted line corresponds to direct computations*

important results are obtained using this technique. Namely, an asymptotic formula (98) for the scattering coefficient R_0 obtained using the evolution equation is the main result of the paper. This formula allows one to calculate Q-factor of Fabry–Perot resonators with staggered mirrors or mirrors of different height under the assumptions of small displacement ($y^*\sqrt{k/a} \ll 1$).

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References

[1] L. A. Weinstein, Open resonators ans open waveguides. Golem Press, Boulder, Colo.,1969.

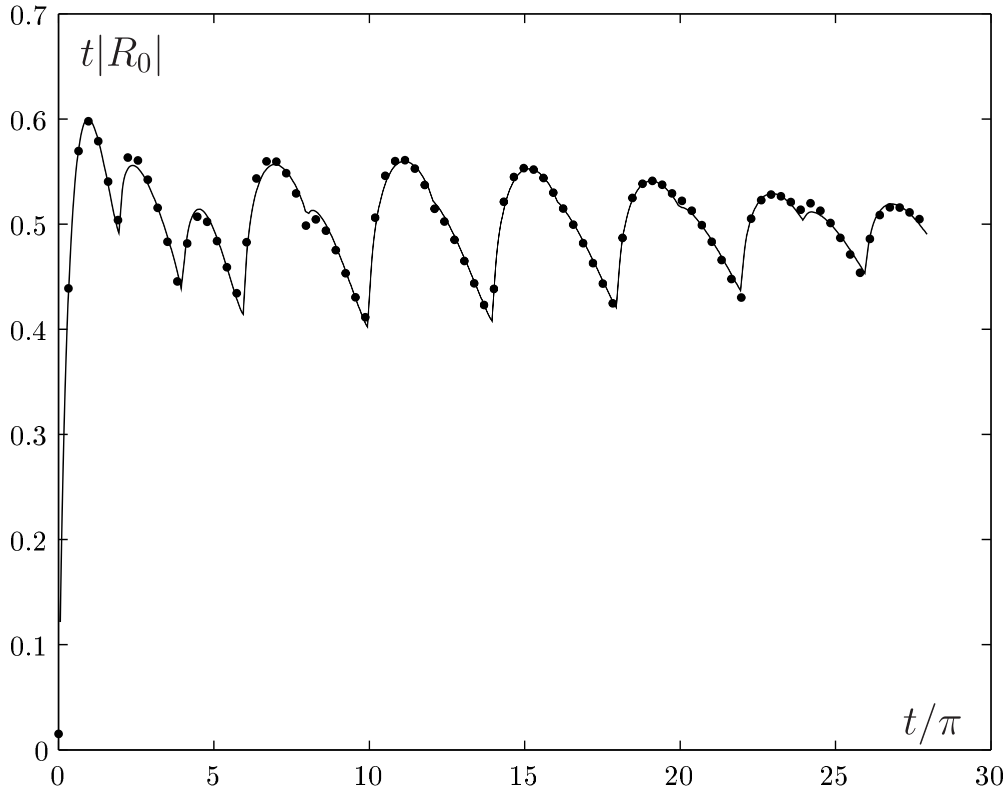

Fig. 7: *Dependence* $p|R_0(p)|$ *on* p *for* $y^*/\sqrt{(a/k)} = 1/3$. *The solid line corresponds to spectral equation method and the dotted line corresponds to direct computations*

[2] L. A. Weinstein, The theory of Diffraction and the Factorization Method. Golem Press, Boulder, Colo.,1969.

[3] E. D. Shabalina, N. V. Shirgina, and A. V. Shanin, High Frequency Modes in a Two Dimensional Rectangular Room with Windows, Acoustical Physics, 56(2010), pp. 525 - 536.

[4] A. V. Shanin, Weinstein's Difraction Problem: Embedding Formula and Spectral Equation in Parabolic Approximation, SIAM J. Appl. Math. (2009), pp. 1201-1218.

[5] A. V. Shanin, Diffraction of a high-frequency grazing wave by a grating with a complicated period, Zap. nauch. sem. POMI RAN, 409(2012), p. 212-239.

[6] J. Boersma, On certain multiple integrals occuring in a waveguide scattering problem, SIAM J. Math. Anal., 9(1978), pp. 377-393.

[7] C. M. Linton, P. McIver, Handbook of Mathematical Techniques for Wave/Structure Interactions, Chapman & Hall/CRC, 2001.